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Sufficient conditions for starlikeness and convexity

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Abstract: In this paper, the differential subordination $\frac{b}{\phi(z)} + c\ \phi(z) + d\ \frac{z\phi'(z)}{\phi^k(z)} \prec s(z), k \geq 1, z \in \mathbb{E}$ is studied by using Lowner Chain. The corresponding result for differential superordination is also obtained to get sandwich type result. Consequently, we obtain sufficient conditions for Starlikeness and Convexity of analytic function f.

Keywords: Analytic function, Univalent function, Starlike function and Convex function.

MSC: 47H10, 54H25.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the open disk $\mathbb{E} = \{z : |z| < 1\}$. For $a \in \mathbb{C}$ (the complex plane) and $n \in \mathbb{N}$ (set of natural numbers), let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots .$$

Let $\mathcal{A}$ denote the class of all analytic functions f which are normalized by the conditions $f(0) = f'(0) - 1 = 0$. Therefore the functions of the class $\mathcal{A}$ are of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

A function $f \in \mathcal{A}$ is starlike of order α if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, 0 \leq \alpha < 1, z \in \mathbb{E}.$$

The class of starlike functions of order α is denoted by $\mathcal{S}^*(\oslash)$. We write $\mathcal{S}^*(0) = \mathcal{S}^*$, the class of univalent starlike functions in $\mathbb{E}$.

A function $f \in \mathcal{A}$ is said to be convex univalent in $\mathbb{E}$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, z \in \mathbb{E}.$$

The class of all convex univalent functions in $\mathbb{E}$ is denoted by $\mathcal{K}$.

A special subclass of $\mathcal{K}$ is the class of convex functions of order α, where $0 \leq \alpha < 1$. This subclass is analytically defined as:

$$\mathcal{K}(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ z \in \mathbb{E} \right\}.$$

It is evident that $\mathcal{K}(0) = \mathcal{K}$.

For further details on these classes, readers may refer to Duren [1]. An analytic function g is said to be subordinate to an analytic function f in $|z| < 1$, denoted as $g \prec f$, if there exists a function $w(z)$, analytic in $|z| < 1$, such that $g(z) = f(w(z))$ and $|w(z)| < 1$ for all z in $|z| < 1$.

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ and let p be an analytic function in $\mathbb{E}$, with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$. Let h be a univalent function in $\mathbb{E}$. Then, the function p satisfies a first-order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \quad \Phi(p(0), 0; 0) = h(0). \tag{1}$$

A univalent function q is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1) is termed the best dominant of the differential subordination (1). The best dominant is unique up to a rotation in $\mathbb{E}$.

Similarly, let $\Psi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be analytic and univalent in $\mathbb{C}^2 \times \mathbb{E}$, h be analytic in $\mathbb{E}$, and p be analytic and univalent in $\mathbb{E}$, with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$. Then, p is a solution of the first-order differential superordination if

$$h(z) \prec \Psi(p(z), zp'(z); z), \quad h(0) = \Psi(p(0), 0; 0). \tag{2}$$

An analytic function q is called a subordinant of the differential superordination (2) if $q \prec p$ for all p satisfying (2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (2) is referred to as the best subordinant of (2). The best subordinant is unique up to a rotation in $\mathbb{E}$. Comprehensive details on the theory of differential superordination can be found in the work by Bulboacă [2].

The classical Loewner theory was introduced in 1923 by the Czech-German mathematician K. Löwner [3]. Subsequent advancements, including the modern parametric representation, were contributed by Kufarev [4] and Pommerenke [5]. In the field of geometric function theory, numerous subordination results have been obtained using the Loewner chain. For instance, works by Gupta et al. [6–8], Singh et al. [9], Miller [10], Miller and Mocanu [11,12], and Singh and Gupta [13,14] have explored applications of the Loewner chain.

In an influential study by Gupta et al. [15], the following subordination involving the Loewner chain was investigated:

$$1 - \frac{\gamma}{p(z)} + \frac{zp'(z)}{p^2(z)} \prec 1 - \frac{\gamma}{q(z)} + \frac{zq'(z)}{q^2(z)}, \quad \gamma \in \mathbb{C},\ \Re(\gamma) \geq 0,$$

where the authors established the following result:

Theorem 1. *Let γ be a complex number with $\Re(\gamma) \geq 0$, and let q be a univalent function such that $\dfrac{zq'(z)}{q^2(z)}$ is starlike in $\mathbb{E}$. If an analytic function p satisfies the differential subordination*

$$1 - \frac{\gamma}{p(z)} + \frac{zp'(z)}{p^2(z)} \prec 1 - \frac{\gamma}{q(z)} + \frac{zq'(z)}{q^2(z)},$$

with $p(0) = q(0) = 1$ and $z \in \mathbb{E}$, then $p(z) \prec q(z)$, and $q(z)$ is the best dominant.

Theorem 2. *Let γ, $\Re\gamma \geq 0$, be a complex number and A and B be real numbers $-1 \leq B < A \leq 1$. Let $f \in \mathcal{A}$ satisfy the differential subordination*

$$\frac{1 - \gamma + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 - \gamma\frac{1+Bz}{1+Az} + \frac{(A-B)z}{(1+Az)^2}, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, z \in \mathbb{E}.$$

The objective of present paper is to obtain certain sufficient conditions for starlike and convex functions $f \in \mathcal{A}$, given a generalized differential subordination

$$\frac{b}{\phi(z)} + c\,\phi(z) + d\,\frac{z\phi'(z)}{\phi^k(z)} \prec \frac{b}{\psi(z)} + c\,\psi(z) + d\,\frac{z\psi'(z)}{\psi^k(z)}, k \geq 1, z \in \mathbb{E}. \tag{3}$$

We also obtain the corresponding results for superordination and consequently get some sandwich type results.

2. Preliminaries

In order to prove our main result, we require the following definitions and lemmas.

Definition 1 ([16], p. 157). A function $L(z,t)$, where $z \in \mathbb{E}$ and $t \geq 0$, is called a *subordination chain* if, for each $t \geq 0$, the function $L(\cdot, t)$ is analytic and univalent in $\mathbb{E}$, $L(z, \cdot)$ is continuously differentiable on $[0, \infty)$ for all $z \in \mathbb{E}$, and

$$L(z,t_1) \prec L(z,t_2) \quad \text{for all } 0 \leq t_1 \leq t_2.$$

Definition 2 ([17]). Let $\mathbb{Q}$ denote the set of all functions $f(z)$ that are analytic and injective on $\overline{\mathbb{E}} \setminus \mathbb{E}(f)$, where

$$\mathbb{E}(f) = \left\{ \zeta \in \partial\mathbb{E} : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and satisfy $f'(\zeta) \neq 0$ for all $\zeta \in \partial\overline{\mathbb{E}} \setminus \mathbb{E}(f)$. The subclass of $\mathbb{Q}$ consisting of functions with $f(0) = a$ is denoted by $\mathbb{Q}(a)$.

Lemma 1 ([11]). *Let F be analytic in $\mathbb{E}$, and let G be analytic and univalent in $\overline{\mathbb{E}}$ except possibly at points $\xi \in \partial\mathbb{E}$ such that*

$$\lim_{z \to \xi} F(z) = \infty,$$

with $F(0) = G(0)$. If $F \prec G$ in $\mathbb{E}$, then there exist $z_0 \in \mathbb{E}$, $\xi_0 \in \partial\mathbb{E}$, and an integer $m \geq 1$ such that

1. $F(\{z : |z| < |z_0|\}) \subset G(\mathbb{E})$,
2. $F(z_0) = G(\xi_0)$, *and*
3. $z_0 F'(z_0) = m\,\xi_0 G'(\xi_0)$.

Lemma 2 ([16], p. 159). *Consider the function*

$$L(z,t) : \mathbb{E} \times [0, \infty) \to \mathbb{C}$$

of the form

$$L(z,t) = a_1(t) z + \cdots$$

where $a_1(t) \neq 0$ for all $t \geq 0$ and $\lim_{z \to \zeta} |a_1(t)| = \infty$. Then $L(z,t)$ is a subordination chain if and only if

$$\Re\left[\frac{z\,\frac{\partial L}{\partial z}}{\frac{\partial L}{\partial t}}\right] > 0 \quad \text{for all } z \in \mathbb{E} \text{ and } t \geq 0.$$

Lemma 3 ([12], Theorem 7). *Let $q \in \mathcal{H}[a,1]$, and let $\phi : \mathbb{C}^2 \to \mathbb{C}$. Define*

$$h(z) = \phi\big(q(z), z\,q'(z)\big).$$

Suppose $L(z;t) = \phi\big(q(z), t\,z\,q'(z)\big)$ is a subordination chain, and let $p \in \mathcal{H}[a,1] \cap \mathbb{Q}$. If

$$h(z) \prec \phi\big(p(z), z\,p'(z)\big),$$

then $q(z) \prec p(z)$. Furthermore, if $\phi\big(q(z), z\,q'(z)\big) = h(z)$ has a univalent solution $q \in \mathbb{Q}$, then q is the best subordinant.

3. Main results

3.1. Subordination theorem

Theorem 3. *Let ψ be univalent in $\mathbb{E}$, with $\psi(0) = 1$, and suppose $\dfrac{z\psi'(z)}{\psi^k(z)}$ is starlike in $\mathbb{E}$. Assume that*

$$\Re\left(-\frac{b}{d}\psi^{k-2}(z) + \frac{c}{d}\psi^k(z)\right) > 0, \quad k \geq 1, \quad \psi(z) \neq 0,$$

where $b, c \in \mathbb{C}$ with $\Re(c - b) > 0$ and $d \in \mathbb{R}$. If $\phi \in \mathcal{H}[1,1]$ satisfies

$$\frac{b}{\phi(z)} + c\phi(z) + d\frac{z\phi'(z)}{\phi^k(z)} \prec \frac{b}{\psi(z)} + c\psi(z) + d\frac{z\psi'(z)}{\psi^k(z)}, \quad k \geq 1, \quad z \in \mathbb{E}, \tag{4}$$

then $\phi(z) \prec \psi(z)$, where ψ is the best dominant of (4).

Proof. Define

$$s(z) = \frac{b}{\psi(z)} + c\psi(z) + d\frac{z\psi'(z)}{\psi^k(z)}, \quad k \geq 1, \quad \psi(z) \neq 0.$$

Since $\psi(z)$ is univalent in $\mathbb{E}$ and $\psi(0) = 1$, we have:

$$\frac{1}{d}\frac{zs'(z)}{\frac{z\psi'(z)}{\psi^k(z)}} = -\frac{b}{d}\psi^{k-2}(z) + \frac{c}{d}\psi^k(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - k\frac{z\psi'(z)}{\psi(z)}.$$

Given that $\Re\left(-\frac{b}{d}\psi^{k-2}(z) + \frac{c}{d}\psi^k(z)\right) > 0$ and $\frac{z\psi'(z)}{\psi^k(z)}$ is starlike, it follows that

$$\frac{1}{d}\Re\frac{zs'(z)}{\frac{z\psi'(z)}{\psi^k(z)}} > 0.$$

Thus, $s(z)$ is close-to-convex and therefore univalent in $\mathbb{E}$. The subordination in (4) is well-defined.

Next, we prove that $\phi \prec \psi$. Assume ψ is analytic and univalent in $\overline{\mathbb{E}}$. If $\phi \not\prec \psi$, by Lemma 1, there exist $z_0 \in \mathbb{E}$ and $\xi_0 \in \partial\mathbb{E}$ such that:

$$\phi(z_0) = \psi(\xi_0), \quad z_0\phi'(z_0) = m\xi_0\psi'(\xi_0), \quad m \geq 1.$$

Substituting into (4), we get:

$$\frac{b}{\phi(z_0)} + c\phi(z_0) + d\frac{z_0\phi'(z_0)}{\phi^k(z_0)} = \frac{b}{\psi(\xi_0)} + c\psi(\xi_0) + dm\frac{\xi_0\psi'(\xi_0)}{\psi^k(\xi_0)}. \tag{5}$$

Define for $t > 0$:

$$L(z,t) = \frac{b}{\psi(z)} + c\psi(z) + d(1+t)\frac{z\psi'(z)}{\psi^k(z)}. \tag{6}$$

This can be expressed as:

$$L(z,t) = s(z) + td\frac{z\psi'(z)}{\psi^k(z)}.$$

The function $L(z,t)$ is regular in $\mathbb{E}$ for all $t \geq 0$ and continuously differentiable with respect to t. Since ψ is univalent, $\psi'(0) \neq 0$. Evaluating:

$$a_1(t) = \psi'(0)[-b + c + d(1+t)].$$

As $t \to \infty$, $|a_1(t)| \to \infty$.

From the hypothesis $\Re\left(-\frac{b}{d}\psi^{k-2}(z)+\frac{c}{d}\psi^{k}(z)\right)>0$ and $\frac{z\psi'(z)}{\psi^{k}(z)}$ is starlike, we have:

$$\Re\left(\frac{z\frac{\partial L}{\partial z}}{\frac{\partial L}{\partial t}}\right)>0.$$

Thus, by Lemma 2, $L(z,t)$ forms a subordination chain. From Definition 1, we have $L(z,t_1)\prec L(z,t_2)$ for $0\le t_1\le t_2$. Since $L(z,0)=s(z)$, we conclude that $L(\xi_0,t)\notin s(\mathbb{E})$ for $|\xi_0|=1$ and $t\ge 0$.

From (5) and (6), we find:

$$L(\xi_0,m)\notin s(\mathbb{E}),$$

which contradicts (4). Therefore, $\phi\prec\psi$ in $\mathbb{E}$. □

3.2. Superordination theorem

Theorem 4. *Let ψ be a univalent function in $\mathbb{E}$, with $\psi(0)=1$ and $\frac{z\psi'(z)}{\psi^{k}(z)}$ being starlike in $\mathbb{E}$. Assume that*

$$\Re\left[-\frac{b}{d}\psi^{k-2}(z)+\frac{c}{d}\psi^{k}(z)\right]>0,\quad k\ge 1,$$

and $\psi(z)\neq 0$, where $b,c\in\mathbb{C}$ such that $\Re(c-b)>0$, and $d\in\mathbb{R}$.

Let $\phi\in\mathcal{H}[\psi(0),1]\cap Q$ such that $\phi(\mathbb{E})\subset\mathbb{D}$ and the function

$$\frac{b}{\phi(z)}+c\phi(z)+d\frac{z\phi'(z)}{\phi^{k}(z)}$$

is univalent in $\mathbb{E}$. If

$$\frac{b}{\psi(z)}+c\psi(z)+d\frac{z\psi'(z)}{\psi^{k}(z)}\prec\frac{b}{\phi(z)}+c\phi(z)+d\frac{z\phi'(z)}{\phi^{k}(z)},\quad k\ge 1,\quad z\in\mathbb{E},\tag{7}$$

then $\psi(z)\prec\phi(z)$, where ψ is the best subordinant.

Proof. Define

$$\xi(\phi(z),z\phi'(z))=\frac{b}{\phi(z)}+c\phi(z)+d\frac{z\phi'(z)}{\phi^{k}(z)}.$$

Eq. (7) can be expressed as

$$\frac{b}{\psi(z)}+c\psi(z)+d\frac{z\psi'(z)}{\psi^{k}(z)}\prec\xi(\phi(z),z\phi'(z)),$$

where $\xi(\phi(z),z\phi'(z))$ is univalent in $\mathbb{E}$. Define

$$L(z,t)=\frac{b}{\psi(z)}+c\psi(z)+dt\frac{z\psi'(z)}{\psi^{k}(z)}.$$

Expanding $L(z,t)$ in a power series yields

$$L(z,t)=b_0+b_1(t)z+\cdots,$$

where

$$b_1(t)=\psi'(0)[-b+c+dt].$$

It follows that

$$\lim_{t\to\infty}|b_1(t)|=\infty.$$

Now, compute

$$\frac{z\frac{\partial L}{\partial z}}{\frac{\partial L}{\partial t}}=-\frac{b}{d}\psi^{k-2}(z)+\frac{c}{d}\psi^{k}(z)+t\left[1+\frac{z\psi''(z)}{\psi'(z)}-k\frac{z\psi'(z)}{\psi(z)}\right].$$

Since $\frac{z\psi'(z)}{\psi^k(z)}$ is starlike and

$$\Re\left[-\frac{b}{d}\psi^{k-2}(z)+\frac{c}{d}\psi^{k}(z)\right]>0,\quad k\geq 1,\quad t\geq 0,$$

it follows that

$$\Re\left[\frac{z\frac{\partial L}{\partial z}}{\frac{\partial L}{\partial t}}\right]>0.$$

By Lemma 2, $L(z,t)$ forms a subordination chain. Applying Lemma 3, we conclude that $\psi(z)\prec\phi(z)$, with ψ being the best subordinant of (7). □

On combining Theorem 3 and Theorem 4, we obtain the sandwich theorem as follows:

3.3. Sandwich theorem

Theorem 5. *Let ψ_1 and ψ_2 be univalent functions in the open unit disk $\mathbb{E}$, satisfying $\psi_1(0)=1=\psi_2(0)$ and $\psi_i(z)\neq 0$ for all $z\in\mathbb{E}$, where $i=1,2$. Suppose that*

$$\frac{z\,\psi_i'(z)}{\psi_i^k(z)}$$

is starlike in $\mathbb{E}$ for each $i=1,2$. Additionally, let $d>0$, $\Re(c-b)>0$, and assume

$$\Re\left[-\frac{b}{d}\,\psi_i^{k-2}(z)+\frac{c}{d}\,\psi_i^{k}(z)\right]>0\quad \text{for } i=1,2.$$

Further, let

$$\phi\in\mathcal{H}[\psi(0),1]\cap Q\quad \text{with}\quad \phi(\mathbb{E})\subset\mathbb{D},$$

and assume that

$$\frac{b}{\phi(z)}+c\,\phi(z)+d\,\frac{z\,\phi'(z)}{\phi^k(z)}$$

is univalent in $\mathbb{E}$. If

$$\frac{b}{\psi_2(z)}+c\,\psi_2(z)+d\,\frac{z\,\psi_2'(z)}{\psi_2^k(z)}\prec\frac{b}{\phi(z)}+c\,\phi(z)+d\,\frac{z\,\phi'(z)}{\phi^k(z)}\prec\frac{b}{\psi_1(z)}+c\,\psi_1(z)+d\,\frac{z\,\psi_1'(z)}{\psi_1^k(z)},$$

then

$$\psi_2(z)\prec\phi(z)\prec\psi_1(z),\quad z\in\mathbb{E},$$

where ψ_1 is the best dominant and ψ_2 is the best subordinant.

3.4. Particular cases

In particular, for $k=1$ in Theorems 3, 4, and 5, one obtains Theorems 6, 7, and 8, respectively, as follows.

Theorem 6. *Let ψ be univalent in $\mathbb{E}$ with $\psi(0)=1$, and suppose*

$$\frac{z\,\psi'(z)}{\psi(z)}$$

is starlike in $\mathbb{E}$. Assume further that

$$\Re\left[-\frac{b}{d}\,\frac{1}{\psi(z)}+\frac{c}{d}\,\psi(z)\right]>0,$$

where $d>0$, $\Re(c-b)>0$, and $q(z)\neq 0$. If

$$\frac{b}{\phi(z)}+c\,\phi(z)+d\,\frac{z\,\phi'(z)}{\phi(z)}\prec\frac{b}{\psi(z)}+c\,\psi(z)+d\,\frac{z\,\psi'(z)}{\psi(z)},\quad z\in\mathbb{E},\tag{8}$$

then $\phi(z) \prec \psi(z)$, *and* $\psi(z)$ *is the best dominant of* (8).

Theorem 7. *Let* ψ *be univalent in* $\mathbb{E}$ *with* $\psi(0) = 1$, *and suppose* $\psi(z) \neq 0$ *for* $z \in \mathbb{E}$. *Assume*

$$\Re\left[-\frac{b}{d}\,\frac{1}{\psi(z)} + \frac{c}{d}\,\psi(z)\right] > 0,$$

where $d > 0$ *and* $\Re(c-b) > 0$. *Let*

$$\phi \in \mathcal{H}[\psi(0),1] \cap Q \quad with \quad \phi(\mathbb{E}) \subset \mathbb{D},$$

and suppose

$$\frac{b}{\phi(z)} + c\,\phi(z) + d\,\frac{z\,\phi'(z)}{\phi(z)}$$

is univalent in $\mathbb{E}$. *If*

$$\frac{b}{\psi(z)} + c\,\psi(z) + d\,\frac{z\,\psi'(z)}{\psi(z)} \prec \frac{b}{\phi(z)} + c\,\phi(z) + d\,\frac{z\,\phi'(z)}{\phi(z)}, \quad z \in \mathbb{E}, \tag{9}$$

then $\psi(z) \prec \phi(z)$, *and* $\psi(z)$ *is the best subordinant of* (9).

Theorem 8. *Let* ψ_1 *and* ψ_2 *be univalent in* $\mathbb{E}$ *with* $\psi_1(0) = 1 = \psi_2(0)$. *Suppose* $\psi_1(z) \neq 0$, $\psi_2(z) \neq 0$ *for all* $z \in \mathbb{E}$, *and*

$$\frac{z\,\psi_1'(z)}{\psi_1(z)}$$

is starlike in $\mathbb{E}$. *Let* $d > 0$ *and assume* $\Re(c-b) > 0$. *Furthermore, suppose*

$$\Re\left[-\frac{b}{d}\,\frac{1}{\psi_i(z)} + \frac{c}{d}\,\psi_i(z)\right] > 0 \quad for\ i = 1,2.$$

Let

$$\phi \in \mathcal{H}[\psi(0),1] \cap Q \quad with \quad \phi(\mathbb{E}) \subset \mathbb{D},$$

and suppose

$$\frac{b}{\phi(z)} + c\,\phi(z) + d\,\frac{z\,\phi'(z)}{\phi(z)}$$

is univalent in $\mathbb{E}$. *If*

$$\frac{b}{\psi_2(z)} + c\,\psi_2(z) + d\,\frac{z\,\psi_2'(z)}{\psi_2(z)} \prec \frac{b}{\phi(z)} + c\,\phi(z) + d\,\frac{z\,\phi'(z)}{\phi(z)} \prec \frac{b}{\psi_1(z)} + c\,\psi_1(z) + d\,\frac{z\,\psi_1'(z)}{\psi_1(z)},$$

then

$$\psi_2(z) \prec \phi(z) \prec \psi_1(z),$$

where ψ_1 *is the best dominant and* ψ_2 *is the best subordinant.*

3.5. Another special case

By setting $k = 2$, $d = 1$, $c = 0$, and $b = -\gamma$ in Theorem 3, we recover the following result proved by Gupta et al. [15].

Theorem 9. *Let* γ *be a complex number with* $\Re(\gamma) \geq 0$. *Let* ψ *be univalent in* $\mathbb{E}$ *such that*

$$\frac{z\,\psi'(z)}{\psi^2(z)}$$

is starlike in $\mathbb{E}$. *Suppose* ϕ *is an analytic function satisfying the differential subordination*

$$1-\frac{\gamma}{\phi(z)}+\frac{z\phi'(z)}{\phi^2(z)} \prec 1-\frac{\gamma}{\psi(z)}+\frac{z\psi'(z)}{\psi^2(z)},$$

with $\phi(0)=\psi(0)=1$ *and* $z\in\mathbb{E}$. *Then* $\phi(z)\prec\psi(z)$, *and* $\psi(z)$ *is the best dominant.*

Further, writing $\phi(z)=\dfrac{zf'(z)}{f(z)}$ in Theorem 9, we get the result as follows:

Corollary 1. *Let* ψ *be univalent in* $\mathbb{E}$, $\psi(0)=1$ *and* $\dfrac{z\psi'(z)}{\psi^2(z)}$ *be starlike in* $\mathbb{E}$ *such that* $\Re\,\gamma>0$ *and* $\psi(z)\neq 0$. *If*

$$-\gamma\frac{f(z)}{zf'(z)}+\frac{f(z)}{zf'(z)}\left[1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right]\prec\frac{-\gamma}{\psi(z)}+\frac{z\psi'(z)}{\psi^2(z)}, z\in\mathbb{E} \tag{10}$$

then

$$\frac{zf'(z)}{f(z)}\prec\psi(z),$$

where ψ *is the best dominant of (11).*

Replacing f by zf' in Corollary 1, we obtain Corollary 2 as given below.

Corollary 2. *Let* ψ *be univalent in* $\mathbb{E}$, $\psi(0)=1$ *and* $\dfrac{z\psi'(z)}{\psi^2(z)}$ *be starlike in* $\mathbb{E}$ *such that* $\Re\,\gamma>0$ *and* $\psi(z)\neq 0$. *If*

$$\frac{-\gamma}{1+\frac{zf''(z)}{f'(z)}}+\left(1+\frac{zf''(z)}{f'(z)}\right)\left[\frac{z^2f'''(z)+2zf''(z)}{zf''(z)+f'(z)}-\frac{zf''(z)}{f'(z)}\right]\prec\frac{-\gamma}{\psi(z)}+\frac{z\psi'(z)}{\psi^2(z)}, z\in\mathbb{E} \tag{11}$$

then

$$1+\frac{zf''(z)}{f'(z)}\prec\psi(z), z\in\mathbb{E}.$$

4. Applications

By choosing

$$\psi(z)=\frac{1+Cz}{1+Dz},\quad -1\leq D<C\leq 1,$$

in Corollary 1, we note that $\psi(z)$ is univalent and that

$$\frac{z\psi'(z)}{\psi^2(z)}$$

is starlike. As a consequence, we recover a result proved by Gupta et al. [15], presented below for completeness.

Corollary 3. *Let* γ *be a complex number such that* $\Re(\gamma)>0$. *If*

$$-\gamma\frac{f(z)}{zf'(z)}+\frac{f(z)}{zf'(z)}\left[1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right]\prec 1-\gamma\frac{1+Cz}{1+Dz}+\frac{(C-D)z}{(1+Cz)^2},\quad z\in\mathbb{E}, \tag{12}$$

then

$$\frac{zf'(z)}{f(z)}\prec\frac{1+Cz}{1+Dz},$$

where $z\in\mathbb{E}$.

Corollary 4. *Let γ be a complex number such that $\Re(\gamma) > 0$. If*

$$\frac{-\gamma}{1+\frac{zf''(z)}{f'(z)}} + \left(1+\frac{zf''(z)}{f'(z)}\right)\left[\frac{z^2 f'''(z)+2zf''(z)}{zf''(z)+f'(z)} - \frac{zf''(z)}{f'(z)}\right] \prec 1-\gamma\frac{1+Cz}{1+Dz} + \frac{(C-D)\,z}{(1+Cz)^2}, \tag{13}$$

then

$$1+\frac{zf''(z)}{f'(z)} \prec \frac{1+Cz}{1+Dz}, \quad z \in \mathbb{E}.$$

Special Case: For $C = 1$ and $D = -1$, Corollary 3 yields the following result.

Corollary 5. *Let γ be a complex number such that $\Re(\gamma) > 0$. If*

$$\gamma\frac{f(z)}{zf'(z)} + \frac{f(z)}{zf'(z)}\left[1+\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right] \prec 1-\gamma\frac{1+z}{1-z} + \frac{2z}{(1+z)^2}, \quad z \in \mathbb{E}, \tag{14}$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{E}.$$

Since

$$\Re\left(\tfrac{1+z}{1-z}\right) > 0 \quad \textit{for} \quad z \in \mathbb{E},$$

it follows that $f \in \mathcal{S}^*$.

Similarly, substituting $C = 1$ and $D = -1$ in Corollary 4 reduces it to the following statement.

Corollary 6. *Let γ be a complex number such that $\Re(\gamma) > 0$. If*

$$\frac{\gamma}{1+\frac{zf''(z)}{f'(z)}} + \left(1+\frac{zf''(z)}{f'(z)}\right)\left[\frac{z^2 f'''(z)+2zf''(z)}{zf''(z)+f'(z)} - \frac{zf''(z)}{f'(z)}\right] \prec 1-\gamma\frac{1+z}{1-z} + \frac{2z}{(1+z)^2}, \tag{15}$$

then

$$1+\frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{E}.$$

Since

$$\Re\left(\tfrac{1+z}{1-z}\right) > 0 \quad \textit{for} \quad z \in \mathbb{E},$$

we conclude that $f \in \mathcal{K}$.

5. Conclusion

In summary, we have derived sufficient conditions for the starlikeness and convexity of analytic functions satisfying the given differential subordination and superordination. These conditions offer a unified framework for proving that certain normalized analytic functions lie in the well-known classes $\mathcal{S}^*$ and $\mathcal{K}$.

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