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Article

Geometric properties of new subclasses of analytic functions defined by Opoola differential operator

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Abstract: In this research, we utilize the Opoola differential operator to define new subclasses of starlike and convex functions within the unit disk U: $S^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma)$, $K^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma)$, $T^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma)$, and $C^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma)$, characterized by parameters α, η, and γ, which denote their order and type. We investigate various geometric properties of these functions, including characterization properties, growth and distortion theorems, arithmetic mean, and radius of convexity. The results obtained generalize many existing findings, forming a foundation for further research in the theory of geometric functions. Additionally, we present several corollaries and remarks to illustrate extensions of our results.

Keywords: Analytic functions, Univalent functions starlike functions , convex functions, close-to-convex functions , differential operator.

MSC: 30C45, 30C50 , 30C55.

1. Introduction

Let A denote the class of functions $f(z)$ that are analytic in the unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}. \tag{1}$$

Let S be the subclass of A consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{2}$$

which are univalent in U. Furthermore, let T be the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad \text{with } a_n \geq 0.$$

The class $S^*(\alpha)$ of starlike functions of order α $(0 \leq \alpha < 1)$ is defined as

$$S^*(\alpha) = \left\{ f \in A : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in U \right\}.$$

In particular, $S^*(0) = S^*$ is the class of starlike functions with respect to the origin.

The class $K(\alpha)$ of convex functions of order α $(0 \leq \alpha < 1)$ is defined by

$$K(\alpha) = \left\{ f \in A : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in U \right\},$$

or equivalently,

$$K(\alpha) = \left\{ f \in A : zf'(z) \in S^*(\alpha), \quad z \in U \right\},$$

as introduced by Robertson [1]. Note that $S^*(0) = S^*$, where S^* represents the class of functions $f \in A$ such that $f(U)$ is starlike with respect to the origin. Similarly, $K(0) = K$, the well-known class of convex functions. It is a well-established fact that $f \in K(\alpha)$ if and only if $zf' \in S^*(\alpha)$.

In [2], Opoola introduced the following differential operator $D^m(\mu, \beta, t) : A \to A$, defined as:

$$D^0(\mu, \beta, t)f(z) = f(z),$$

$$D^1(\mu, \beta, t)f(z) = zD_tf(z) = tzf'(z) - z(\beta - \mu)t + [1 + (\beta - \mu - 1)t]\, f(z),$$

and, for $m \in \mathbb{N}$,

$$D^m(\mu, \beta, t)f(z) = zD_t\big(D^{m-1}(\mu, \beta, t)f(z)\big). \tag{3}$$

If $f(z)$ is given by (1), then from (3) we have

$$D^m(\mu, \beta, t)f(z) = z + \sum_{n=2}^{\infty} [1 + (n + \beta - \mu - 1)\, t]^m\, a_n z^n, \tag{4}$$

where $0 \le \mu \le \beta$, $\beta \ge 0$, $t \ge 0$, and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

- When $\beta = \mu$ and $t = 1$, $D^n(\mu, \mu, 1)f(z) = D^n f(z)$ was introduced by Salagean [3].
- When $\beta = \mu$ and $t = \lambda$, $D^n(\mu, \mu, \lambda)f(z) = D^n_\lambda f(z)$ was defined by Al-Oboudi [4].

The study of geometric properties of analytic functions plays a vital role in complex analysis. Researchers have extensively investigated properties such as radii of starlikeness, convexity, close-to-convexity, and growth and distortion theorems.

Gupta and Jain [5] introduced the subclasses $S(\alpha, \beta)$ and $C(\alpha, \beta)$ for $f \in A$ and established geometric properties such as coefficient inequalities, growth and distortion theorems, closure under arithmetic mean and linear combinations, integral representations, and extreme point theorems. Kulkarni [6] further extended this by introducing the subclass $S(\alpha, \beta, \eta)$, proving similar geometric properties and deriving integral representations.

Sambo and Opoola [7] studied a subclass of analytic functions, obtaining characterization properties such as radii of starlikeness, convexity, close-to-convexity, and growth and distortion results.

Motivated by these works and contributions from [8–26], we define new subclasses of analytic functions and derive their geometric properties, including characterization properties, growth and distortion theorems, arithmetic means, and radii of convexity. Our results generalize several existing findings. We introduce new subclasses of starlike and convex functions as follows:

Definition 1. A function $f(z)$ of the form (1) is said to belong to the class $S^{m,t}_{\beta,\mu}(\alpha, \eta, \gamma)$ if it satisfies the following condition:

$$\left| \frac{\frac{z(D^m(\mu,\beta,t)f(z))'}{D^m(\mu,\beta,t)f(z)} - 1}{(2\gamma - 1)\frac{z(D^m(\mu,\beta,t)f(z))'}{D^m(\mu,\beta,t)f(z)} + (1 - 2\gamma\alpha)} \right| < \eta, \tag{5}$$

where $0 \le \alpha < 1$, $0 < \eta \le 1$, $\frac{1}{2} < \gamma \le 1$, $\beta \ge 0$, $0 \le \mu \le \beta$, and $D^m(\mu, \beta, t)f(z)$ is the Opoola differential operator defined in (4).

Definition 2. A function $f(z)$ of the form (1) is said to belong to the class $K^{m,t}_{\beta,\mu}(\alpha, \eta, \gamma)$ if it satisfies the following condition:

$$\left| \frac{\frac{z(D^m(\mu,\beta,t)f(z))''}{(D^m(\mu,\beta,t)f(z))'}}{(2\gamma - 1)\frac{z(D^m(\mu,\beta,t)f(z))''}{(D^m(\mu,\beta,t)f(z))'} + 2\gamma(1 - \alpha)} \right| < \eta, \tag{6}$$

where $0 \le \alpha < 1$, $0 < \eta \le 1$, $\frac{1}{2} < \gamma \le 1$, $\beta \ge 0$, $0 \le \mu \le \beta$, and $D^m(\mu, \beta, t)f(z)$ is the Opoola differential operator defined in (4).

Let $T^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma)$ and $C^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma)$ denote subclasses of T in (2), defined as:

$$T^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma) = S^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma) \cap T,$$
$$C^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma) = K^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma) \cap T.$$

Remark 1.
1. $S^{0,t}_{\beta,\mu}(\alpha,\eta,\gamma) = S(\alpha,\eta,\gamma)$, studied by Kulkarni [6].
2. $K^{0,t}_{\beta,\mu}(\alpha,\eta,\gamma) = K(\alpha,\eta,\gamma)$, examined by Joshi and Shelake [27].
3. $S^{0,t}_{\beta,\mu}(\alpha,\eta,1) = S(\alpha,\eta)$ and $K^{0,t}_{\beta,\mu}(\alpha,\eta,1) = K(\alpha,\eta)$ are well-known subclasses of starlike and convex functions of order α and type β, respectively, introduced by Gupta and Jain [5].
4. $S^{0,t}_{\beta,\mu}(\alpha,1,1) = S^*(\alpha)$ and $K^{0,t}_{\beta,\mu}(\alpha,1,1) = K(\alpha)$ are well-known subclasses of starlike and convex functions of order α, introduced by Robertson [1], MacGregor [28], and Schild [29].
5. $T^{0,t}_{\beta,\mu}(\alpha,\eta,\gamma) = T^*(\alpha,\gamma,\beta)$ and $C^{0,t}_{\beta,\mu}(\alpha,\eta,\gamma) = C(\alpha,\gamma,\beta)$, as studied by Shelake et al. [30].
6. $T^{0,t}_{\beta,\mu}(\alpha,1,1) = T^*(\alpha)$ and $C^{0,t}_{\beta,\mu}(\alpha,1,1) = C(\alpha)$ are subclasses of starlike and convex functions of order α with negative coefficients, introduced by Silverman [31].

2. Main Results

This section presents the main results of this study.

2.1. Characterization Properties of the Classes: $S^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma), T^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma), K^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma), C^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma)$

Theorem 3. *A function $f(z)$ of the form (1) belongs to the class $S^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma)$ if*

$$\sum_{n=2}^{\infty} [n-1+\eta(1-n+2\gamma n-2\gamma\alpha)]\, M^m_n(\mu,\beta,t)\,|a_n| \le 2\eta\gamma(1-\alpha), \tag{7}$$

where $M^m_n(\mu,\beta,t) = [1+(n+\beta-\mu-1)t]^m$. The result in (7) is sharp for functions of the form

$$f(z) = z + \frac{2\eta\gamma(1-\alpha)}{[n-1+\eta(1-n+2\gamma n-2\gamma\alpha)]\, M^m_n(\mu,\beta,t)} z^n, \quad n \ge 2. \tag{8}$$

Proof. Suppose that (7) holds. For $|z| = 1$, we have

$$\begin{aligned}
&\left| z\left(D^m(\mu,\beta,t)f(z)\right)' - D^m(\mu,\beta,t)f(z) \right| - \eta \left| (2\gamma-1)z\left(D^m(\mu,\beta,t)f(z)\right)' + (1-2\gamma\alpha)D^m(\mu,\beta,t)f(z) \right| \\
&\le \left| z\left(D^m(\mu,\beta,t)f(z)\right)' - D^m(\mu,\beta,t)f(z) - \eta \left\{ (2\gamma-1)z\left(D^m(\mu,\beta,t)f(z)\right)' + (1-2\gamma\alpha)D^m(\mu,\beta,t)f(z) \right\} \right| \\
&= \left| z\left(1+\sum_{n=2}^{\infty} nM^m_n(\mu,\beta,t)a_n z^{n-1}\right) - \left(z+\sum_{n=2}^{\infty} M^m_n(\mu,\beta,t)a_n z^n\right) \right. \\
&\quad \left. -\eta\left\{(2\gamma-1)z\left(1+\sum_{n=2}^{\infty} nM^m_n(\mu,\beta,t)a_n z^{n-1}\right) + (1-2\gamma\alpha)\left(z+\sum_{n=2}^{\infty} M^m_n(\mu,\beta,t)a_n z^n\right)\right\} \right| \\
&= \left| \sum_{n=2}^{\infty} [(n-1)M^m_n(\mu,\beta,t)a_n z^n + (\eta-2\eta\gamma)] - (2\eta\gamma-\eta)\sum_{n=2}^{\infty} nM^m_n(\mu,\beta,t)a_n z^n + (2\eta\gamma\alpha-\eta)z \right. \\
&\quad \left. + (\eta-2\eta\gamma\alpha)\sum_{n=2}^{\infty} M^m_n(\mu,\beta,t)a_n z^n \right| \\
&\le \sum_{n=2}^{\infty} [n-1+\eta(1-n+2\gamma n-2\gamma\alpha)]\, M^m_n(\mu,\beta,t)\,|a_n| - 2\eta\gamma(1-\alpha) \le 0.
\end{aligned} \tag{9}$$

By the maximum modulus theorem,

$$\sum_{n=2}^{\infty} [n-1+\eta(1-n+2\gamma n-2\gamma\alpha)]\, M^m_n(\mu,\beta,t)\,|a_n| - 2\eta\gamma(1-\alpha) < 0, \tag{10}$$

which implies

$$\left|z\left(D^m(\mu,\beta,t)f(z)\right)' - D^m(\mu,\beta,t)f(z)\right| - \eta\left|(2\gamma-1)z\left(D^m(\mu,\beta,t)f(z)\right)' + (1-2\gamma\alpha)D^m(\mu,\beta,t)f(z)\right| < 0.$$

Therefore, $f(z) \in S^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma)$. This completes the proof. □

Theorem 4. *If a function $f(z)$ of the form (2) is in the class $T^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma)$, then*

$$\sum_{n=2}^{\infty}[n-1+\eta(1-n+2\gamma n-2\gamma\alpha)]M_n^m(\mu,\beta,t)a_n \leq 2\eta\gamma(1-\alpha). \tag{11}$$

The result in (11) is sharp for functions of the form

$$f(z) = z - \frac{2\eta\gamma(1-\alpha)}{[n-1+\eta(1-n+2\gamma n-2\gamma\alpha)]M_n^m(\mu,\beta,t)}z^n, \qquad n \geq 2. \tag{12}$$

Proof. It suffices to prove and only if part. If $f(z)$ in (2) belongs to the class $T^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma)$, then

$$\begin{aligned}
&\left|\frac{\frac{z(D^m(\mu,\beta,t)f(z))'}{D^m(\mu,\beta,t)f(z)}-1}{(2\gamma-1)\frac{z(D^m(\mu,\beta,t)f(z))'}{D^m(\mu,\beta,t)f(z)}+(1-2\gamma\alpha)}\right| \\
&= \left|\frac{-\sum_{n=2}^{\infty}nM_n^m(\mu,\beta,t)\left|a_n\right|z^{n-1}+\sum_{n=2}^{\infty}M_n^m(\mu,\beta,t)\left|a_n\right|z^{n-1}}{(2\gamma-1)(1-\sum_{n=2}^{\infty}nM_n^m(\mu,\beta,t)\left|a_n\right|z^{n-1})+(1-2\gamma\alpha)(1-\sum_{n=2}^{\infty}M_n^m(\mu,\beta,t)\left|a_n\right|z^{n-1})}\right| \\
&= \left|\frac{\sum_{n=2}^{\infty}nM_n^m(\mu,\beta,t)\left|a_n\right|z^{n-1}-\sum_{n=2}^{\infty}M_n^m(\mu,\beta,t)\left|a_n\right|z^{n-1}}{(2\gamma-1)(1-\sum_{n=2}^{\infty}nM_n^m(\mu,\beta,t)\left|a_n\right|z^{n-1})+(1-2\gamma\alpha)(1-\sum_{n=2}^{\infty}M_n^m(\mu,\beta,t)\left|a_n\right|z^{n-1})}\right| < \eta.
\end{aligned} \tag{13}$$

We know that $\Re z \leq |z|$, so that

$$\Re\left\{\frac{\sum_{n=2}^{\infty}nM_n^m(\mu,\beta,t)\left|a_n\right|z^{n-1}-\sum_{n=2}^{\infty}M_n^m(\mu,\beta,t)\left|a_n\right|z^{n-1}}{(2\gamma-1)(1-\sum_{n=2}^{\infty}nM_n^m(\mu,\beta,t)\left|a_n\right|z^{n-1})+(1-2\gamma\alpha)(1-\sum_{n=2}^{\infty}M_n^m(\mu,\beta,t)\left|a_n\right|z^{n-1})}\right\} < \eta. \tag{14}$$

Taking values of z on the real line and making $z \to 1$, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}nM_n^m(\mu,\beta,t)\left|a_n\right| - \sum_{n=2}^{\infty}M_n^m(\mu,\beta,t)\left|a_n\right| \\
&\leq \eta\left\{(2\gamma-1)\left(1-\sum_{n=2}^{\infty}nM_n^m(\mu,\beta,t)\left|a_n\right|\right)+(-2\gamma\alpha)\left(1-\sum_{n=2}^{\infty}M_n^m(\mu,\beta,t)\left|a_n\right|\right)\right\} \\
&\Rightarrow \sum_{n=2}^{\infty}nM_n^m(\mu,\beta,t)\left|a_n\right| - \sum_{n=2}^{\infty}M_n^m(\mu,\beta,t)\left|a_n\right| + (2\eta\gamma-\eta)\sum_{n=2}^{\infty}nM_n^m(\mu,\beta,t)\left|a_n\right| \\
&\quad + (\eta-2\eta\gamma\alpha)\sum_{n=2}^{\infty}M_n^m(\mu,\beta,t)\left|a_n\right| \leq (2\eta\gamma-\eta)+(\eta-2\eta\gamma\alpha) \\
&\Rightarrow \sum_{n=2}^{\infty}\left[n-1+\eta(1-n+2\gamma n-2\gamma\alpha)\right]M_n^m(\mu,\beta,t)a_n \leq 2\eta\gamma(1-\alpha).
\end{aligned} \tag{15}$$

Which is the required result. □

Theorem 5. *A function $f(z)$ of the form (1) is in the class $K^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma)$ if*

$$\sum_{n=2}^{\infty}n[n-1+\eta(1-n+2\gamma n-2\gamma\alpha)]M_n^m(\mu,\beta,t)\left|a_n\right| \leq 2\eta\gamma(1-\alpha). \tag{16}$$

The result in (16) is sharp for functions of the form

$$f(z) = z + \frac{2\eta\gamma(1-\alpha)}{n[n-1+\eta(1-n+2\gamma n-2\gamma\alpha)]M_n^m(\mu,\beta,t)} z^n, \qquad n \geq 2. \tag{17}$$

Proof. Suppose (16) holds. Taking $|z| = 1$, and using the inequality $|A| - |B| \leq |A - B|$, we have:

$$\begin{aligned}
&\left|z(D^m(\mu,\beta,t)f(z))''\right| - \eta\left|(2\gamma-1)z(D^m(\mu,\beta,t)f(z))'' + 2\gamma(1-\alpha)(D^m(\mu,\beta,t)f(z))'\right| \\
&\leq \left|z(D^m(\mu,\beta,t)f(z))'' - \eta\left\{(2\gamma-1)z(D^m(\mu,\beta,t)f(z))'' + 2\gamma(1-\alpha)(D^m(\mu,\beta,t)f(z))'\right\}\right| \\
&= \Bigg|z\left(\sum_{n=2}^{\infty} n(n-1)M_n^m(\mu,\beta,t)a_n z^{n-2}\right) \\
&\quad -\eta\left\{(2\gamma-1)z\left(\sum_{n=2}^{\infty} n(n-1)M_n^m(\mu,\beta,t)a_n z^{n-2}\right) + 2\gamma(1-\alpha)\left(1+\sum_{n=2}^{\infty} nM_n^m(\mu,\beta,t)a_n z^{n-1}\right)\right\}\Bigg| \\
&\leq \sum_{n=2}^{\infty} n(n-1)M_n^m(\mu,\beta,t)|a_n||z|^{n-1} + (2\eta\gamma-\eta)\sum_{n=2}^{\infty} n(n-1)M_n^m(\mu,\beta,t)|a_n||z|^{n-1} \\
&\quad + 2\eta\gamma(\alpha-1) + 2\eta\gamma(1-\alpha)\sum_{n=2}^{\infty} nM_n^m(\mu,\beta,t)|a_n||z|^{n-1} \\
&= \sum_{n=2}^{\infty} n\left[n-1+\eta(1-n+2\gamma n-2\gamma\alpha)\right]M_n^m(\mu,\beta,t)|a_n| - 2\eta\gamma(1-\alpha) \\
&\leq 0.
\end{aligned} \tag{18}$$

By the Maximum Modulus Theorem, we obtain:

$$\sum_{n=2}^{\infty} n\left[n-1+\eta(1-n+2\gamma n-2\gamma\alpha)\right]M_n^m(\mu,\beta,t)|a_n| - 2\eta\gamma(1-\alpha) < 0. \tag{19}$$

Hence, we have:

$$\left|z(D^m(\mu,\beta,t)f(z))''\right| - \eta\left|(2\gamma-1)z(D^m(\mu,\beta,t)f(z))'' + 2\gamma(1-\alpha)(D^m(\mu,\beta,t)f(z))'\right| < 0.$$

Therefore, $f(z) \in K_{\beta,\mu}^{m,t}(\alpha,\eta,\gamma)$. The proof is complete. □

Theorem 6. *If a function $f(z)$ of the form (2) is in the class $C_{\beta,\mu}^{m,t}(\alpha,\eta,\gamma)$, then*

$$\sum_{n=2}^{\infty} n[n-1+\eta(1-n+2\gamma n-2\gamma\alpha)]M_n^m(\mu,\beta,t)a_n \leq 2\eta\gamma(1-\alpha). \tag{20}$$

The result in (20) is sharp for functions of the form

$$f(z) = z - \frac{2\eta\gamma(1-\alpha)}{n[n-1+\eta(1-n+2\gamma n-2\gamma\alpha)]M_n^m(\mu,\beta,t)} z^n, \qquad n \geq 2. \tag{21}$$

Proof. It suffices to prove the if part only. Assume $f(z)$ in (2) belongs to the class $C_{\beta,\mu}^{m,t}(\alpha,\eta,\gamma)$, then

$$\begin{aligned}
&\left|\frac{\frac{z(D^m(\mu,\beta,t)f(z))''}{(D^m(\mu,\beta,t)f(z))'}}{(2\gamma-1)\frac{z(D^m(\mu,\beta,t)f(z))''}{(D^m(\mu,\beta,t)f(z))'} + 2\gamma(1-\alpha)}\right| \\
&= \left|\frac{-\sum_{n=2}^{\infty} n(n-1)M_n^m(\mu,\beta,t)\,|a_n|\,z^{n-1}}{(2\gamma-1)(-\sum_{n=2}^{\infty} n(n-1)M_n^m(\mu,\beta,t)\,|a_n|\,z^{n-1}) + 2\gamma(1-\alpha)(1-\sum_{n=2}^{\infty} nM_n^m(\mu,\beta,t)\,|a_n|\,z^{n-1})}\right| \\
&= \left|\frac{\sum_{n=2}^{\infty} n(n-1)M_n^m(\mu,\beta,t)\,|a_n|\,z^{n-1}}{(2\gamma-1)(-\sum_{n=2}^{\infty} n(n-1)M_n^m(\mu,\beta,t)\,|a_n|\,z^{n-1}) + 2\gamma(1-\alpha)(1-\sum_{n=2}^{\infty} nM_n^m(\mu,\beta,t)\,|a_n|\,z^{n-1})}\right| < \eta.
\end{aligned} \tag{22}$$

We know that $\Re z \leq |z|$, so

$$\Re\left\{\frac{\sum_{n=2}^{\infty} n(n-1)M_n^m(\mu,\beta,t)\,|a_n|\,z^{n-1}}{(2\gamma-1)(-\sum_{n=2}^{\infty} n(n-1)M_n^m(\mu,\beta,t)\,|a_n|\,z^{n-1})+2\gamma(1-\alpha)(1-\sum_{n=2}^{\infty} nM_n^m(\mu,\beta,t)\,|a_n|\,z^{n-1})}\right\} < \eta.$$

Taking values of z on the real line and making $z \to 1$, we have

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)M_n^m(\mu,\beta,t)\,|a_n| \leq \eta \Bigg\{ &(2\gamma-1)\left(-\sum_{n=2}^{\infty} n(n-1)M_n^m(\mu,\beta,t)\,|a_n|\right) \\ &+2\gamma(1-\alpha)\left(1-\sum_{n=2}^{\infty} nM_n^m(\mu,\beta,t)\,|a_n|\right)\Bigg\} \\ &= \sum_{n=2}^{\infty} n(n-1)M_n^m(\mu,\beta,t)\,|a_n| + (2\eta\gamma-\eta)\sum_{n=2}^{\infty} n(n-1)M_n^m(\mu,\beta,t)\,|a_n| \\ &+(2\eta\gamma-2\eta\gamma\alpha)\sum_{n=2}^{\infty} nM_n^m(\mu,\beta,t)\,|a_n| \\ &\leq 2\eta\gamma(1-\alpha), \\ &= \sum_{n=2}^{\infty} n\left[n-1+\eta(1-n+2\gamma n-2\gamma\alpha)\right]M_n^m(\mu,\beta,t)a_n \\ &\leq 2\eta\gamma(1-\alpha).\end{aligned}$$

Which is the required result. □

Remark 2.
1. When $m=0$ in Theorems 3 and 4 the results reduce to results obtained by Kulkarni [6].
2. When $m=0, \gamma=1$ in Theorem 4 the results reduce to results obtained by Gupta and Jain [5].
3. When $m=0$ in Theorems 5 and 6 the results reduce to results obtained by Joshi and Shelake [27].
4. When $m=0, \gamma=1$ in Theorem 6 the results reduce to results obtained by Gupta and Jain [5].

2.2. Growth and Distortion Theorem

Theorem 7. *If $f(z) \in T_{\beta,\mu}^{m,t}(\alpha,\eta,\gamma)$, then for $|z|=r$,*

$$r-\frac{2\eta\gamma(1-\alpha)}{[1-\eta+2\eta\gamma(2-\alpha)][1+(\beta-\mu+1)t]^m}r^2 \leq |f(z)| \leq r+\frac{2\eta\gamma(1-\alpha)}{[1-\eta+2\eta\gamma(2-\alpha)][1+(\beta-\mu+1)t]^m}r^2,$$
$$1-r\frac{4\eta\gamma(1-\alpha)}{[1-\eta+2\eta\gamma(2-\alpha)][1+(\beta-\mu+1)t]^m} \leq |f'(z)| \leq 1+r\frac{4\eta\gamma(1-\alpha)}{[1-\eta+2\eta\gamma(2-\alpha)][1+(\beta-\mu+1)t]^m}.$$

Proof. Let $|z|=r<1$. By Theorem 4, we have

$$\sum_{n=2}^{\infty}|a_n| \leq \frac{2\eta\gamma(1-\alpha)}{[1-\eta+2\eta\gamma(2-\alpha)][1+(\beta-\mu+1)t]^m},$$

and

$$\sum_{n=2}^{\infty}n|a_n| \leq \frac{4\eta\gamma(1-\alpha)}{[1-\eta+2\eta\gamma(2-\alpha)][1+(\beta-\mu+1)t]^m}.$$

Hence,

$$|f(z)| \leq |z|+\sum_{n=2}^{\infty}|a_n||z|^n,$$

and

$$|f(z)| \geq |z|-\sum_{n=2}^{\infty}|a_n||z|^n.$$

Therefore,

$$|f(z)| \leq r+r^2\sum_{n=2}^{\infty}|a_n| = r+r^2\frac{2\eta\gamma(1-\alpha)}{[1-\eta+2\eta\gamma(2-\alpha)][1+(\beta-\mu+1)t]^m},$$

and

$$|f(z)| \geq r - r^2 \sum_{n=2}^{\infty} |a_n| = r - r^2 \frac{2\eta\gamma(1-\alpha)}{[1-\eta+2\eta\gamma(2-\alpha)][1+(\beta-\mu+1)t]^m}.$$

Also,

$$|f'(z)| \leq 1 + \sum_{n=2}^{\infty} n|a_n||z|^{n-1},$$

and

$$|f'(z)| \geq 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1}.$$

Thus,

$$|f'(z)| \leq 1 + r \sum_{n=2}^{\infty} n|a_n| = 1 + r \frac{4\eta\gamma(1-\alpha)}{[1-\eta+2\eta\gamma(2-\alpha)][1+(\beta-\mu+1)t]^m},$$

and

$$|f'(z)| \geq 1 - r \sum_{n=2}^{\infty} n|a_n| = 1 - r \frac{4\eta\gamma(1-\alpha)}{[1-\eta+2\eta\gamma(2-\alpha)][1+(\beta-\mu+1)t]^m}.$$

Hence, the proof is complete. □

Theorem 8. *If $f(z) \in C_{\beta,\mu}^{m,t}(\alpha,\eta,\gamma)$, then for $|z| = r$,*

$$r - \frac{\eta\gamma(1-\alpha)}{[1-\eta+2\eta\gamma(2-\alpha)][1+(\beta-\mu+1)t]^m} r^2 \leq |f(z)| \leq r + \frac{\eta\gamma(1-\alpha)}{[1-\eta+2\eta\gamma(2-\alpha)][1+(\beta-\mu+1)t]^m} r^2,$$
$$1 - r \frac{2\eta\gamma(1-\alpha)}{[1-\eta+2\eta\gamma(2-\alpha)][1+(\beta-\mu+1)t]^m} \leq |f'(z)| \leq 1 + r \frac{2\eta\gamma(1-\alpha)}{[1-\eta+2\eta\gamma(2-\alpha)][1+(\beta-\mu+1)t]^m}.$$

Proof. Let $|z| = r < 1$. By Theorem 6, we have

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{\eta\gamma(1-\alpha)}{[1-\eta+2\eta\gamma(2-\alpha)][1+(\beta-\mu+1)t]^m}$$

and

$$\sum_{n=2}^{\infty} n|a_n| \leq \frac{2\eta\gamma(1-\alpha)}{[1-\eta+2\eta\gamma(2-\alpha)][1+(\beta-\mu+1)t]^m}.$$

Hence, we have the following inequalities for $f(z)$:

$$|f(z)| \leq |z| + \sum_{n=2}^{\infty} |a_n||z|^n \quad \text{and} \quad |f(z)| \geq |z| - \sum_{n=2}^{\infty} |a_n||z|^n.$$

Therefore,

$$|f(z)| \leq r + r^2 \sum_{n=2}^{\infty} |a_n| = r + r^2 \frac{\eta\gamma(1-\alpha)}{[1-\eta+2\eta\gamma(2-\alpha)][1+(\beta-\mu+1)t]^m}$$

and

$$|f(z)| \geq r - r^2 \sum_{n=2}^{\infty} |a_n| = r - r^2 \frac{\eta\gamma(1-\alpha)}{[1-\eta+2\eta\gamma(2-\alpha)][1+(\beta-\mu+1)t]^m}.$$

For the derivative $f'(z)$, we have

$$|f'(z)| \leq 1 + \sum_{n=2}^{\infty} n|a_n||z|^{n-1} \quad \text{and} \quad |f'(z)| \geq 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1}.$$

Thus,

$$|f'(z)| \leq 1 + r \sum_{n=2}^{\infty} n|a_n| = 1 + r \frac{2\eta\gamma(1-\alpha)}{[1-\eta+2\eta\gamma(2-\alpha)][1+(\beta-\mu+1)t]^m}$$

and

$$|f'(z)| \geq 1 - r\sum_{n=2}^{\infty} n|a_n| = 1 - r\frac{2\eta\gamma(1-\alpha)}{[1-\eta+2\eta\gamma(2-\alpha)][1+(\beta-\mu+1)t]^m}.$$

Hence, the proof is complete. □

2.3. Arithmetic Mean

We assert that the classes $T^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma)$ and $C^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma)$ are closed under arithmetic mean.

Theorem 9. *If $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ and $g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$ are in the class $T^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma)$ then,*

$$h(z) = z - \frac{1}{2}\sum_{n=2}^{\infty} |a_n + b_n| z^n \in T^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma).$$

Proof. Since $f(z)$ and $g(z) \in T^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma)$, from Theorem 4, we have

$$\sum_{n=2}^{\infty} [n-1+\eta(1-n+2\gamma n-2\gamma\alpha)]\, M^m_n(\mu,\beta,t)|a_n| \leq 2\eta\gamma(1-\alpha),$$
$$\sum_{n=2}^{\infty} [n-1+\eta(1-n+2\gamma n-2\gamma\alpha)]\, M^m_n(\mu,\beta,t)|b_n| \leq 2\eta\gamma(1-\alpha).$$

Therefore, we obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{n=2}^{\infty} [n-1+\eta(1-n+2\gamma n-2\gamma\alpha)]\, M^m_n(\mu,\beta,t)|a_n+b_n| \\
&\quad \leq \frac{1}{2}\sum_{n=2}^{\infty} [n-1+\eta(1-n+2\gamma n-2\gamma\alpha)]\, M^m_n(\mu,\beta,t)|a_n| \\
&\qquad + \frac{1}{2}\sum_{n=2}^{\infty} [n-1+\eta(1-n+2\gamma n-2\gamma\alpha)]\, M^m_n(\mu,\beta,t)|b_n| \\
&\quad = 2\eta\gamma(1-\alpha).
\end{aligned}$$

Hence, $h(z) \in T^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma)$, and the proof is complete. □

Theorem 10. *If $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ and $g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$ are in the class $C^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma)$ then,*

$$h(z) = z - \frac{1}{2}\sum_{n=2}^{\infty} |a_n + b_n| z^n \in C^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma).$$

Proof. The proof holds same as Theorem 9. □

2.4. Radius of convexity

Theorem 11. *If $f(z) \in T^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma)$, then $f(z)$ is convex of order ρ in $|z| < r(\alpha,\eta,\gamma,\rho)$, where*

$$r(\alpha,\eta,\gamma,\rho) = \inf_{n\geq 2}\left\{\frac{[n-1+\eta(1-n+2\gamma n-2\gamma\alpha)]M^m_n(\mu,\beta,t)(1-\rho)}{n(n-\rho)2\eta\gamma(1-\alpha)}\right\}^{\frac{1}{n-1}}.$$

Proof. Let $f(z) \in T^{m,t}_{\beta,\mu}(\alpha,\eta,\gamma)$. Then, we have the inequality

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1-\rho.$$

Now, consider
$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{z\left(-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}\right)}{1-\sum_{n=2}^{\infty} na_n z^{n-1}}\right| = \left|\frac{-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} na_n z^{n-1}}\right|.$$
Taking the modulus, we get
$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{\sum_{n=2}^{\infty} n(n-1)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty} n|a_n||z|^{n-1}} < 1-\rho.$$
Rearranging, we obtain
$$\sum_{n=2}^{\infty} n(n-1)|a_n||z|^{n-1} < (1-\rho)\left(1-\sum_{n=2}^{\infty} n|a_n||z|^{n-1}\right).$$
Simplifying further:
$$\sum_{n=2}^{\infty} n(n-\rho)|a_n||z|^{n-1} < 1-\rho.$$
From Theorem 3, we know that
$$\sum_{n=2}^{\infty} [n-1+\eta(1-n+2\gamma n-2\gamma\alpha)]\, M_n^m(\mu,\beta,t)|a_n| \leq 2\eta\gamma(1-\alpha).$$
This inequality holds if
$$\frac{n(n-\rho)|z|^{n-1}}{1-\rho} < \frac{[n-1+\eta(1-n+2\gamma n-2\gamma\alpha)]\, M_n^m(\mu,\beta,t)}{2\eta\gamma(1-\alpha)}.$$
Thus,
$$|z| < \left\{\frac{[n-1+\eta(1-n+2\gamma n-2\gamma\alpha)]\, M_n^m(\mu,\beta,t)(1-\rho)}{n(n-\rho)2\eta\gamma(1-\alpha)}\right\}^{\frac{1}{n-1}} \quad (n \geq 2),$$
which completes the proof. □

Remark 3.
1. When $m = 0$ in Theorems 7, 9, and 11, we recover the results of Kulkarni [6].
2. When $m = 0$ and $\gamma = 1$ in Theorems 7–11, we recover the results of Gupta and Jain [5].

3. Conclusion

It is clear that the new classes studied in this work generalise some well-known classes of analytic and univalent functions. Also, the results in this study generally reduce to some well-known and new results with appropriate variations of the involved parameters. The new classes however, apparently generalised many existing ones and the results from this research extended many known and new ones when the underlying parameters are varied. Invariably, these results augment those that are already existing in literature.

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