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On affine Riemann surfaces

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Abstract: We show that the universal covering space of a connected component of a regular level set of a smooth complex valued function on $\mathbb{C}^2$, which is a smooth affine Riemann surface, is $\mathbb{R}^2$. This implies that the orbit space of the action of the covering group on $\mathbb{R}^2$ is the original affine Riemann surface.

Keywords: universal covering space, affine Riemann surface.

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1. Introduction

Let

$$F : \mathbb{C}^2 \to \mathbb{C} : (z, w) \mapsto u + \mathrm{i}\, v = \mathrm{Re}\, F + \mathrm{i}\, \mathrm{Im}\, F$$

be a smooth function. Let X_F be the holomorphic Hamiltonian vector field on $(\mathbb{C}^2, dz \wedge dw)$ corresponding to F, that is, $X_F \lrcorner\, (dz \wedge dw) = dF$. On $\mathbb{C}^2 = \mathbb{R}^4 = (\mathrm{Re}\, z, \mathrm{Im}\, z, \mathrm{Re}\, w, \mathrm{Im}\, w)$ with real symplectic form $\Omega = \mathrm{Re}(dz \wedge dw)$ we have real Hamiltonian vector fields X_u and X_v. Then

$$\begin{aligned}(X_u + \mathrm{i}\, X_v) \lrcorner\, \mathrm{Re}(dz \wedge dw) &= du + \mathrm{i}\, dv = dF = X_F \lrcorner\, (dz \wedge dw) \\ &= (\mathrm{Re}\, X_F + \mathrm{i}\, \mathrm{Im}\, X_F) \lrcorner\, \big(\mathrm{Re}\,(dz \wedge dw) + \mathrm{i}\, \mathrm{Im}\,(dz \wedge dw)\big) \\ &= \big(\mathrm{Re}\, X_F + \mathrm{i}\, \mathrm{Im}\, X_F\big) \lrcorner\, \mathrm{Re}(dz \wedge dw) + \big(-\mathrm{Im}\, X_F + \mathrm{i}\, \mathrm{Re}\, X_F\big) \lrcorner\, \mathrm{Im}(dz \wedge dw)\end{aligned}$$

So

$$(X_u + \mathrm{i}\, X_v) \lrcorner\, \mathrm{Re}(dz \wedge dw) = \big(\mathrm{Re}\, X_F + \mathrm{i}\, \mathrm{Im}\, X_F\big) \lrcorner\, \mathrm{Re}(dz \wedge dw),$$

since the 2-forms $\mathrm{Re}(dz \wedge dw)$ and $\mathrm{Im}(dz \wedge dw)$ are linearly independent. This implies

$$X_{\mathrm{Re}\, F} = X_u = \mathrm{Re}\, X_F \quad \text{and} \quad X_{\mathrm{Im}\, F} = X_v = \mathrm{Im}\, X_F,$$

since $\mathrm{Re}(dz \wedge dw)$ is nondegenerate.

Proposition 1. *Let S be a connected component of $F^{-1}(c)$, where $c \in \mathbb{C}$ is a regular value of F, which lies in its image. Then the universal covering space of S is $\mathbb{R}^2$.*

Proof. S is a connected smooth 1 dimensional complex manifold. Our argument constructs global coordinates on the universal covering space of S. We begin. For every $(z, w) \in S$ the complex tangent space to S at (z, w) is $\ker dF(z, w)$, where

$$(0, 0) \neq dF(z, w) = du_{|S}(z, w) + \mathrm{i}\, dv_{|S}(z, w) = (du + \mathrm{i}\, dv)_{|S}(z, w)$$

for every $(z, w) \in S$. Thus the nonzero vector field $X_F = (X_u + \mathrm{i}\, X_v)_{|S}$ spans the complex tangent space of S at each point of S. Because X_F is nonzero on S, the real vector fields $X_{u|S}$ and $X_{v|S}$ are linearly independent at each point of S. To see this we argue as follows. Suppose that the real vector fields $X_{u_{|S}}$ and $X_{v_{|S}}$ are linearly dependent at some point $(z, w) \in S$. Then $\mathrm{span}_{\mathbb{R}}\{X_{u_{|S}}(z, w), X_{v_{|S}}(z, w)\}$ has real dimension 1. Thus $(X_{u_{|S}} + \mathrm{i}\, X_{v_{|S}})(z, w)$ does not span the complex tangent space to S at (z, w), which is a contradiction.

Consider the 2-form $\Omega_{|S}$ on S. Since Ω is closed, it follows that $\Omega_{|S}$ is closed. Because $X_{u|S}$ and $X_{v|S}$ are linearly independent vector fields on S and Ω is nondegenerate on $\mathbb{R}^4$, it follows that $\Omega_{|S}$ is nondegenerate on $\mathrm{span}_{\mathbb{R}}\{X_{u|S}(z,w), X_{v|S}(z,w)\}$ for every $(z,w) \in S$. To see this from $(X_u + \mathrm{i}\, X_v) \lrcorner\, \Omega = dF$ and the fact that $dF \neq (0,0)$ on S we get $\Omega(X_u, X_v) \neq 0$ on S. Hence $\Omega_{|S}$ is a symplectic form on S.

Let M be the universal covering space of S with covering mapping $\rho : M \to S$. Because ρ is a local diffeomorphism, the 2-form $\omega = \rho^*(\Omega_{|S})$ on M is symplectic. Consider the smooth functions $U = \rho^*(u_{|S})$ and $V = \rho^*(v_{|S})$ on (M, ω). The corresponding Hamiltonian vector fields X_U and X_V on (M, ω) are given by $dU = X_U \lrcorner\, \omega$ and $dV = X_V \lrcorner\, \omega$. Since

$$X_U \lrcorner\, \omega = dU = d\left(\rho^* u_{|S}\right) = \rho^*(du_{|S}) = \rho^*(X_{u_{|S}} \lrcorner\, \Omega_{|S})$$
$$= \rho^*(X_{u_{|S}}) \lrcorner\, \rho^*(\Omega_{|S}) = \rho^*(X_{u_{|S}}) \lrcorner\, \omega,$$

it follows that $X_U = \rho^*(X_{u_{|S}})$, because ω is nondegenerate. Similarly, $X_V = \rho^*(X_{v_{|S}})$. Since ρ is a local diffeomorphism and the vector fields $X_{u_{|S}}$ and $X_{v_{|S}}$ are linearly independent at each point of S, the vector fields X_U and X_V are linearly independent at each point of M. Thus the 1-forms dU and dV on M are linearly independent at each point of M, because ω is nondegenerate. So the vector fields $\frac{\partial}{\partial U}$ and $\frac{\partial}{\partial V}$ are linearly independent at each point of M.

Consider the nonzero 2-form $\varpi = dV \wedge dU$ on M. Since M is 2-dimensional, the de Rham cohomology group of 2-forms on M has dimension 1. Thus $\varpi = a\omega$ for some nonzero real number a.[1] Because $\{\frac{\partial}{\partial U}, \frac{\partial}{\partial V}\}$ is a basis of the tangent space of M at each point of M, we may write $X_U = A\frac{\partial}{\partial U} + B\frac{\partial}{\partial V}$. Then

$$dU = X_U \lrcorner\, \omega = \frac{1}{a} X_U \lrcorner\, \varpi = \frac{1}{a}(BdU - AdV),$$

which implies $X_U = a\frac{\partial}{\partial V}$. A similiar argument shows that $X_V = -a\frac{\partial}{\partial U}$.

The pair of functions (U, V) are coordinates on M, since the vector fields $\frac{1}{a}X_U = \frac{\partial}{\partial V}$ and $-\frac{1}{a}X_V = \frac{\partial}{\partial U}$ are linearly independent at each point of M and commute. This latter assertion follows because

$$\{u, v\} = L_{X_v} u = L_{X_{\mathrm{Im}\, F}}(\mathrm{Re}\, F) = L_{\frac{1}{2\mathrm{i}}(X_{F - \mathrm{i}\, F})} \tfrac{1}{2}(F + \mathrm{i}\, F)$$
$$= \frac{1}{4i}[L_{X_F} F + \mathrm{i}\, L_{X_F} F - \mathrm{i}\, L_{X_F} F + L_{X_F} F] = 0$$

implies $[X_v, X_u] = X_{\{u,v\}} = 0$. From

$$T\rho\,[X_U, X_V] = [X_u|S, X_v|S] \circ \rho = [X_u, X_v]_{|S} \circ \rho = 0,$$

we get $[X_U, X_V] = 0$, because ρ is a local diffeomorphism. Hence $[\frac{\partial}{\partial U}, \frac{\partial}{\partial V}] = 0$. Thus we may identify M with $\mathbb{R}^2$. □

Corollary 1. *(Bates and Cushman [1]). The image of the linear flow of the vector field $X_{U+\mathrm{i}\,V}$ on $\mathbb{C}$ under the covering map ρ is the flow of the vector field X_F on S.*

Proof. The flow of $X_{U+\mathrm{i}\,V}$ on $\mathbb{C}$ is $U(t) + \mathrm{i}\, V(t) = \big(U(0) + \mathrm{i}at\big) + \big(\mathrm{i}V(0) - at\big)$, since $X_U = a\frac{\partial}{\partial V}$ and $X_V = -a\frac{\partial}{\partial U}$. Hence an integral curve of $X_{U+\mathrm{i}\,V}$ starting at $U(0) + \mathrm{i}\, V(0)$ is $t \mapsto \big(U(0) + \mathrm{i}\, V(0)\big) + a(-t + \mathrm{i}t)$, which is a straight line in $\mathbb{C}$. Thus the flow of $X_{U+\mathrm{i}V}$ is linear. Since

$$T\rho X_{U+\mathrm{i}\,V} = T\rho(X_U + \mathrm{i}\, X_V) = T\rho X_U + \mathrm{i}\, T\rho X_V$$
$$= X_{u_{|S}} \circ \rho + \mathrm{i}\, X_{v_{|S}} \circ \rho = X_{(u+\mathrm{i}v)_{|S}} \circ \rho = X_{F|S} \circ \rho,$$

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[1] We compute a as follows. Let $D \subseteq \mathbb{R}^2$ be the unit disk in $(\mathbb{R}^2, \varpi = dV \wedge dU)$ with Euclidean inner product. Orient D so that its boundary is traversed clockwise. Then $\pi = \int_D \varpi = a \int_D \omega$, that is, $a = \pi / \int_D \omega$.

the image of the flow of $X_{U+\mathrm{i}V}$ under the covering map ρ is the flow of X_F. □

Define a Riemannian metric E on $\mathbb{R}^2$ by $\mathrm{E} = dU \odot dU + dV \odot dV$. Since $\mathrm{E}(\frac{\partial}{\partial U}, \frac{\partial}{\partial U}) = 1 = \mathrm{E}(\frac{\partial}{\partial V}, \frac{\partial}{\partial V})$ and $\mathrm{E}(\frac{\partial}{\partial U}, \frac{\partial}{\partial V}) = 0$, we find that E is the Euclidean inner product on $T_{(U,V)}\mathbb{R}^2 = \mathbb{R}^2$ for every $(U,V) \in \mathbb{R}^2$. The metric E is flat, since it is indendent of $(U,V) \in \mathbb{R}^2$. Let G be the group of covering transformations of S. Then G is a discrete subgroup of the two dimensional Euclidean group. G acts properly on $\mathbb{R}^2$. Since each element of G leaves no point of $\mathbb{R}^2$ fixed, we obtain the

Corollary 2. *The orbit space $\mathbb{R}^2/G$ of the action of the covering group G on the universal covering space $\mathbb{R}^2$ of the affine Riemann surface S is diffeomorphic to S.*

2. Example[3]

Let

$$F : \mathbb{C}^2 \to \mathbb{C} : (z,w) \mapsto w^2 + z^6. \tag{1}$$

Then 1 is a regular value of F, since $(0,0) = dF(z,w) = (6z^5, 2w)$ if and only if $z = w = 0$. But $(0,0) \notin F^{-1}(1) = S$. Thus S is a smooth affine Riemann surface, which is connected. Let $\pi : \mathbb{C}^2 \to \mathbb{C} : (z,w) \mapsto z$. Then $\pi_{|S} : S \subseteq \mathbb{C}^2 \to \mathbb{C}$ is a branched covering map of S with branch points $B = \{(z_k = \mathrm{e}^{2\pi \mathrm{i} k/6}, 0) \in S \,\big|\, \text{for } k = 0,1,\ldots,5\}$ and branch values $V = \{z_k \,\big|\, k = 0,1,\ldots,5\}$. The map $\pi_{|S}$ is smooth on $S \setminus B$ with image $\mathbb{C} \setminus V$. The sheets S_ℓ of the branched covering map $\pi_{|S}$ are defined by $w_\ell = \mathrm{e}^{2\pi \mathrm{i}\ell/2}(1-z^6)^{1/2}$ for $\ell = 0,1$, where $z \in \mathbb{C}$, that is, S_ℓ is a connected component of $(\pi_{|S})^{-1}(\mathbb{C}) = \coprod_{\ell=0,1} S_\ell$.

Let $\rho : \mathbb{R}^2 \to S$ be the universal covering map of S. The sheets of the covering map ρ are $\Sigma_\ell = \rho^{-1}(S_\ell)$ for $\ell = 0,1$. The group G of covering transformations of S is the collection of isometries of $(\mathbb{R}^2, \mathrm{E})$, where E is the Euclidean inner product on $\mathbb{R}^2$, which permute the sheets Σ_ℓ of ρ. Consider the group G' of diffeomorphisms of S generated by the transformations

$$\mathcal{R} : S \subseteq \mathbb{C}^2 \to S \subseteq \mathbb{C}^2 : (z,w) \mapsto (\mathrm{e}^{2\pi \mathrm{i}/6} z, w)$$

and

$$\mathcal{U} : S \subseteq \mathbb{C}^2 \to S \subseteq \mathbb{C}^2 : (z,w) \mapsto (\overline{z}, \overline{w}).$$

Since $\mathcal{R}^6 = \mathcal{U}^2 = \mathrm{id}$ and $\mathcal{R}\mathcal{U} = \mathcal{U}\mathcal{R}^{-1}$, the group G' is isomorphic to the dihedral group on 6 letters.[3] Because $\mathcal{R}(S_\ell) = S_\ell$ for $\ell = 0,1$ and $\mathcal{U}(S_0) = S_1$, the map $\mathcal{R}$ induces the identity permutation of the sheets of the covering map ρ; while the map $\mathcal{U}$ transposes the sheets of ρ. Thus $\mathcal{R}$ and $\mathcal{U}$ generate the covering group G.

We want to describe the action of G, as a subgroup of the Euclidean group of $(\mathbb{R}^2, \mathrm{E})$. We will need some preliminary results. Let

$$f : \mathbb{C} \setminus V \to \mathbb{C} : z \mapsto \int_0^z \frac{1}{2w}\, dz, \tag{2}$$

where $w = \sqrt{1-z^6}$. Then f is a local diffeomorphism, because $df = \frac{1}{2w} dz$ is nonvanishing on $\mathbb{C} \setminus V$. We have

Proposition 2. *Up to a coordinate transformation $\lambda : \mathbb{C} \to \mathbb{C}$, the map*

$$\delta : S \subseteq \mathbb{C}^2 \to \mathbb{C} : (z,w) \mapsto \zeta = \alpha (f \circ \pi_{|S})(z,w), \tag{3}$$

where $\alpha = \sqrt{2}\mathrm{e}^{3\pi \mathrm{i}/4}$, is a right inverse of the universal covering map ρ, that is, $\rho \circ \lambda \circ \delta = \mathrm{id}_S$.

To prove Proposition 2 we need:

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2 See Cushman [2].

3 The group G' is also generated by the reflections $\{R^k U, k = 0,1,\ldots 5 \,\big|\, R^6 = U^2 = \mathrm{id}\}$. Thus G' is the Weyl group of the complex simple Lie algebra $\mathbf{A}_5$.

Lemma 1. *The image under the map δ (3) of an integral curve of the vector field $(X_F)_{|S}$ on S is an integral curve of the vector field $\alpha \frac{\partial}{\partial \zeta}$ on $\mathbb{C}$.*

Proof. It suffices to show that for every $(z,w) \in S$

$$T_{(z,w)}\delta \, X_F(z,w) = \alpha \frac{\partial}{\partial \zeta}\Big|_{\zeta=\delta(z,w)}. \tag{4}$$

This we do as follows. Using the definition of the map $\pi_{|S}$ and the vector field $(X_F)_{|S} = 2w\frac{\partial}{\partial z} - 6w^5\frac{\partial}{\partial w}$, for every $(z,w) \in S$ we get

$$T_{(z,w)}\pi_{|S} \, X_F(z,w) = T_{(z,w)}\pi_{|S}(2w\frac{\partial}{\partial z} - 6w^5\frac{\partial}{\partial w}) = 2w\frac{\partial}{\partial z}.$$

By definition of the function f (2) we have $df = \frac{1}{2w}dz$, which implies $T_z f\big(2w\frac{\partial}{\partial z}\big) = \frac{\partial}{\partial \zeta}$. Thus for every $(z,w) \in S$

$$T_{(z,w)}\delta \, X_F(z,w) = \alpha T_z f\Big(T_{(z,w)}\pi_{|S}\big(X_F(z,w)\big)\Big) = \alpha \frac{\partial}{\partial \zeta},$$

which establishes Eq. (4). □

Corollary 3. *The map δ (3) is a local diffeomorphism.*

Proof. This follows from Eq. (4), which shows that the tangent map of δ is injective at each point of S. □

Proof of Proposition 2. Let $U + \mathrm{i}V = \rho^*(\mathrm{Re}\, F) + \mathrm{i}\, \rho^*(\mathrm{Im}\, F)$. By Proposition 1, $U + \mathrm{i}\, V$ is a coordinate on $\mathbb{C}$. Define the diffeomorphism

$$\lambda : \mathbb{C} \to \mathbb{C} : \zeta \mapsto U + \mathrm{i}\, V$$

by requiring $\lambda_*\big(\alpha\frac{\partial}{\partial \zeta}\big) = X_U + \mathrm{i}\, X_V$, that is, set $U = \lambda(\mathrm{Re}\, \zeta)$ and $V = \lambda(\mathrm{Im}\, \zeta)$. By construction we have $\alpha\frac{\partial}{\partial \zeta} = \lambda^*\rho^*\big((X_F)_{|S}\big)$, see the proof of Proposition 1. By Eq. (4) we have $\alpha\frac{\partial}{\partial \zeta} = \delta_*\big((X_F)_{|S}\big)$. Thus $\delta_* = \lambda^*\rho^*$, which implies $\rho \circ \lambda \circ \delta = \mathrm{id}_S$. To see this suppose that $\rho \circ \lambda \circ \delta \neq \mathrm{id}_S$. Then $\delta^* \circ (\rho \circ \lambda)^* \neq \mathrm{id}_{TS}$. Hence $\lambda^*\rho^* \neq \delta_*$, which is a contradiction.

Let

$$R : \mathbb{C} \to \mathbb{C} : z \mapsto \mathrm{e}^{2\pi\mathrm{i}/6} z. \tag{5}$$

Then $f(Rz) = Rf(z)$, where f is the function defined in (2). To see this we compute.

$$\begin{aligned} f(Rz) &= \int_0^{Rz} \frac{d\xi}{2w(\xi)}, \text{ where } w(\xi) = \sqrt{1-\xi^6} \\ &= \int_0^{z} \frac{R dz}{2w(z)}, \text{ using } \xi = Rz \text{ and } w(Rz) = w(z) \\ &= Rf(z). \end{aligned}$$

Thus up to a dialation the image under f (2) of the closed equilateral triangle

$$T' = \{z = r'\mathrm{e}^{\mathrm{i}\theta'} \in \mathbb{C} \big| \, 0 \le r' \le 1 \;\&\; 0 \le \theta' \le 2\pi/6\}$$

with vertex at the origin and one edge of length 1 along the real axis is the equilateral triangle

$$T = f(T') = \{\zeta = r\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \big| \, 0 \le r \le C \;\&\; 2\pi/6 \le \theta \le 4\pi/6\} = CR(T'),$$

where $C = \int_0^1 \frac{dz}{\sqrt{1-z^6}}$. Hence f maps a regular hexagon into another. In particular, it sends the closed regular hexagon H' with center at the origin O and edge length 1 onto the regular hexagon H with center at O and edge length C. Since H' is simply connected and is contained in the unit disk $\{|z| \le 1\}$, the complex square root $\sqrt{1-z^6}$ is single valued for all $z \in H'$. Thus H' is the image under $\pi_{|S}$ of a domain $\mathcal{D} \subseteq S$, which is contained in some sheet $S_{\ell'}$ of the covering map ρ of S.

Let

$$U : \mathbb{C} \to \mathbb{C} : z \mapsto \overline{z}. \tag{6}$$

The regular hexagon H is invariant under the action of the group $\mathcal{G}$, generated be the rotation R and the reflection RU in the diagonal of H, which is an edge of the triangle T with the orgin as an end point that is not the real axis. The map δ (3) intertwines the action of the group G' generated by $\mathcal{R}$ and $\mathcal{RU}$ on S with the action of the $\mathcal{G}$ on H. Thus the domain $\mathcal{D}$ contains a fundamental domain of the action of the covering group G on $\mathbb{R}^2$.

Let $\mathcal{T}$ be the abelian group generated by the translations

$$\tau_k : \mathbb{C} \to \mathbb{C} : z \mapsto z + u_k, \text{ for } k = 0, 1, \ldots, 5.$$

Here $u_k = \sqrt{3}C\, \mathrm{e}^{2\pi \mathrm{i}(1/12+k/6)}$, which is perpendicular to an edge of the equilateral triangle $R^k(T)$ that lies on the boundary of the hexagon H. The action of $\mathcal{T}$ on $\mathbb{C}$ has fundamental domain H. To see this recall that in [2] it is shown that

$$\bigcup_{n\geq 0} \bigcup_{\ell_1+\cdots+\ell_k=n} \tau_1^{\ell_1} \circ \cdots \circ \tau_k^{\ell_k}(K) = \mathbb{C},$$

where K is the closed stellated hexagon formed by placing an equilateral triangle of edge length C on each bounding edge of H. But

$$K = H \cup \bigcup_{k=0}^{5} \tau_k(R^{(4+k) \bmod 6} T).$$

So H is the fundamental domain of the $\mathcal{T}$ action on $\mathbb{C}$. Because applying an element of G' to the domain $\mathcal{D} \subseteq S$ gives a domain whose boundary has a nonempty intersection with the boundary of $\mathcal{D}$, it follows that under the mapping δ (3) the corresponding element of the group of motions in $\mathbb{C}$ sends the hexagon H to a hexagon which has an edge in common with H. Thus this group of motions is the group $\mathcal{T}$. Because the mapping δ intertwines the G' action on S with the $\mathcal{T}$ action on $\mathbb{C}$ and sends the domain $\mathcal{D} \subseteq S_{\ell'}$ diffeomorphically onto H, it follows that $\mathcal{D}$ is a fundamental domain for the action of G' on S. Consider $\lambda(H)$, which is a regular hexagon with center at the origin, since the coordinate change λ maps straight lines to straight lines. From proposition 2.1 we deduce that $\lambda(H)$ is a fundamental domain for the action of the covering group G on $\mathbb{C} = \mathbb{R}^2$ of the affine Riemann surface S. Hence $S = \mathbb{R}^2/\mathcal{T}$. □

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