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Some log −series and applications

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Abstract: The purpose of this paper is to apply the concept of log −series in order to determine the sum of certain power series, where the n-th terms involves the factorial mapping, the generalized harmonic numbers and the reciprocals of factorial sums.

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1. Introduction

The theory of dual numbers was originally introduced by Clifford [1] in 1873. In 1891, E. Study [2] recognized that the associative algebra of dual numbers was well-suited for describing the group of motions in three-dimensional space. At the turn of the 20th century, Kotelnikov [3] expanded this theory by developing the concepts of dual vectors and dual quaternions. The algebraic properties of dual numbers have been extensively studied and documented in various papers, including [4–7].

The study of functions involving dual variables has also garnered significant attention, as seen in [8,9]. This concept finds applications in numerous fields of fundamental sciences and engineering; for further details, see [3,10–16].

Building on this foundation, multidual numbers were introduced by Messelmi in [17] as a natural extension of dual numbers to higher dimensions. Messelmi further explored functions of multidual variables in this context.

In the domain of multidual analysis, Messelmi introduced the novel concepts of log-series and log-functions in [18]. His approach involved replacing natural powers with multidual integers and coefficients with multidual sequences in real power series. However, his study was limited to elementary log-functions, representing specific extensions of classical real functions. These log-functions were utilized to expand certain special functions expressed as integrals involving the nth power of logarithmic functions and harmonic numbers.

The main objective of the present paper is to investigate certain log-series and log-functions. Specifically, the second section begins with an introduction to basic concepts in dual analysis, including the holomorphy of dual functions, generalized Cauchy-Riemann formulas, and a continuation principle for real functions in the dual algebra. Furthermore, we define and examine the properties of the ring of dual integers, $\mathbb{Z}(\varepsilon)$. Additionally, certain notions related to relative numbers, such as the factorial map, generalized harmonic numbers, and the sum of reciprocals of factorials, are extended to dual integers.

The third section focuses on the study of log-series and log-functions, restricted to the dual case. Within this framework, several significant results are established. These findings allow for the determination of sums for classes of power series involving generalized harmonic numbers and sums of reciprocals of factorials.

2. Preliminaries

A dual number z is defined as an ordered pair of real numbers $(x, y,)$ associated with the real unit 1 and the dual unit ε, where ε is an nilpotent number i.e. $\varepsilon^2 = 0$. Indeed, a dual number is usually denoted in the form

$$z = x + y\varepsilon. \tag{1}$$

for which, we admit that $\varepsilon^0 = 1$.

The set of dual numbers denonted by $\mathbb{D}$ is given by

$$\mathbb{D} = \left\{ z = x + y\varepsilon \mid x, y \in \mathbb{R} \text{ where } \varepsilon^2 = 0 \right\}. \tag{2}$$

If $z = x + y\varepsilon$ is a dual number, we will denote by real (z) the real part of z given by

$$\text{real}\,(z) = x_0. \tag{3}$$

The dual numbers form a commutative ring with characteristic 0. Moreover the inherited multiplication gives the dual numbers the structure of 2−dimensional Clifford Algebra, see for more details regarding dual numbers the references [2,4–7,9,19]. In abstract algebra terms, the dual ring can be obtained as the quotient of the polynomial ring $\mathbb{R}[X]$ by the ideal generated by the polynomial X^2, i.e.

$$\mathbb{D} \simeq \frac{\mathbb{R}[X]}{\langle X^2 \rangle}. \tag{4}$$

There are many ways to choose the dual unit number ε. The fundamental example can be given by the matrix

$$\varepsilon = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

It is also important to point out that every dual number possess a matrix representation that can be formulated as follows.

Let us denote by $\mathcal{G}_2(\mathbb{R})$ the subset of $\mathcal{M}_2(\mathbb{R})$ given by

$$\mathcal{G}_2(\mathbb{R}) = \left\{ A \in \mathcal{M}_2(\mathbb{R}) \mid A = \begin{bmatrix} a_0 & 0 \\ a_1 & a_0 \end{bmatrix} \right\}. \tag{5}$$

It is clear that $\mathcal{G}_2(\mathbb{R})$ is a subring of $\mathcal{M}_2(\mathbb{R})$, it has also a structure of 2−dimensional associative, commutative and unitary algebra. If $a_0 \neq 0$, the set $\mathcal{G}_2(\mathbb{R})$ can be seen as a subgroup of $GL(2)$.

Introducing now the following mapping

$$\begin{cases} \mathcal{R} : \mathbb{D} \longrightarrow \mathcal{G}_2(\mathbb{R}), \\ \mathcal{R}(x + y\varepsilon) = \begin{bmatrix} x & 0 \\ y & x \end{bmatrix} \end{cases} \tag{6}$$

The result below shows the relationship between the sets $\mathbb{D}$ and $\mathcal{G}_2(\mathbb{R})$.

Proposition 1 ([8]). *$\mathcal{R}$ is an isomorphism of algebras.*

If z is a dual number, the conjugate of z denoted by $\bar{z}$ is the dual number given by

$$\bar{z} = x - y\varepsilon. \tag{7}$$

Hence, $z = x + y\varepsilon$ has a unique conjugate if and only if $x \neq 0$. If $x = 0$ the number $y\varepsilon$ is a divisor of zero. The set of zero divisors of the ring $\mathbb{D}$ is denoted by D and we have

$$D = \{ y\varepsilon \mid y \in \mathbb{R} \}. \tag{8}$$

For the sequel we admit that $\mathbb{D}$ is endowed with the usual topology of $\mathbb{R}^2$. We recall now, according to the work [8], some concepts and results regarding dual functions.

Let Ω be an open subset of $\mathbb{D}$, $z = x + y\varepsilon \in \Omega$ and $f : \Omega \longrightarrow \mathbb{D}$ a dual function. The Cauchy-Riemann conditions can be generalized for dual function as follows.

Proposition 2. *Let f be a dual function in $\Omega \subset \mathbb{D}_2$, which can be written in terms of its real and dual parts as*

$$f(z) = p(x,y) + q(x,y)\varepsilon. \tag{9}$$

and suppose that the partial derivatives of f exist. Then,

1. f is holomorphic in Ω if and only if the following formulas hold

$$\begin{cases} \dfrac{\partial p}{\partial x} = \dfrac{\partial q}{\partial y}, \\ \dfrac{\partial p}{\partial y} = 0. \end{cases} \tag{10}$$

2. f is holomorphic in Ω if and only if its partial derivatives satisfy

$$\frac{\partial f}{\partial y} = \varepsilon \frac{\partial f}{\partial x}. \tag{11}$$

We deduce in particular that if the function f is holomorphic, then

$$\frac{df}{dz} = \frac{\partial f}{\partial x}. \tag{12}$$

A dual function defined in $\Omega \subset \mathbb{D}$ is said to be homogeneous if

$$f(\text{real}(z)) \in \mathbb{R}. \tag{13}$$

The following proposition ensures that every regular real function can be extended to the algebra of dual numbers.

Proposition 3 (Continuation of real functions)**.** *Let $f : O \longrightarrow \mathbb{R}$ be a real function, where O is an open connected domain of $\mathbb{R}$. Denote by the set $\Omega_O = \{z = x + y\varepsilon \in \mathbb{D} \mid x \in O\}$.*

1. Suppose that $f \in \mathcal{C}^2(O)$. Then, there exists a unique homogeneous holomorphic dual function $\widetilde{f} : \Omega_O \subset \mathbb{D} \longrightarrow \mathbb{D}$ satisfying

$$\widetilde{f}(x) = f(x) \quad \forall x \in O, \tag{14}$$

where

$$\widetilde{f}(x) = f(x) - f'(x)\, y\varepsilon. \tag{15}$$

If in addition, if $f \in \mathcal{C}^q(O)$, $q \geq 2$, then $\widetilde{f} \in \mathcal{C}^{q-2}(\Omega_O)$. In Particular, if $f \in \mathcal{C}^\infty(O)$, then $\widetilde{f} \in \mathcal{C}^\infty(\Omega_O)$, we say in such case that f is an analytic function in Ω_O.

Further, as stated in the paper [5], the set of dual integers $\mathbb{Z}(\varepsilon)$ can be defined as

$$\mathbb{Z}(\varepsilon) = \{m = m_0 + m_1\varepsilon \mid m_0,\ m_1 \in \mathbb{Z}\}. \tag{16}$$

The set $\mathbb{Z}(\varepsilon)$ can be seen as a generated $\mathbb{Z}-$module having $(1, \varepsilon)$ as system of generators. It is worth noting that $\mathbb{Z}(\varepsilon)$ can be also obtained as the quotient of the polynomial ring $\mathbb{Z}[X]$ by the ideal generated by the polynomial X^2, i.e.

$$\mathbb{Z}(\varepsilon) \simeq \frac{\mathbb{Z}[X]}{\langle X^2 \rangle}. \tag{17}$$

The set of zero divisors of the ring $\mathbb{Z}(\varepsilon)$ denoted by $D(\varepsilon)$ coincides with the ideal generated by ε. This means that

$$D(\varepsilon) = \varepsilon\mathbb{Z}(\varepsilon) = \{m = m_0\varepsilon \mid m_0 \in \mathbb{Z}\}. \tag{18}$$

A dual integer $m = m_0 + m_1\varepsilon$ is said to be positive if $m_0 > 0$. The set of positive dual integers given by

$$\mathbb{Z}^+(\varepsilon) = \{m = m_0 + m_1\varepsilon \in \mathbb{Z}(\varepsilon) \mid m_0 > 0\}, \tag{19}$$

forms a commutative monoid under multiplication. We suggests here a generalization of the factorial map for positive dual integers. To this end, let us introduce the following definition.

Definition 1. The factorial of the integer $m = m_0 + m_1\varepsilon \in \mathbb{Z}^+(\varepsilon)$ is defined by the formula

$$m! = \prod_{n=1}^{m_0} (n + m_1\varepsilon). \tag{20}$$

As consequence, we have

$$(m_0 + 1 + m_1\varepsilon)! = (m_0 + m_1\varepsilon)!\,(m_0 + 1 + m_1\varepsilon). \tag{21}$$

In the following statement, we provide an expression of the dual factorial using harmonic numbers.

Proposition 4. *Let $m = m_0 + m_1\varepsilon \in \mathbb{Z}^+(\varepsilon)$. We have*

$$m! = m_0!\left(1 + m_1 H_{m_0,1}\varepsilon\right). \tag{22}$$

Proof. Let $m = m_0 + m_1\varepsilon \in \mathbb{Z}^+(\varepsilon)$, in view of (20) we can write

$$m! = m_0! \prod_{n=1}^{m_0} \left(1 + \frac{m_1}{n}\varepsilon^i\right).$$

Thus

$$m! = m_0! e^{m_0 \sum_{n=1}^{m_0} \frac{1}{n}\varepsilon} = m_0!\left(1 + m_0 \sum_{n=1}^{m_0} \frac{1}{n}\varepsilon\right).$$

Then, the desired outcome is achieved. □

Moreover, the $p-$th generalized harmonic number, denoted here by $H_{p,q}$, is defined as, see [20]

$$H_{p,q} = \sum_{n=1}^{p} \frac{1}{n^q}. \tag{23}$$

The $p-$th generalized harmonic number can be also generalized for dual integers as follows.

$$H_{m_0+m_1\varepsilon,q} = \sum_{n=1}^{m_0} \frac{1}{(n + m_1\varepsilon)^q}. \tag{24}$$

In the below proposition we provide an expression of $H_{m_0+m_1\varepsilon,q}$ with respect the real m_0-th generalized harmonic numbers.

Proposition 5. *The $(m_0 + m_1\varepsilon)-$th generalized harmonic number $H_{m_0+m_1\varepsilon,q}$ satisfy the formula*

$$H_{m_0+m_1\varepsilon,q} = H_{m_0,q} - m_1 q H_{m_0,q+1}\varepsilon. \tag{25}$$

Proof. Let us proceed as follows. We have

$$H_{m_0+m_1\varepsilon,q} = \sum_{n=1}^{m_0} \frac{1}{(n + m_1\varepsilon)^q} = \sum_{n=1}^{m_0} \frac{1}{n^q\left(1 + q\frac{m_1}{n}\varepsilon\right)} = \sum_{n=1}^{m_0} \frac{1}{n^q} - q m_1 \sum_{n=1}^{m_0} \frac{1}{n^{q+1}}\varepsilon.$$

This allows us to accomplish the proof. □

Denote now by F_{m_0} the sum of reciprocals of factorials up to m_0, i.e.

$$F_{m_0} = \sum_{n=0}^{m_0} \frac{1}{n!}. \tag{26}$$

This map can be also generalized for dual integers as

$$F_{m_0+m_1\varepsilon} = \sum_{n=1}^{m_0} \frac{1}{(n+m_1\varepsilon)!}, \tag{27}$$

where $m_0 \geq 1$. Taking into account (22), one easily finds

$$F_{m_0+m_1\varepsilon} = F_{m_0} - 1 - m_1\varepsilon \sum_{n=1}^{m_0} \frac{H_{n,1}}{n!}. \tag{28}$$

Throughout this paper, we will use the following special functions.

$$\begin{cases} \sigma^+(x) = \int_0^x e^s \log s ds, \\ \sigma^-(x) = \int_0^x e^{-s} \log s ds. \end{cases} \tag{29}$$

Notice that

$$\sigma^-(x) = \frac{\partial \gamma}{\partial a}(1, x), \tag{30}$$

where γ is the lower incomplete gamma function, see [19], given by

$$\gamma(a, x) = \int_0^x s^{a-1} e^{-s} ds.$$

Furthermore, it is straightforward to verify that the functions σ^+ and σ^- can be also expressed as follows

$$\begin{cases} \sigma^+(x) = (e^x - 1)\log x - \int_0^x \frac{e^s-1}{s} ds, \\ \sigma^-(x) = (1 - e^{-x})\log x + \int_0^x \frac{e^{-s}-1}{s} ds. \end{cases} \tag{31}$$

In order to establish some theorems in this paper the following lemma is required.

Lemma 1. *Let us consider the sequence of functions* $(f_p(x))_{p\geq 1}$ *given by the belwo reccurence relation*

$$\frac{df_p}{dx} = \frac{f_{p-1}(x)}{x} \quad for \quad p \geq 1, \tag{32}$$

where

$$f_0(x) \overset{v(0)}{\sim} x. \tag{33}$$

Then, the following formula holds

$$f_p(x) = \frac{1}{(p-1)!} \int_0^x (\log x - \log s)^{p-1} \frac{f_0(s)}{s} ds \quad for \quad p \geq 1. \tag{34}$$

Proof. Formula (32) gives for $p = 1$

$$f_1(x) = \int_0^x \frac{f_0(s)}{s} ds.$$

Making use again (32), we can infer for $p = 2$

$$f_2(x) = \int_0^x \frac{1}{t} \int_0^t \frac{f_0(s)}{s} ds dt.$$

Thus, by integration by parts we obtain

$$f_2(x) = \log x \int_0^x \frac{f_0(s)}{s} ds - \int_0^x \log s \frac{f_0(s)}{s} ds = \int_0^x (\log x - \log s) \frac{f_0(s)}{s} ds.$$

Thus, the formula (34) is satisfied for the both cases $p = 1$ and $p = 2$. In order to prove the formula for every $p \geq 1$, we use a proof by induction. To do this, we suppose that the statement is truth for some $p \geq 1$ and show that it remains true for $p + 1$. Indeed, (34) can be written

$$f_p(x) = \frac{1}{(p-1)!} \sum_{k=0}^{p-1} \binom{p-1}{k} (-1)^{p-k-1} \log^k x \int_0^x \log^{p-k-1} s \frac{f_0(s)}{s} ds. \tag{35}$$

Furthermore, (32) implies that

$$f_{p+1}(x) = \int_0^x \frac{f_p(s)}{s} ds. \tag{36}$$

Making together (35) and (36), we can infer

$$f_{p+1}(x) = \frac{1}{(p-1)!} \sum_{k=0}^{p-1} (-1)^{p-k-1} \binom{p-1}{k} \int_0^x \frac{\log^k t}{t} \int_0^t \log^{p-k-1} s \frac{f_0(s)}{s} ds dt.$$

We employ now an integration by parts, designating for this purpose

$$u = \int_0^t \log^{p-k-1} s \frac{f_0(s)}{s} ds \text{ and } dv = \frac{\log^k t}{t} dt.$$

Then

$$du = \log^{p-k-1} t \frac{f_0(t)}{t} \text{ and } v = \frac{1}{k+1} \log^{k+1} t.$$

This leads to

$$\begin{aligned} f_{p+1}(x) &= \frac{1}{(p-1)!} \sum_{k=0}^{p-1} (-1)^{p-k-1} \binom{p-1}{k} \left[\frac{1}{k+1} \log^{k+1} x \int_0^x \log^{p-k-1} s \frac{f_0(s)}{s} ds \right. \\ &\quad \left. - \frac{1}{k+1} \int_0^x \log^p s \frac{f_0(s)}{s} ds \right] \\ &= \frac{1}{p!} \sum_{k=0}^{p-1} (-1)^{p-k-1} \binom{p}{k+1} \left[\log^{k+1} x \int_0^x \log^{p-k-1} s \frac{f_0(s)}{s} ds - \int_0^x \log^p s \frac{f_0(s)}{s} ds \right] \\ &= \frac{1}{p!} \sum_{k'=1}^{p} (-1)^{p-k'} \binom{p}{k'} \left[\log^{k'} x \int_0^x \log^{p-k'} s \frac{f_0(s)}{s} ds - \int_0^x \log^p s \frac{f_0(s)}{s} ds \right] \\ &= \frac{1}{p!} \sum_{k'=0}^{p} (-1)^{p-k'} \binom{p}{k'} \log^{k'} x \int_0^x \log^{p-k'} s \frac{f_0(s)}{s} ds. \end{aligned}$$

This yields

$$f_{p+1} = \frac{1}{p!} \int_{0}^{x} (\log x - \log s)^{p} \frac{f_0(s)}{s} ds.$$

That is, the statement holds true for $p+1$, establishing the induction proof. □

3. Applications of log −Series

In the context of multidual numbers, the theory of log −Series was initially introduced in the reference [18]. This paper will specifically focus on the study of log −Series within the algebra of dual numbers. We will start by recalling foundational concepts of the theory in question, as outlined in reference [18].

Definition 2. A log −series of the real variable x is an infinite series of the form

$$\sum_{m_0=1}^{+\infty} \left(p\left(m_0, m_1\right) + q\left(m_0, m_1\right)\varepsilon\right) x^{m_0+m_1\varepsilon}. \tag{37}$$

Here, the dual sequence $p(m_0, m_1) + q(m_0, m_1)\varepsilon$ represents the coefficient of the term $x^{m_0+m_1\varepsilon}$ in the series.

Furthermore, the log −series (40) can be also written

$$x^{m_1\varepsilon} \sum_{m_0=1}^{+\infty} \left(p\left(m_0, m_1\right) + q\left(m_0, m_1\right)\varepsilon\right) x^{m_0}. \tag{38}$$

So, the log −series converges if and only the below real power series $p(m_0, m_1)$ and $q(m_0, m_1)$ converge simultaneously.

In addition, it is well known, as noted in [8], that the term $x^{m_1\varepsilon}$ is defined for $x \geq 0$, in such a manner that

$$x^{m_1\varepsilon} = \begin{cases} 0 & \text{if } x = 0, \\ 1 + m_1\varepsilon \log x & \text{if } x > 0. \end{cases} \tag{39}$$

Denote by R_i, $i = 1, 2$, the radius of convergence of the real power series $p(m_0, m_1)$ and $q(m_0, m_1)$, respectively. Obviously, the log −series (40) converges for every $x \in [0, R[$, where

$$R = \min(R_1, R_2), \tag{40}$$

On the other hand, we will have, making use (39)

$$x^{m_1\varepsilon} \sum_{m_0=1}^{+\infty} \left(p\left(m_0, m_1\right) + q\left(m_0, m_1\right)\varepsilon\right) x^{m_0} = \left(1 + m_1\varepsilon \log x\right) \sum_{m_0=1}^{+\infty} \left(p\left(m_0, m_1\right) + q\left(m_0, m_1\right)\varepsilon\right) x^{m_0}.$$

This yields

$$\lim_{x \longrightarrow 0} \sum_{m_0=1}^{+\infty} \left(p\left(m_0, m_1\right) + q\left(m_0, m_1\right)\varepsilon\right) x^{m_0+m_1\varepsilon} = 0. \tag{41}$$

We conclude that if the log −series (37) converges then its limit is continuous at 0.

Moreover, if the log −series converges its sum will be referred to as a log −function. We will investigate in this paper several log −series and we will applicate the obtained results to determinate the sums of certain series involves the factorial map, the generalized harmonic numbers and the sums of reciprocals of factorials.

3.1. Application 1

Let us introduce for $p \geq 0$ the two log −series given by

$$\sum_{m_0=1}^{+\infty} \frac{x^{m_0+m_1\varepsilon}}{(m_0+m_1\varepsilon)^p (m_0+m_1\varepsilon)!}, \tag{42}$$

$$\sum_{m_0=1}^{+\infty} (-1)^{m_0} \frac{x^{m_0+m_1\varepsilon}}{(m_0+m_1\varepsilon)^p (m_0+m_1\varepsilon)!}. \tag{43}$$

It can be easily demonstrated that the both log −series converge for every $x \in [0,+\infty[$. Denote by $\varphi^{+}_{p,m_1}(x)$ and $\varphi^{-}_{p,m_1}(x)$ their sums, respectively.

The below statement provides the analytical expression of the functions $\varphi^{+}_{p,m_1}(x)$ and $\varphi^{-}_{p,m_1}(x)$.

Theorem 1. *The following formulas hold*

$$\varphi^{+}_{0,m_1}(x) = e^x - 1 + e^x \sigma^-(x)\, m_1\varepsilon, \tag{44}$$

$$\varphi^{-}_{0,m_1}(x) = e^{-x} - 1 - e^{-x} \sigma^+(x)\, m_1\varepsilon. \tag{45}$$

If $p \geq 1$, then

$$\varphi^{+}_{p,m_1}(x) = \frac{1}{(p-1)!}\int_0^x (\log x - \log s)^{p-1} \frac{e^s-1}{s} ds + \frac{m_1\varepsilon}{(p-1)!}\int_0^x (\log x - \log s)^{p-1} \frac{e^s\sigma^-(s)}{s} ds, \tag{46}$$

$$\varphi^{-}_{p,m_1}(x) = \frac{1}{(p-1)!}\int_0^x (\log x - \log s)^{p-1} \frac{e^{-s}-1}{s} ds + \frac{m_1\varepsilon}{(p-1)!}\int_0^x (\log x - \log s)^{p-1} \frac{e^{-s}\sigma^+(s)}{s} ds. \tag{47}$$

Proof. Suppose first that $p = 0$, i.e.

$$\varphi^{+}_{0,m_1}(x) = \sum_{m_0=1}^{+\infty} \frac{x^{m_0+m_1\varepsilon}}{(m_0+m_1\varepsilon)!}, \tag{48}$$

$$\varphi^{-}_{0,m_1}(x) = \sum_{m_0=1}^{+\infty} (-1)^{m_0} \frac{x^{m_0+m_1\varepsilon}}{(m_0+m_1\varepsilon)!}, \tag{49}$$

So, we find

$$\varphi^{+}_{0,m_1}(x) = \exp_{m_1}(x) - 1.$$

For further details regarding the exponential log −function $\exp_{m_1}$ refer to the cited source [18]. Hence, we get

$$\begin{aligned} \varphi^{+}_{0,m_1}(x) &= \exp_{m_1}(x) - 1 \\ &= e^x - 1 + e^x\sigma^-(x)\, m_1\varepsilon. \end{aligned}$$

On the other hand, by calculating the derivative of $\varphi^{-}_{0,m_1}(x)$, it follows

$$\frac{d\varphi^{-}_{0,m_1}}{dx} = -x^{m_1\varepsilon} - \varphi^{-}_{0,m_1}(x). \tag{50}$$

It is straightforward to verify that the solution of the homogeneous ODE corresponding to (50) is given by

$$\varphi^{-}_{0,m_1}(x) = Ce^{-x},$$

where C represents a dual constant.

Using the method of variation of the parameter, the below equation holds

$$
\begin{aligned}
C'(x)\,e^{-x} &= -x^{m_1\varepsilon} \\
&= -1 - m_1\varepsilon \log x.
\end{aligned}
$$

Then, since $\varphi^{-}_{0,m_1}(0) = 0$, (45) immediately follows. Let us now assume that $p \geq 1$. By calculating the derivatives of both functions φ^{+}_{p,m_1} and φ^{-}_{p,m_1}, yields

$$
\begin{aligned}
\frac{d\varphi^{+}_{p,m_1}}{dx} &= \frac{x^{m_1\varepsilon}}{(1+m_1\varepsilon)^p} + \sum_{m_0=2}^{+\infty} \frac{x^{m_0-1+m_1\varepsilon}}{(m_0+m_1\varepsilon)^p\,(m_0-1+m_1\varepsilon)!}, \\
\frac{d\varphi^{-}_{p,m_1}}{dx} &= \frac{-x^{m_1\varepsilon}}{(1+m_1\varepsilon)^p} + \sum_{m_0=2}^{+\infty} (-1)^{m_0} \frac{x^{m_0-1+m_1\varepsilon}}{(m_0+m_1\varepsilon)^p\,(m_0-1+m_1\varepsilon)!}.
\end{aligned}
$$

Which leads to

$$
\begin{aligned}
\frac{d\varphi^{+}_{p,m_1}}{dx} &= \frac{x^{m_1\varepsilon}}{(1+m_1\varepsilon)^p} + \frac{1}{x}\sum_{m_0=2}^{+\infty} \frac{x^{m_0+m_1\varepsilon}}{(m_0+m_1\varepsilon)^{p-1}\,(m_0+m_1\varepsilon)!}, \\
\frac{d\varphi^{-}_{p,m_1}}{dx} &= \frac{-x^{m_1\varepsilon}}{(1+m_1\varepsilon)^p} + \frac{1}{x}\sum_{m_0=2}^{+\infty} (-1)^{m_0} \frac{x^{m_0+m_1\varepsilon}}{(m_0+m_1\varepsilon)^{p-1}\,(m_0+m_1\varepsilon)!}.
\end{aligned}
$$

Then, we can infer that

$$
\begin{aligned}
\frac{d\varphi^{+}_{p,m_1}}{dx} &= \frac{\varphi^{+}_{p,m_1}(x)}{x}, \\
\frac{d\varphi^{-}_{p,m_1}}{dx} &= \frac{\varphi^{+}_{p,m_1}(x)}{x}.
\end{aligned}
$$

Consequently, Lemma 5 allows us to deduce that

$$
\begin{aligned}
\frac{d\varphi^{+}_{p,m_1}}{dx} &= \frac{1}{(p-1)!}\int_0^x (\log x - \log s)^{p-1}\,\frac{\varphi^{+}_{0,m_1}(x)}{s}ds, \\
\frac{d\varphi^{-}_{p,m_1}}{dx} &= \frac{1}{(p-1)!}\int_0^x (\log x - \log s)^{p-1}\,\frac{\varphi^{-}_{0,m_1}(x)}{s}ds.
\end{aligned}
$$

Therefore, to conclude the proof it is enough to substitute the expressions of the functions φ^{+}_{0,m_1} and φ^{-}_{0,m_1}, provided by formulas (44) and (45), into (46) and (47), respectively. □

Theorem 2. *We have*

$$
\sum_{m_0=1}^{+\infty} \frac{H_{m_0,1}}{m_0!}x^{m_0} = (e^x-1)\log x + e^x\sigma^{-}(x), \tag{51}
$$

$$
\sum_{m_0=1}^{+\infty} (-1)^{m_0}\frac{H_{m_0,1}}{m_0!}x^{m_0} = \left(e^{-x}-1\right)\log x - e^{-x}\sigma^{+}(x). \tag{52}
$$

If $p \geq 1$, the below formulas hold

$$
\sum_{m_0=1}^{+\infty} \frac{x^{m_0}}{m_0^p m_0!} = \frac{1}{(p-1)!}\int_0^x (\log x - \log s)^{p-1}\,\frac{e^s-1}{s}ds, \tag{53}
$$

$$\sum_{m_0=1}^{+\infty} (-1)^{m_0} \frac{x^{m_0}}{m_0^p m_0!} = \frac{1}{(p-1)!} \int_0^x (\log x - \log s)^{p-1} \frac{e^{-s}-1}{s} ds, \tag{54}$$

and

$$\sum_{m_0=1}^{+\infty} \frac{H_{m_0,1}}{m_0^p m_0!} x^{m_0} = \frac{1}{(p-1)!} \int_0^x (\log x - \log s)^{p-1} \left(\frac{e^s-1}{s} \log s - \frac{e^s \sigma^-(s)}{s} \right) ds, \tag{55}$$

$$\sum_{m_0=1}^{+\infty} (-1)^{m_0} \frac{H_{m_0,1}}{m_0^p m_0!} x^{m_0} = \frac{1}{(p-1)!} \int_0^x (\log x - \log s)^{p-1} \left(\frac{e^{-s}-1}{s} \log s - \frac{e^{-s} \sigma^+(s)}{s} \right) ds. \tag{56}$$

Proof. Let us first start with the case $p = 0$. The first assertion (51) has been proved in the paper [18]. In order to prove (52), we follow these steps. We have

$$\begin{aligned}
\varphi^-_{0,m_1}(x) &= \sum_{m_0=1}^{+\infty} (-1)^{m_0} \frac{x^{m_0+m_1\varepsilon}}{(m_0+m_1\varepsilon)!} \\
&= (1 + m_1 \varepsilon \log x) \sum_{m_0=1}^{+\infty} (-1)^{m_0} \frac{x^{m_0}}{m_0! \,(1 + H_{m_0,1} m_1 \varepsilon)} \\
&= (1 + m_1 \varepsilon \log x) \sum_{m_0=1}^{+\infty} \frac{(-1)^{m_0}}{m_0!} (1 - H_{m_0,1} m_1 \varepsilon)\, x^{m_0} \\
&= \sum_{m_0=1}^{+\infty} \frac{(-1)^{m_0}}{m_0!} \left(1 + (\log x - H_{m_0,1})\, m_1 \varepsilon\right) x^{m_0}.
\end{aligned}$$

Hence, in view of (44), we find

$$\sum_{m_0=1}^{+\infty} \frac{(-1)^{m_0}}{m_0!} x^{m_0} = e^{-x} - 1,$$

as well as

$$\left(e^{-x} - 1\right) \log x - \sum_{m_0=1}^{+\infty} (-1)^{m_0} \frac{H_{m_0,1}}{m_0!} x^{m_0} = e^{-x} \sigma^+(x).$$

Thus, the case $p = 0$ is accomplished. If $p \geq 1$ the function $\varphi^+_{p,m_1}(x)$ becomes

$$\begin{aligned}
\varphi^+_{p,m_1}(x) &= \sum_{m_0=1}^{+\infty} \frac{x^{m_0+m_1\varepsilon}}{(m_0+m_1\varepsilon)^p (m_0+m_1\varepsilon)!} \\
&= (1 + m_1 \varepsilon \log x) \sum_{m_0=1}^{+\infty} \frac{x^{m_0}}{m_0^p m_0! \left(1 + p\frac{m_1}{m_0}\varepsilon\right) (1 + H_{m_0,1} m_1 \varepsilon)} \\
&= (1 + m_1 \varepsilon \log x) \sum_{m_0=1}^{+\infty} \frac{1}{m_0^p m_0!} \left(1 - \left(H_{m_0,1} + \frac{p}{m_0}\right) m_1 \varepsilon\right) x^{m_0} \\
&= \sum_{m_0=1}^{+\infty} \frac{1}{m_0^p m_0!} \left(1 + \left(\log x - H_{m_0,1} - \frac{p}{m_0}\right) m_1 \varepsilon\right) x^{m_0}.
\end{aligned}$$

As consequence, Theorem 1 permits us to find

$$\sum_{m_0=1}^{+\infty} \frac{x^{m_0}}{m_0^p m_0!} = \frac{1}{(p-1)!} \int_0^x (\log x - \log s)^{p-1} \frac{e^s - 1}{s} ds, \tag{57}$$

and

$$\sum_{m_0=1}^{+\infty}\frac{1}{m_0^p m_0!}\left(\log x-H_{m_0,1}-\frac{p}{m_0}\right)x^{m_0}=\frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^s\sigma^-(s)}{s}ds. \quad (58)$$

Combining formula (57) and (58), we obtain

$$\begin{aligned}\sum_{m_0=1}^{+\infty}\frac{H_{m_0,1}}{m_0^p m_0!}x^{m_0} &= -p\sum_{m_0=1}^{+\infty}\frac{1}{m_0^{p+1}m_0!}x^{m_0}+\frac{\log x}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^s-1}{s}ds\\ &\quad -\frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^s\sigma^-(s)}{s}ds.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{m_0=1}^{+\infty}\frac{H_{m_0,1}}{m_0^p m_0!}x^{m_0} &= -p\sum_{m_0=1}^{+\infty}\frac{1}{m_0^{p+1}m_0!}x^{m_0}+\frac{\log x}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^s-1}{s}ds\\ &\quad -\frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^s\sigma^-(s)}{s}ds-\frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p}\frac{e^s-1}{s}ds\\ &= \frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\left(\frac{e^s-1}{s}\log s-\frac{e^s\sigma^-(s)}{s}\right).\end{aligned}$$

Additionally, the following formula can be readily established

$$\begin{aligned}\varphi^-_{p,m_1}(x) &= \sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{x^{m_0+m_1\varepsilon}}{(m_0+m_1\varepsilon)^p(m_0+m_1\varepsilon)!}\\ &= \sum_{m_0=1}^{+\infty}\frac{(-1)^{m_0}}{m_0^p m_0!}\left(1+\left(\log x-H_{m_0,1}-\frac{p}{m_0}\right)m_1\varepsilon\right)x^{m_0}. \quad (59)\end{aligned}$$

Making together formulas (47) and (59), one has

$$\sum_{m_0=1}^{+\infty}\frac{(-1)^{m_0}}{m_0^p m_0!}x^{m_0}=\frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^{-s}-1}{s}ds,$$

and

$$\begin{aligned}\frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^{-s}\sigma^+(s)}{s}ds &= \frac{\log x}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^{-s}-1}{s}ds\\ &\quad -\frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p}\frac{e^{-s}\sigma^+(s)}{s}ds-\sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{H_{m_0,1}}{m_0^p m_0!}x^{m_0}.\end{aligned}$$

Which eventually gives (56) and enables us to complete the proof. □

3.2. Application 2

Given for every $p\geq 1$ the following log −series

$$\sum_{m_0=1}^{+\infty}\frac{H_{m_0+m_1\varepsilon,p}}{(m_0+m_1\varepsilon)!}x^{m_0+m_1\varepsilon}, \quad (60)$$

$$\sum_{m_0=1}^{+\infty} (-1)^{m_0} \frac{H_{m_0+m_1\varepsilon,p}}{(m_0+m_1\varepsilon)!} x^{m_0+m_1\varepsilon}. \tag{61}$$

It is straightforward to show that the both log −series converge for every $x \in [0, +\infty[$. Denote by $\psi^{+}_{q,m_1}(x)$ and $\psi^{-}_{q,m_1}(x)$ their sums, respectively.

Theorem 3. *We have*

$$\psi^{+}_{1,m_1}(x) = e^{x} \int_0^x \frac{1-e^{-s}}{s} ds + m_1\varepsilon e^{x} \int_0^x \frac{\sigma^{-}(s)}{s} ds, \tag{62}$$

$$\psi^{-}_{1,m_1}(x) = e^{-x} \int_0^x \frac{1-e^{s}}{s} ds - m_1\varepsilon e^{-x} \int_0^x \frac{\sigma^{+}(s)}{s} ds. \tag{63}$$

For every $p \geq 2$, we have

$$\begin{aligned} \psi^{+}_{p,m_1}(x) =& \frac{e^{x}}{(p-2)!} \int_0^x \int_0^t \frac{e^{-t}}{t} (\log t - \log s)^{p-2} \frac{e^{s}-1}{s} ds dt \\ &+ m_1\varepsilon \frac{e^{x}}{(p-2)!} \int_0^x \int_0^t \frac{e^{-t}}{t} (\log t - \log s)^{p-2} \frac{e^{s}\sigma^{-}(s)}{s} ds dt, \end{aligned} \tag{64}$$

$$\begin{aligned} \psi^{-}_{p,m_1}(x) =& \frac{e^{-x}}{(p-2)!} \int_0^x \int_0^t \frac{e^{t}}{t} (\log t - \log s)^{p-2} \frac{e^{-s}-1}{s} ds dt \\ &- m_1\varepsilon \frac{e^{-x}}{(p-2)!} \int_0^x \int_0^t \frac{e^{t}}{t} (\log t - \log s)^{p-2} \frac{e^{-s}\sigma^{+}(s)}{s} ds dt. \end{aligned} \tag{65}$$

Proof. The functions $\psi^{+}_{p,m_1}(x)$ and $\psi^{-}_{p,m_1}(x)$ can be written

$$\begin{aligned} \psi^{+}_{p,m_1}(x) &= \sum_{m_0=1}^{+\infty} \left(\sum_{n=1}^{m_0} \frac{1}{(n+m_1\varepsilon)^{p}} \right) \frac{x^{m_0+m_1\varepsilon}}{(m_0+m_1\varepsilon)!}, \\ \psi^{-}_{p,m_1}(x) &= \sum_{m_0=1}^{+\infty} (-1)^{m_0} \left(\sum_{n=1}^{m_0} \frac{1}{(n+m_1\varepsilon)^{p}} \right) \frac{x^{m_0+m_1\varepsilon}}{(m_0+m_1\varepsilon)!}. \end{aligned}$$

Which implies by computing the derivatives

$$\frac{d\psi^{+}_{p,m_1}}{dx} = \frac{x^{m_1\varepsilon}}{(1+m_1\varepsilon)^{p}} + \psi^{+}_{p,m_1}(x) + \lambda^{+}_{p,m_1}(x), \tag{66}$$

$$\frac{d\psi^{-}_{p,m_1}}{dx} = -\frac{x^{m_1\varepsilon}}{(1+m_1\varepsilon)^{p}} - \psi^{-}_{p,m_1}(x) - \lambda^{-}_{p,m_1}(x), \tag{67}$$

where the functions λ^{+}_{p,m_1} and λ^{-}_{p,m_1} are given by

$$\lambda^{+}_{p,m_1}(x) = \sum_{m_0=1}^{+\infty} \frac{x^{m_0+m_1\varepsilon}}{(m_0+m_1\varepsilon)!\,(m_0+1+m_1\varepsilon)^{p}}, \tag{68}$$

$$\lambda^{-}_{p,m_1}(x) = \sum_{m_0=1}^{+\infty} (-1)^{m_0} \frac{x^{m_0+m_1\varepsilon}}{(m_0+m_1\varepsilon)!\,(m_0+1+m_1\varepsilon)^{p}}. \tag{69}$$

Clearly, there are two cases to deal with separately. The first one corresponds to $p = 1$. Here, the functions λ^+_{1,m_1} and λ^+_{1,m_1} can be expressed as

$$\begin{aligned}\lambda^+_{1,m_1}(x) &= \sum_{m_0=1}^{+\infty} \frac{x^{m_0+m_1\varepsilon}}{(m_0+1+m_1\varepsilon)!}\\ &= \frac{1}{x}\left(e^x - 1 + e^x\sigma^+(x)\,m_1\varepsilon - \frac{x^{1+m_1\varepsilon}}{1+m_1\varepsilon}\right),\end{aligned}$$

and

$$\begin{aligned}\lambda^-_{1,m_1}(x) &= \sum_{m_0=1}^{+\infty} (-1)^{m_0} \frac{x^{m_0+m_1\varepsilon}}{(m_0+1+m_1\varepsilon)!}\\ &= \frac{-1}{x}\left(\sum_{m_0=1}^{+\infty} (-1)^{m_0} \frac{x^{m_0+m_1\varepsilon}}{(m_0+m_1\varepsilon)!} - \frac{x^{1+m_1\varepsilon}}{1+m_1\varepsilon}\right)\\ &= \frac{-1}{x}\left(e^{-x} - 1 - e^{-x}\sigma^+(x)\,m_1\varepsilon + \frac{x^{1+m_1\varepsilon}}{1+m_1\varepsilon}\right).\end{aligned}$$

The preceding formulas result in

$$\frac{d\psi^+_{1,m_1}}{dx} = \psi^+_{1,m_1}(x) + \frac{1}{x}\left(e^x - 1 + e^{-x}\sigma^-(x)\,m_1\varepsilon\right), \tag{70}$$

$$\frac{d\psi^-_{1,m_1}}{dx} = -\psi^-_{1,m_1}(x) + \frac{1}{x}\left(e^{-x} - 1 - e^{-x}\sigma^+(x)\,m_1\varepsilon\right). \tag{71}$$

The solutions of the homogenious ODE corresponding to (70) and (71) are given by

$$\psi^+_{1,m_1}(x) = C_1 e^x \text{ and } \psi^-_{1,m_1}(x) = C_2 e^{-x},$$

where C_1 and C_2 are two dual constants. We now apply the method of variation of parameters to obtain

$$\begin{aligned}C_1(x) &= \int_0^x \frac{1-e^{-s}}{s}ds + m_1\varepsilon\int_0^x \frac{\sigma^-(s)}{s}ds + C_1,\\ C_1(x) &= \int_0^x \frac{1-e^{s}}{s}ds - m_1\varepsilon\int_0^x \frac{\sigma^+(s)}{s}ds + C_2,\end{aligned}$$

Thus, since $\psi^-_{1,m_1}(0) = 0$ and $\psi^+_{1,m_1}(0) = 0$, the desired results follow.

The second case regarding $p \geq 2$. We observe here that

$$\begin{aligned}\lambda^+_{p,m_1}(x) &= \frac{d\varphi^+_{p,m_1}}{dx} - \frac{x^{m_1\varepsilon}}{(1+m_1\varepsilon)^p},\\ \lambda^-_{p,m_1}(x) &= -\frac{d\varphi^-_{p,m_1}}{dx} - \frac{x^{m_1\varepsilon}}{(1+m_1\varepsilon)^p},\end{aligned}$$

This allows us to obtain, keeping in mind (66) and (67)

$$\begin{aligned}\frac{d\psi^+_{p,m_1}}{dx} &= \psi^+_{p,m_1}(x) + \frac{d\varphi^+_{p,m_1}}{dx},\\ \frac{d\psi^-_{p,m_1}}{dx} &= -\psi^-_{p,m_1}(x) + \frac{d\varphi^-_{p,m_1}}{dx},\end{aligned}$$

On the other hand, using the Theorem 1, we can assert that

$$\begin{aligned}\frac{d\varphi^{+}_{p,m_1}}{dx} &= \frac{1}{(p-2)!x}\int_0^x (\log x-\log s)^{p-2}\frac{e^s-1}{s}ds+\frac{m_1\varepsilon}{(p-2)!x}\int_0^x(\log x-\log s)^{p-2}\frac{e^s\sigma^-(s)}{s}ds.\\ \frac{d\varphi^{-}_{p,m_1}}{dx} &= \frac{1}{(p-2)!x}\int_0^x (\log x-\log s)^{p-1}\frac{e^{-s}-1}{s}ds+\frac{m_1\varepsilon}{(p-2)!x}\int_0^x(\log x-\log s)^{p-1}\frac{e^{-s}\sigma^+(s)}{s}ds.\end{aligned}$$

This yields

$$\begin{aligned}\frac{d\psi^{+}_{p,m_1}}{dx} &= \psi_{p,m_1}(x)+\frac{1}{(p-2)!x}\int_0^x (\log x-\log s)^{p-2}\frac{e^s-1}{s}ds\\ &\quad+\frac{m_1\varepsilon}{(p-2)!x}\int_0^x(\log x-\log s)^{p-2}\frac{e^s\sigma^-(s)}{s}ds; \qquad (72)\\ \frac{d\psi^{-}_{p,m_1}}{dx} &= -\psi^{-}_{p,m_1}(x)+\frac{1}{(p-2)!x}\int_0^x (\log x-\log s)^{p-1}\frac{e^{-s}-1}{s}ds\\ &\quad+\frac{m_1\varepsilon}{(p-2)!x}\int_0^x(\log x-\log s)^{p-1}\frac{e^{-s}\sigma^+(s)}{s}ds. \qquad (73)\end{aligned}$$

Obviously, the homogeneous ODE corresponding to (72) and (73) admit the following functions as solutions

$$\psi_{p,m_1}(x)=C_1e^x \text{ and } \psi_{p,m_1}(x)=C_2e^{-x},$$

where C_1 and C_2 are two dual constants. By the method of variation of the parameter. It comes

$$\begin{aligned}C_1'(x) &= \frac{e^{-x}}{(p-2)!x}\int_0^x(\log x-\log s)^{p-2}\left(\frac{e^s-1}{s}+\frac{e^s\sigma^-(s)}{s}m_1\varepsilon\right)ds,\\ C_2'(x) &= \frac{e^{x}}{(p-2)!x}\int_0^x(\log x-\log s)^{p-2}\left(\frac{e^{-s}-1}{s}+\frac{e^{-s}\sigma^+(s)}{s}m_1\varepsilon\right)ds,\end{aligned}$$

So, since $\psi^{-}_{p,m_1}(0)=0$ and $\psi^{+}_{p,m_1}(0)=0$, formulas (64) and (65) are satisfied, thereby completing the proof. □

Theorem 4. *We have*

$$\sum_{m_0=1}^{+\infty}\frac{H^2_{m_0,1}+H_{m_0,2}}{m_0!}x^{m_0}=e^x\log x\int_0^x\frac{1-e^{-s}}{s}ds-e^x\int_0^x\frac{\sigma^-(s)}{s}ds, \qquad (74)$$

$$\sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{H^2_{m_0,1}+H_{m_0,2}}{m_0!}x^{m_0}=e^{-x}\log x\int_0^x\frac{1-e^{s}}{s}ds-e^{-x}\int_0^x\frac{\sigma^+(s)}{s}ds. \qquad (75)$$

If $p\geq 2$, the following formulas are fulfilled

$$\sum_{m_0=1}^{+\infty}\frac{H_{m_0,p}}{m_0!}x^{m_0}=\frac{e^x}{(p-2)!}\int_0^x\int_0^t\frac{e^{-t}}{t}(\log t-\log s)^{p-2}\frac{e^s-1}{s}dsdt, \qquad (76)$$

$$\sum_{m_0=1}^{+\infty}\frac{H_{m_0,1}H_{m_0,p}}{m_0!}x^{m_0}=\frac{1}{(p-2)!}e^x\log x\int_0^x\int_0^t\frac{e^{-t}}{t}(\log t-\log s)^{p-2}\frac{e^s-1}{s}dsdt$$

$$- \frac{p}{(p-1)!} e^{x} \int_0^x \int_0^t \frac{e^{-t}}{t} (\log t - \log s)^{p-1} \frac{e^{s}-1}{s} ds dt$$

$$- \frac{1}{(p-2)!} e^{x} \int_0^x \int_0^t \frac{e^{-t}}{t} (\log t - \log s)^{p-2} \frac{e^{s}\sigma^{-}(s)}{s} ds dt, \tag{77}$$

and

$$\sum_{m_0=1}^{+\infty} (-1)^{m_0} \frac{H_{m_0,1}^2 + H_{m_0,2}}{m_0!} x^{m_0} = e^{-x} \log x \int_0^x \frac{1-e^{s}}{s} ds - e^{-x} \int_0^x \frac{\sigma^{+}(s)}{s} ds, \tag{78}$$

$$\sum_{m_0=1}^{+\infty} (-1)^{m_0} \frac{H_{m_0,1} H_{m_0,p}}{m_0!} x^{m_0} = \frac{e^{-x} \log x}{(p-2)!} \int_0^x \int_0^t \frac{e^{t}}{t} (\log t - \log s)^{p-2} \frac{e^{-s}-1}{s} ds dt$$

$$- \frac{p}{(p-1)!} e^{-x} \int_0^x \int_0^t \frac{e^{t}}{t} (\log t - \log s)^{p-1} \frac{e^{-s}-1}{s} ds dt$$

$$- \frac{1}{(p-2)!} e^{-x} \int_0^x \int_0^t \frac{e^{t}}{t} (\log t - \log s)^{p-2} \frac{e^{-s}\sigma^{+}(s)}{s} ds dt. \tag{79}$$

Proof. Formulas (60) and (61) imply using (22) and (25)

$$\psi_{p,m_1}^{+}(x) = \sum_{m_0=1}^{+\infty} \frac{H_{m_0,p} - m_1 p H_{m_0,p+1}\varepsilon}{m_0! \left(1 + m_1 H_{m_0,1}\varepsilon\right)} x^{m_0 + m_1\varepsilon}$$

$$= (1 + m_1 \varepsilon \log x) \sum_{m_0=1}^{+\infty} \frac{\left(H_{m_0,p} - m_1 p H_{m_0,p+1}\varepsilon\right)\left(1 - m_1 H_{m_0,1}\varepsilon\right)}{m_0!} x^{m_0}$$

$$\psi_{p,m_1}^{-}(x) = \sum_{m_0=1}^{+\infty} (-1)^{m_0} \frac{H_{m_0,p} - m_1 p H_{m_0,p+1}\varepsilon}{m_0! \left(1 + m_1 H_{m_0,1}\varepsilon\right)} x^{m_0 + m_1\varepsilon}$$

$$= (1 + m_1 \varepsilon \log x) \sum_{m_0=1}^{+\infty} (-1)^{m_0} \frac{\left(H_{m_0,p} - m_1 p H_{m_0,p+1}\varepsilon\right)\left(1 - m_1 H_{m_0,1}\varepsilon\right)}{m_0!} x^{m_0}.$$

So, the following formulas fold

$$\psi_{p,m_1}^{+}(x) = \sum_{m_0=1}^{+\infty} \frac{1}{m_0!} \left(H_{m_0,p} + \left(H_{m_0,p} \log x - H_{m_0,p} H_{m_0,1} - p H_{m_0,p+1}\right) m_1 \varepsilon\right) x^{m_0}, \tag{80}$$

$$\psi_{p,m_1}^{-}(x) = \sum_{m_0=1}^{+\infty} \frac{(-1)^{m_0}}{m_0!} \left(H_{m_0,p} + \left(H_{m_0,p} \log x - H_{m_0,p} H_{m_0,1} - p H_{m_0,p+1}\right) m_1 \varepsilon\right) x^{m_0}, \tag{81}$$

Let us start with the case $p = 1$. One can easily obtain making use (62), (63), (80) and (81)

$$\sum_{m_0=1}^{+\infty} \frac{H_{m_0,1}}{m_0!} x^{m_0} = e^{x} \int_0^x \frac{1-e^{-s}}{s} ds,$$

$$\sum_{m_0=1}^{+\infty} (-1)^{m_0} \frac{H_{m_0,1}}{m_0!} x^{m_0} = e^{-x} \int_0^x \frac{1-e^{s}}{s} ds,$$

and

$$\sum_{m_0=1}^{+\infty} \frac{H_{m_0,1}^2 + H_{m_0,2}}{m_0!} x^{m_0} = e^{x} \log x \int_0^x \frac{1-e^{-s}}{s} ds - e^{x} \int_0^x \frac{\sigma^{-}(s)}{s} ds,$$

$$\sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{H_{m_0,1}^2+H_{m_0,2}}{m_0!}x^{m_0} = e^{-x}\log x\int_0^x\frac{1-e^s}{s}ds-e^{-x}\int_0^x\frac{\sigma^+(s)}{s}ds,$$

Moreover, for $p\geq 2$, we can infer taking into account (64), (65), (80) and (81)

$$\frac{e^x}{(p-2)!}\int_0^x\int_0^t\frac{e^{-t}}{t}(\log t-\log s)^{p-2}\frac{e^s-1}{s}dsdt+m_1\varepsilon\frac{e^x}{(p-2)!}\int_0^x\int_0^t\frac{e^{-t}}{t}(\log t-\log s)^{p-2}\frac{e^s}{s}\sigma^-(s)\,dsdt$$
$$=\sum_{m_0=1}^{+\infty}\frac{1}{m_0!}\left(H_{m_0,p}+\left(H_{m_0,p}\log x-H_{m_0,p}H_{m_0,1}-pH_{m_0,p+1}\right)m_1\varepsilon\right)x^{m_0},$$

and

$$\frac{e^{-x}}{(p-2)!}\int_0^x\int_0^t\frac{e^{t}}{t}(\log t-\log s)^{p-2}\frac{e^{-s}-1}{s}dsdt+m_1\varepsilon\frac{e^{-x}}{(p-2)!}\int_0^x\int_0^t\frac{e^{t}}{t}(\log t-\log s)^{p-2}\frac{e^{-s}\sigma^+(s)}{s}dsdt$$
$$=\sum_{m_0=1}^{+\infty}\frac{(-1)^{m_0}}{m_0!}\left(H_{m_0,p}+\left(H_{m_0,p}\log x-H_{m_0,p}H_{m_0,1}-pH_{m_0,p+1}\right)m_1\varepsilon\right)x^{m_0}.$$

Thus, we deduce that

$$\sum_{m_0=1}^{+\infty}\frac{H_{m_0,p}}{m_0!}x^{m_0} = \frac{e^x}{(p-2)!}\int_0^x\int_0^t\frac{e^{-t}}{t}(\log t-\log s)^{p-2}\frac{e^s-1}{s}dsdt,$$
$$\sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{H_{m_0,p}}{m_0!}x^{m_0} = \frac{e^{-x}}{(p-2)!}\int_0^x\int_0^t\frac{e^{t}}{t}(\log t-\log s)^{p-2}\frac{e^{-s}-1}{s}dsdt,$$

and

$$\begin{aligned}\sum_{m_0=1}^{+\infty}\frac{H_{m_0,p}H_{m_0,1}}{m_0!}x^{m_0} =&\frac{e^x\log x}{(p-2)!}\int_0^x\int_0^t\frac{e^{-t}}{t}(\log t-\log s)^{p-2}\frac{e^s-1}{s}dsdt\\&-p\frac{e^x}{(p-1)!}\int_0^x\int_0^t\frac{e^{-t}}{t}(\log t-\log s)^{p-1}\frac{e^s-1}{s}dsdt\\&-\frac{e^x}{(p-2)!}\int_0^x\int_0^t\frac{e^{-t}}{t}(\log t-\log s)^{p-2}\frac{e^s\sigma^-(s)}{s}dsdt,\end{aligned}$$

and

$$\begin{aligned}\sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{H_{m_0,p}H_{m_0,1}}{m_0!}x^{m_0} =&\frac{e^{-x}\log x}{(p-2)!}\int_0^x\int_0^t\frac{e^{t}}{t}(\log t-\log s)^{p-2}\frac{e^{-s}-1}{s}dsdt\\&-p\frac{e^{-x}}{(p-1)!}\int_0^x\int_0^t\frac{e^{t}}{t}(\log t-\log s)^{p-1}\frac{e^{-s}-1}{s}dsdt\\&-\frac{e^{-x}}{(p-2)!}\int_0^x\int_0^t\frac{e^{t}}{t}(\log t-\log s)^{p-2}\frac{e^{-s}\sigma^+(s)}{s}dsdt.\end{aligned}$$

□

3.3. Application 3

Given $m = \sum_{i=0}^{n} m_i \varepsilon^i \in \mathbb{Z}_n^+(\varepsilon)$ and consider the two log −series given by

$$\sum_{m_0=1}^{+\infty} \frac{F_{m_0+m_1\varepsilon}}{(m_0+m_1\varepsilon)^p} x^{m_0+m_1\varepsilon}, \tag{82}$$

$$\sum_{m_0=1}^{+\infty} (-1)^{m_0} \frac{F_{m_0+m_1\varepsilon}}{(m_0+m_1\varepsilon)^p} x^{m_0+m_1\varepsilon}. \tag{83}$$

It easy to verify that the both log −series converge for every $x \in [0,1[$. Denote by $\mu_{p,m_1}^+(x)$ and $\mu_{p,m_1}^-(x)$ their sums, respectively.

Theorem 5. *We have*

$$\mu_{0,m_1}^+(x) = \frac{1}{1-x}\left(e^x - 1 + e^x\sigma^-(x)\,m_1\varepsilon\right), \tag{84}$$

$$\mu_{0,m_1}^-(x) = \frac{1}{1+x}\left(e^{-x} - 1 - e^{-x}\sigma^+(x)\,m_1\varepsilon\right). \tag{85}$$

For every $p \geq 1$, we have

$$\begin{aligned}\mu_{p,m_1}^+(x) &= \frac{1}{(p-1)!}\int_0^x (\log x - \log s)^{p-1} \frac{e^s-1}{s(1-s)} ds \\ &\quad + m_1\varepsilon \frac{1}{(p-1)!}\int_0^x (\log x - \log s)^{p-1} \frac{e^s\sigma^-(s)}{s(1-s)} ds,\end{aligned} \tag{86}$$

$$\begin{aligned}\mu_{p,m_1}^-(x) &= \frac{1}{(p-1)!}\int_0^x (\log x - \log s)^{p-1} \frac{e^{-s}-1}{s(1+s)} ds \\ &\quad + m_1\varepsilon \frac{1}{(p-1)!}\int_0^x (\log x - \log s)^{p-1} \frac{e^{-s}\sigma^+(s)}{s(1+s)} ds.\end{aligned} \tag{87}$$

Proof. Suppose first that $p = 0$. We have

$$\begin{aligned}\mu_{0,m_1}^+(x) &= \sum_{m_0=1}^{+\infty} F_{m_0+m_1\varepsilon} x^{m_0+m_1\varepsilon}, \\ \mu_{0,m_1}^-(x) &= \sum_{m_0=1}^{+\infty} (-1)^{m_0} F_{m_0+m_1\varepsilon} x^{m_0+m_1\varepsilon},\end{aligned}$$

Then,

$$\begin{aligned}\mu_{0,m_1}^+(x) &= x^{m_1\varepsilon}\sum_{m_0=1}^{+\infty}\left(\sum_{n=1}^{m_0}\frac{1}{(n+m_1\varepsilon)!}\right)x^{m_0} \\ &= x^{m_1\varepsilon}\sum_{m_0=1}^{+\infty}\frac{1}{(m_0+m_1\varepsilon)!}\sum_{n=m_0}^{+\infty}x^n \\ &= \frac{x^{m_1\varepsilon}}{1-x}\left(\sum_{m_0=1}^{+\infty}\frac{x^{m_0}}{(m_0+m_1\varepsilon)!}\right) \\ &= \frac{1}{1-x}\sum_{m_0=1}^{+\infty}\frac{x^{m_0+m_1\varepsilon}}{(m_0+m_1\varepsilon)!} \\ &= \frac{1}{1-x}\left(e^x - 1 + m_1 e^x\sigma^-(x)\,\varepsilon\right),\end{aligned}$$

and

$$\begin{aligned}
\mu^{-}_{0,m_1}(x) &= x^{m_1\varepsilon}\sum_{m_0=1}^{+\infty}(-1)^{m_0}\left(\sum_{n=1}^{m_0}\frac{1}{(n+m_1\varepsilon)!}\right)x^{m_0} \\
&= x^{m_1\varepsilon}\sum_{m_0=1}^{+\infty}\frac{1}{(m_0+m_1\varepsilon)!}\sum_{n=m_0}^{+\infty}(-1)^n x^n \\
&= \frac{x^{m_1\varepsilon}}{1+x}\left(\sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{x^{m_0}}{(m_0+m_1\varepsilon)!}\right) \\
&= \frac{1}{1+x}\sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{x^{m_0+m_1\varepsilon}}{(m_0+m_1\varepsilon)!} \\
&= \frac{1}{1+x}\left(e^{-x}-1-e^{-x}\sigma^{+}(x)\,m_1\varepsilon\right).
\end{aligned}$$

These eventually yield (84) and (85). For $p \geq 1$, we proceed as follows. Note that

$$\begin{aligned}
\mu^{+}_{p,m_1}(x) &= \sum_{m_0=1}^{+\infty}\frac{F_{m_0+m_1\varepsilon}}{(m_0+m_1\varepsilon)^p}x^{m_0+m_1\varepsilon} \\
&= \sum_{m_0=1}^{+\infty}\left(\sum_{n=1}^{m_0}\frac{1}{(n+m_1\varepsilon)!}\right)\frac{x^{m_0+m_1\varepsilon}}{(m_0+m_1\varepsilon)^p},
\end{aligned}$$

and

$$\begin{aligned}
\mu^{-}_{p,m_1}(x) &= \sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{F_{m_0+m_1\varepsilon}}{(m_0+m_1\varepsilon)^p}x^{m_0+m_1\varepsilon} \\
&= \sum_{m_0=1}^{+\infty}(-1)^{m_0}\left(\sum_{n=1}^{m_0}\frac{1}{(n+m_1\varepsilon)!}\right)\frac{x^{m_0+m_1\varepsilon}}{(m_0+m_1\varepsilon)^p}.
\end{aligned}$$

The above two formulas imply by calculating the derivatives of the two hand sides, respectively, that

$$\begin{aligned}
\frac{d\mu^{+}_{p,m_1}}{dx} &= \sum_{m_0=1}^{+\infty}\left(\sum_{n=1}^{m_0}\frac{1}{(n+m_1\varepsilon)!}\right)\frac{x^{m_0-1+m_1\varepsilon}}{(m_0+m_1\varepsilon)^{p-1}} \\
&= \frac{\mu^{+}_{p-1,m_1}(x)}{x},
\end{aligned}$$

and

$$\begin{aligned}
\frac{d\mu^{-}_{p,m_1}}{dx} &= \sum_{m_0=1}^{+\infty}(-1)^{m_0}\left(\sum_{n=1}^{m_0}\frac{1}{(n+m_1\varepsilon)!}\right)\frac{x^{m_0-1+m_1\varepsilon}}{(m_0+m_1\varepsilon)^{p-1}} \\
&= \frac{\mu^{-}_{p-1,m_1}(x)}{x}.
\end{aligned}$$

Thus, Lemma 5 permits us to find

$$\mu^{+}_{p,m_1}(x) = \frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{\mu^{+}_{0,m_1}(x)}{s}ds,$$

and

$$\mu^{-}_{p,m_1}(x) = \frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{\mu^{-}_{0,m_1}(x)}{s}ds.$$

These permit us to conclude exploiting (84) and (85)

$$\begin{aligned}\mu^{+}_{p,m_1}(x) &= \frac{1}{(p-1)!}\int_0^x \frac{(\log x-\log s)^{p-1}}{s(1-s)}\left(e^s-1+e^s\sigma^-(s)\,m_1\varepsilon\right)ds\\ &= \frac{1}{(p-1)!}\int_0^x \frac{e^s-1}{s(1-s)}(\log x-\log s)^{p-1}ds\\ &\quad+\frac{m_1\varepsilon}{(p-1)!}\int_0^x \frac{e^s}{s(1-s)}(\log x-\log s)^{p-1}\sigma^-(s)\,ds,\end{aligned}$$

and

$$\begin{aligned}\mu^{-}_{p,m_1}(x) &= \frac{1}{(p-1)!}\int_0^x \frac{(\log x-\log s)^{p-1}}{s(1+s)}\left(e^{-s}-1-e^{-s}\sigma^+(s)\,m_1\varepsilon\right)ds\\ &= \frac{1}{(p-1)!}\int_0^x \frac{e^{-s}-1}{s(1+s)}(\log x-\log s)^{p-1}ds\\ &\quad+\frac{m_1\varepsilon}{(p-1)!}\int_0^x \frac{e^{-s}}{s(1+s)}(\log x-\log s)^{p-1}\sigma^+(s)\,ds.\end{aligned}$$

Which allows us to finalize the proof. □

Theorem 6. *We have*

$$\sum_{m_0=1}^{+\infty} F_{m_0}x^{m_0} = \frac{e^x+x-1}{1-x}, \tag{88}$$

$$\sum_{m_0=1}^{+\infty} (-1)^{m_0} F_{m_0}x^{m_0} = \frac{e^{-x}-x-1}{1+x}, \tag{89}$$

and

$$\sum_{m_0=1}^{+\infty}\left(\sum_{n=1}^{m_0}\frac{H_{n,1}}{n!}\right)x^{m_0} = \frac{e^x-1}{1-x}\log x-\frac{e^x\sigma^-(x)}{1-x}, \tag{90}$$

$$\sum_{m_0=1}^{+\infty}(-1)^{m_0}\left(\sum_{n=1}^{m_0}\frac{H_{n,1}}{n!}\right)x^{m_0} = \frac{e^{-x}-1}{1+x}\log x+\frac{e^{-x}\sigma^+(x)}{1+x}, \tag{91}$$

For every $p\geq 1$, we have

$$\sum_{m_0=1}^{+\infty}\frac{F_{m_0}}{m_0^p}x^{m_0} = Li_p(x)+\frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^s-1}{s(1-s)}ds, \tag{92}$$

$$\sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{F_{m_0}}{m_0^p}x^{m_0} = Li_p(-x)+\frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^{-s}-1}{s(1+s)}ds, \tag{93}$$

$$\sum_{m_0=1}^{+\infty}\frac{1}{m_0^p}\left(\sum_{n=1}^{m_0}\frac{H_{n,1}}{n!}\right)x^{m_0} = \frac{1}{(p-1)!}\int_0^x\frac{(\log x-\log s)^{p-1}}{s(1-s)}\left((e^s-1)\log s-e^s\sigma^-(s)\right)ds, \tag{94}$$

$$\sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{1}{m_0^p}\left(\sum_{n=1}^{m_0}\frac{H_{n,1}}{n!}\right)x^{m_0} = \frac{1}{(p-1)!}\int_0^x\frac{(\log x-\log s)^{p-1}}{s(1+s)}\left((e^{-s}-1)\log s-e^{-s}\sigma^+(s)\right)ds, \tag{95}$$

where Li_p represents the polylogarithm function given by

$$Li_p(x) = \sum_{n=1}^{+\infty} \frac{x^k}{k^p}. \tag{96}$$

For further details regarding the function Li_p, see the references [21,22].

Proof. For every $p \geq 0$, we have

$$\begin{aligned}
\mu^+_{p,m_1}(x) &= \sum_{m_0=1}^{+\infty} \frac{1}{m_0^p} \left(F_{m_0} - 1 - m_1\varepsilon \sum_{n=1}^{m_0} \frac{H_{n,1}}{n!} \right) \left(1 - p\frac{m_1}{m_0}\varepsilon \right) x^{m_0+m_1\varepsilon} \\
&= \sum_{m_0=1}^{+\infty} \frac{F_{m_0}-1}{m_0^p} x^{m_0} + m_1\varepsilon \sum_{m_0=1}^{+\infty} \frac{1}{m_0^p} \left((F_{m_0}-1)\log x - p\frac{F_{m_0}-1}{m_0} - \sum_{n=1}^{m_0} \frac{H_{n,1}}{n!} \right) x^{m_0},
\end{aligned} \tag{97}$$

and

$$\begin{aligned}
\mu^-_{p,m_1}(x) &= \sum_{m_0=1}^{+\infty} \frac{(-1)^{m_0}}{m_0^p} \left(F_{m_0} - 1 - m_1\varepsilon \sum_{n=1}^{m_0} \frac{H_{n,1}}{n!} \right) \left(1 - p\frac{m_1}{m_0}\varepsilon \right) x^{m_0+m_1\varepsilon} \\
&= \sum_{m_0=1}^{+\infty} (-1)^{m_0} \frac{F_{m_0}-1}{m_0^p} x^{m_0} + m_1\varepsilon \sum_{m_0=1}^{+\infty} \frac{(-1)^{m_0}}{m_0^p} \left((F_{m_0}-1)\log x - p\frac{F_{m_0}-1}{m_0} - \sum_{n=1}^{m_0} \frac{H_{n,1}}{n!} \right) x^{m_0}.
\end{aligned} \tag{98}$$

Suppose first that $p = 0$. We can infer using (97) and (98) that

$$\mu^+_{0,m_1}(x) = \sum_{m_0=1}^{+\infty} (F_{m_0}-1) x^{m_0} + m_1\varepsilon \sum_{m_0=1}^{+\infty} \left((F_{m_0}-1)\log x - \sum_{n=1}^{m_0} \frac{H_{n,1}}{n!} \right) x^{m_0},$$

and

$$\mu^-_{0,m_1}(x) = \sum_{m_0=1}^{+\infty} (-1)^{m_0} (F_{m_0}-1) x^{m_0} + m_1\varepsilon \sum_{m_0=1}^{+\infty} (-1)^{m_0} \left((F_{m_0}-1)\log x - \sum_{n=1}^{m_0} \frac{H_{n,1}}{n!} \right) x^{m_0}.$$

So, we deduce keeping in mind (84) and (85) that

$$\begin{aligned}
\sum_{m_0=1}^{+\infty} F_{m_0} x^{m_0} &= \frac{e^x - 1}{1-x} + \sum_{m_0=1}^{+\infty} x^{m_0} \\
&= \frac{e^x + x - 1}{1-x},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{m_0=1}^{+\infty} (-1)^{m_0} F_{m_0} x^{m_0} &= \frac{e^{-x} - 1}{1+x} + \sum_{m_0=1}^{+\infty} (-1)^{m_0} x^{m_0} \\
&= \frac{e^{-x} - x - 1}{1+x}.
\end{aligned}$$

Also, the following formulas come

$$\begin{aligned}
\sum_{m_0=1}^{+\infty} \left(\sum_{n=1}^{m_0} \frac{H_{n,1}}{n!} \right) x^{m_0} &= \log x \sum_{m_0=1}^{+\infty} \frac{1}{m_0^p} (F_{m_0}-1) x^{m_0} - \frac{e^x}{1-x}\sigma^-(x) \\
&= \frac{1}{1-x} \left((e^x - 1)\log x - e^x \sigma^-(x) \right),
\end{aligned}$$

and

$$\sum_{m_0=1}^{+\infty}(-1)^{m_0}\left(\sum_{n=1}^{m_0}\frac{H_{n,1}}{n!}\right)x^{m_0} = \log x\sum_{m_0=1}^{+\infty}(-1)^{m_0}\left(F_{m_0}-1\right)x^{m_0}-\frac{e^{-x}}{1+x}\sigma^{+}(x)$$
$$= \frac{1}{1+x}\left(\left(e^{-x}-1\right)\log x+e^{-x}\sigma^{+}(x)\right).$$

Considering now the case $p\geq 1$. Combining (86) and (97) we find

$$= \sum_{m_0=1}^{+\infty}\frac{F_{m_0}-1}{m_0^p}x^{m_0}-\sum_{m_0=1}^{+\infty}\frac{1}{m_0^p}\left(\left(F_{m_0}-1\right)\log x-p\frac{F_{m_0}-1}{m_0}-\sum_{n=1}^{m_0}\frac{H_{n,1}}{n!}\right)x^{m_0}m_1\varepsilon$$
$$= \frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^s-1}{s(1-s)}ds+m_1\varepsilon\frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^s\sigma^{-}(s)}{s(1-s)}ds.$$

This yields

$$\sum_{m_0=1}^{+\infty}\frac{F_{m_0}}{m_0^p}x^{m_0} = \sum_{m_0=1}^{+\infty}\frac{1}{m_0^p}x^{m_0}+\frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^s-1}{s(1-s)}ds$$
$$= Li_p(x)+\frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^s-1}{s(1-s)}ds,$$

and

$$\sum_{m_0=1}^{+\infty}\frac{1}{m_0^p}\left(\sum_{n=1}^{m_0}\frac{H_{n,1}}{n!}\right)x^{m_0} = \log x\sum_{m_0=1}^{+\infty}\frac{F_{m_0}-1}{m_0^p}x^{m_0}-p\sum_{m_0=1}^{+\infty}\frac{F_{m_0}-1}{m_0^{p+1}}x^{m_0}$$
$$-\frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^s\sigma^{-}(s)}{s(1-s)}ds$$
$$= \frac{1}{(p-1)!}\log x\int_0^x(\log x-\log s)^{p-1}\frac{e^s-1}{s(1-s)}ds$$
$$-\frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p}\frac{e^s-1}{s(1-s)}ds$$
$$-\frac{e^x}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^s\sigma^{-}(s)}{s(1-s)}ds.$$

Hence, we get

$$\sum_{m_0=1}^{+\infty}\frac{1}{m_0^p}\left(\sum_{n=1}^{m_0}\frac{H_{n,1}}{n!}\right)x^{m_0} = \frac{\log x}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^s-1}{s(1-s)}ds$$
$$-\frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p}\frac{e^s-1}{s(1-s)}ds$$
$$-\frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^s\sigma^{-}(s)}{s(1-s)}ds.$$

This provides (92) and (94). To establish (93) and (95) it is enough to combine (87) and (98) in order to derive the following formulas

$$\sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{F_{m_0}}{m_0^p}x^{m_0}=Li_p(-x)+\frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^{-s}-1}{s(1+s)}ds,$$

and

$$\sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{1}{m_0^p}\left((F_{m_0}-1)\log x-p\frac{F_{m_0}-1}{m_0}-\sum_{n=1}^{m_0}\frac{H_{n,1}}{n!}\right)x^{m_0}$$
$$=\frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^{-s}\sigma^{+}(s)}{s(1+s)}ds.$$

So, we get

$$\begin{aligned}\sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{1}{m_0^p}\left(\sum_{n=1}^{m_0}\frac{H_{n,1}}{n!}\right)x^{m_0} &= \log x\sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{F_{m_0}-1}{m_0^p}x^{m_0}-p\sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{F_{m_0}-1}{m_0^{p+1}}x^{m_0}\\ &\quad-\frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^{-s}}{s(1+s)}\sigma^{+}(s)\,ds\\ &= \frac{\log x}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^{-s}-1}{s(1+s)}ds\\ &\quad-\frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p}\frac{e^{-s}-1}{s(1+s)}ds\\ &\quad-\frac{e^x}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^{-s}\sigma^{+}(s)}{s(1+s)}ds.\end{aligned}$$

Hence, we find

$$\begin{aligned}\sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{1}{m_0^p}\left(\sum_{n=1}^{m_0}\frac{H_{n,1}}{n!}\right)x^{m_0} &= \frac{1}{(p-1)!}\log x\int_0^x(\log x-\log s)^{p-1}\frac{e^{-s}-1}{s(1+s)}ds\\ &\quad-\frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p}\frac{e^{-s}-1}{s(1+s)}ds\\ &\quad-\frac{1}{(p-1)!}\int_0^x(\log x-\log s)^{p-1}\frac{e^{-s}}{s(1+s)}\sigma^{+}(s)\,ds.\end{aligned}$$

Which eventually yields the desired results and completes the proof of the theorem. □

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References

[1] Clifford. (1871). Preliminary sketch of biquaternions. *Proceedings of the London Mathematical Society, 1*(1), 381-395.

[2] E. Study, *Geometrie der dynamen*, Teubner, Leipzig, 1901.

[3] Kotelnikov, A. P. (1895). Screw calculus and some applications to geometry and mechanics. *Annals of the Imperial University of Kazan*, 24.

[4] Ercan, Z., & Yuce, S. (2011). On properties of the dual quaternions. *European Journal of Pure and Applied Mathematics, 4*(2), 142-146.

[5] Messelmi, F. (2019). Ring of Miltidual Integers. *International Journal of Mathematics, Game Theory, and Algebra, 28*(4), 349-360.

[6] Kandasamy, W. B. V., & Smarandache, F. (2012). *Dual numbers*, ZIP Publishing, Ohio.

[7] Soulé, C. (1980). Rational K-theory of the dual numbers of a ring of algebraic integers. *Algebraic K-theory, Evanston,* 402-408.

[8] Messelmi, F. (2013). Analysis of dual functions. *Annual Review of Chaos Theory, Bifurcations and Dynamical Systems, 4*, 37-54.

[9] Kramer, E. E. (1930). Polygenic functions of the dual variable w= u+ jv. *American Journal of Mathematics, 52*(2), 370-376.

[10] condurache, D. (2017). Dual algebra solutions to the extended Wahba problem. *Romanian Journal of Mechanics, 1*(1), 31-44.

[11] Condurache, D. (2017). Poisson-Darboux problems's extended in dual Lie algebra. In *Advances in the Astronautical Sciences, 162*, 3345-3364.

[12] Condurache, D., & Burlacu, A. (2016). Fractional order Cayley transforms for dual quaternions based pose representation. *Advances in the Astronautical Sciences, 156*, 1317-1339.

[13] Fike, J., & Alonso, J. (2011, January). The development of hyper-dual numbers for exact second-derivative calculations. In 49th *AIAA Aerospace Sciences Meeting Including the New Horizons Forum and Aerospace Exposition* (p. 886).

[14] Pennestrì, E., & Stefanelli, R. (2007). Linear algebra and numerical algorithms using dual numbers. *Multibody System Dynamics, 18*, 323-344.

[15] Veldkamp, G. R. (1976). On the use of dual numbers, vectors and matrices in instantaneous, spatial kinematics. *Mechanism and Machine Theory, 11*(2), 141-156.

[16] Yang, A. T., & Freudenstein, F. (1964). Application of dual-number quaternion algebra to the analysis of spatial mechanisms. *Journal of Applied Mechanics, 31*(2), 300-308.

[17] Messelmi, F. A. R. I. D. (2015). Multidual numbers and their multidual functions. *Electron. J. Math. Anal. Appl, 3*(2), 154-172.

[18] Messelmi, F. (2023). log-Series and log-Functions as application of multidual analysis. *Hilbert Journal of Mathematical Analysis, 2*(1), 046-066.

[19] Antosiewicz, H. A., Abramowitz, M., & Stegun, I. A. (1964). Handbook of mathematical functions. *Handbook of Mathematical Functions, 437.*

[20] Conway, J. H., & Guy, R. (1998). The Book of Numbers. Springer Science & Business Media.

[21] Cvijović, D., & Klinowski, J. (1997). Continued-fraction expansions for the Riemann zeta function and polylogarithms. *Proceedings of the American Mathematical Society, 125*(9), 2543-2550.

[22] Fowler, R. H. (1939). *Statistical Thermodynamics*. CUP Archive.