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The geometry and norm-attainability of operators in operator ideals: the role of singular values and compactness

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Abstract: This paper investigates the geometry and norm-attainability of operators within various operator ideals, with a particular focus on the role of singular values and compactness. We explore the behavior of norm-attainable operators in the context of classical operator ideals, such as trace-class and Hilbert-Schmidt operators, and examine how their geometric and algebraic properties are influenced by membership in these ideals. A key result of this study is the connection between the singular values of trace-class operators and their operator norm, establishing a foundational relationship for understanding norm-attainment. Additionally, we explore the conditions under which weakly compact and compact operators can attain their operator norm, providing further insights into the structural properties that govern norm-attainability in operator theory. The findings contribute to a deeper understanding of the interplay between operator ideals and norm-attainability, with potential applications in functional analysis and related fields.

Keywords: Norm-Attainability, Operator Ideals, Singular Values, Trace-Class Operator

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1. Introduction

In operator theory, the study of norm-attainability plays a central role in understanding the geometry and functional properties of operators in various operator ideals. An operator T is said to be *norm-attainable* if there exists a vector x in the Hilbert space H such that the operator norm $\|T\|$ is achieved at x, i.e., $\|T(x)\| = \|T\|$ [1–6]. This property not only provides insights into the operator's functional behavior but also bridges concepts such as compactness, weak compactness, and singular value decay with norm-attainability. The current research delves into the norm-attainability of operators within various operator ideals, including compact operators ($\mathcal{K}$), weakly compact operators ($\mathcal{W}_c$), trace-class operators ($\mathcal{T}_1$), and Hilbert-Schmidt operators ($\mathcal{H}_2$) [7–10]. We explore fundamental properties that link norm-attainability with the decay of singular values, compactness, and inclusion relationships among operator ideals. Key results establish conditions under which operators in these classes attain their norm, emphasizing the impact of singular value behavior and operator structure. For instance, the rapid decay of singular values in compact operators often precludes norm-attainability, while weak compactness guarantees norm-attainability under suitable constraints. Furthermore, the relationships between operator ideals are shown to preserve norm-attainability, shedding light on how operators behave under transitions between nested ideals [11–15]. Through a sequence of lemmas, propositions, and theorems, this paper characterizes norm-attainability in operator theory, providing novel connections and results. Highlights include:

- The relationship between compactness and norm-attainability, demonstrating that compact operators in operator ideals can achieve their operator norm under certain conditions.
- The role of singular value decay, with results illustrating how rapid decay prevents norm-attainability.
- Norm-attainability for trace-class and Hilbert-Schmidt operators, showing explicit links between operator norms and singular values.

- The preservation of norm-attainability across nested operator ideals, offering a robust framework for analyzing operator behavior.

By unifying these concepts, this study contributes to a deeper understanding of operator theory and its geometric implications, providing a foundation for further research into functional and spectral properties of operators. The results presented herein have broad applications in mathematics, including functional analysis, quantum mechanics, and numerical analysis, where the interplay between compactness, operator ideals, and norm-attainability is of critical importance.

2. Preliminaries

In this section, we introduce the foundational concepts and notation that underpin the results discussed in this paper. These preliminaries focus on operator ideals, compactness, weak compactness, singular values, and norm-attainability, setting the stage for the subsequent analysis.

2.1. Hilbert Spaces and Operators

Let H denote a separable Hilbert space over $\mathbb{C}$ or $\mathbb{R}$, equipped with the inner product $\langle \cdot, \cdot \rangle$ and norm $\|x\| = \sqrt{\langle x, x \rangle}$ for $x \in H$. A bounded linear operator $T : H \to H$ satisfies $\|T(x)\| \leq \|T\|\|x\|$ for all $x \in H$, where $\|T\| = \sup\{\|T(x)\| : x \in H, \|x\| = 1\}$ is the operator norm of T.

2.2. Operator Ideals

An operator ideal $\mathcal{I}$ is a class of bounded linear operators on H with specific algebraic and topological properties. Common examples include:

- *Compact Operators ($\mathcal{K}$)*: Operators $T \in \mathcal{K}$ for which the image of any bounded set in H is relatively compact.
- *Weakly Compact Operators ($\mathcal{W}_c$)*: Operators $T \in \mathcal{W}_c$ such that the image of any bounded set is relatively compact in the weak topology.
- *Trace-Class Operators ($\mathcal{T}_1$)*: Operators $T \in \mathcal{T}_1$ for which the sum of singular values $\sum_{i=1}^{\infty} \sigma_i(T)$ is finite.
- *Hilbert-Schmidt Operators ($\mathcal{H}_2$)*: Operators $T \in \mathcal{H}_2$ such that $\sum_{i=1}^{\infty} \sigma_i(T)^2$ is finite.

2.3. Singular Values and Compactness

For a compact operator $T : H \to H$, the singular values $\{\sigma_i(T)\}_{i=1}^{\infty}$ are the eigenvalues of $|T| = \sqrt{T^*T}$ arranged in decreasing order. Compactness implies $\sigma_i(T) \to 0$ as $i \to \infty$. The norm of T is given by $\|T\| = \sigma_1(T)$, the largest singular value.

2.4. Norm-Attainability

An operator $T \in \mathcal{B}(H)$, where $\mathcal{B}(H)$ denotes the space of all bounded linear operators on H, is *norm-attainable* if there exists $x \in H$ with $\|x\| = 1$ such that $\|T(x)\| = \|T\|$. The existence of such a vector x links the geometry of the operator to its functional properties and is central to the study of operator ideals.

2.5. Weak Compactness and Singular Value Decay

Weakly compact operators exhibit behavior closely tied to their singular values. Trace-class operators $\mathcal{T}_1$, for instance, are weakly compact if and only if their singular values converge to zero. Rapid decay of singular values can, however, prevent norm-attainability for compact operators, as detailed in later sections.

2.6. Inclusion Relationships Among Ideals

If $\mathcal{I}_1 \subseteq \mathcal{I}_2$, an operator $T \in \mathcal{I}_1$ retains its norm-attainability in $\mathcal{I}_2$. This inclusion relationship allows us to extend results about norm-attainability across different operator ideals, facilitating a unified approach to operator behavior. These preliminaries provide the necessary background for understanding the results on norm-attainability in operator ideals, as discussed in subsequent sections.

3. Main Results and Discussions

Compact operators in certain operator ideals are shown to have specific properties that can influence their ability to achieve the operator norm. In particular, we focus on compact operators in general operator ideals and their norm-attainability. The following lemma establishes the existence of a point where the operator norm is attained, showing that compact operators in operator ideals retain norm-attainability.

Lemma 1. *Let $T \in \mathcal{I}$ be a compact operator, where $\mathcal{I}$ is any operator ideal. If T is norm-attainable, then there exists a point $x \in H$ such that $\|T(x)\| = \|T\|$, where H is a Hilbert space.*

Proof. Let $T \in \mathcal{I}$ be a compact operator, where $\mathcal{I}$ is an operator ideal. Since T is norm-attainable, by definition, there exists some element $x_0 \in H$ such that $\|T(x_0)\| = \|T\|$, where H is a Hilbert space. Let $\{x_n\}$ be a sequence in H such that $\|T(x_n)\| \to \|T\|$ as $n \to \infty$. This is possible because of the assumption that T is norm-attainable. Since T is a compact operator, it maps bounded sets to relatively compact sets. The set $\{x_n\}$ is bounded (because $\|T(x_n)\| \leq \|T\|$ for all n), and thus by the definition of compactness, the sequence $\{x_n\}$ has a convergent subsequence. Without loss of generality, assume that $x_n \to x_0$ for some $x_0 \in H$. By the continuity of T, we have $T(x_n) \to T(x_0)$. Thus, $\|T(x_n)\| \to \|T(x_0)\|$. Since $\|T(x_n)\| \to \|T\|$, it follows that $\|T(x_0)\| = \|T\|$. Therefore, there exists a point $x_0 \in H$ such that $\|T(x_0)\| = \|T\|$, completing the proof. □

Weakly compact operators often exhibit interesting geometric properties that relate to norm-attainability. The next lemma illustrates the condition under which weakly compact operators can achieve their norm, tying the concept of weak compactness to norm-attainability.

Lemma 2. *Let $T \in \mathcal{W}_c$ be a weakly compact operator. If T is norm-attainable, then there exists an element $x \in H$ such that $\|T(x)\| = \|T\|$.*

Proof. Let $T \in \mathcal{W}_c$ be a weakly compact operator on a Hilbert space H. We are given that T is norm-attainable, meaning that there exists an element $x_0 \in H$ such that

$$\|T(x_0)\| = \|T\|.$$

We aim to show that there exists an element $x \in H$ such that $\|T(x)\| = \|T\|$. By the definition of weak compactness, T is weakly continuous, meaning that the image of any bounded set under T is relatively weakly compact. In particular, since T is weakly compact, the set $T(B_H)$ is weakly compact, where B_H denotes the unit ball of H. Next, let x_0 be the point where the norm of T is attained, i.e., $\|T(x_0)\| = \|T\|$. We now apply the Banach-Alaoglu Theorem, which ensures that the unit ball in the dual space H^* is weak*-compact. Therefore, the weak closure of $T(B_H)$ is a compact set in the weak topology of H. Since $T(B_H)$ is weakly compact and since $\|T\|$ is the maximum value that $\|T(x)\|$ can attain, we conclude that the norm is attained for some $x \in H$ such that

$$\|T(x)\| = \|T\|.$$

Thus, the result follows. □

The singular values of trace-class operators are fundamental in determining their operator norm and norm-attainability. These values directly relate to the operator's behavior, offering key insights into the conditions necessary for norm-attainment. Understanding this connection helps establish the groundwork for further research into the norm-attainability of trace-class operators.

Lemma 3. *Let $T \in \mathcal{T}_1$ be a trace-class operator. Then the operator norm $\|T\|$ is given by the sum of the singular values of T:*

$$\|T\| = \sum_{i=1}^{\infty} \sigma_i(T),$$

where $\sigma_i(T)$ are the singular values of T.

Proof. Let $T \in \mathcal{T}_1$ be a trace-class operator. The operator norm of T is defined as

$$\|T\| = \sup_{\|x\|=1} \|T(x)\|,$$

where $\|x\|$ is the norm of x in the Hilbert space H. For trace-class operators, the singular values $\sigma_i(T)$ of T are the eigenvalues of the positive operator $|T| = (T^*T)^{1/2}$, where T^* is the adjoint of T. By the spectral theorem, the operator $|T|$ has a countable sequence of non-negative eigenvalues $\sigma_1(T), \sigma_2(T), \ldots$, with

$$\|T\| = \sup_{\|x\|=1} \|T(x)\| = \sum_{i=1}^{\infty} \sigma_i(T),$$

where the sum of singular values $\sum_{i=1}^{\infty} \sigma_i(T)$ converges due to the fact that T is trace-class. This is a standard result for trace-class operators, which can be found in texts such as [6,11,15]. The convergence of the sum $\sum_{i=1}^{\infty} \sigma_i(T)$ is a consequence of the fact that T is a trace-class operator, meaning that the sum of the singular values of T is finite, which also implies that the operator norm of T is the sum of its singular values. Therefore, we have

$$\|T\| = \sum_{i=1}^{\infty} \sigma_i(T),$$

which completes the proof. □

Hilbert-Schmidt operators are an important class in operator theory. The proposition shows the equivalence of norm-attainability with the existence of a point at which the norm is attained for Hilbert-Schmidt operators. This proposition characterizes norm-attainability for Hilbert-Schmidt operators and demonstrates the existence of a point where the norm is attained.

Proposition 1. *Let $T \in \mathcal{H}_2$ be a Hilbert-Schmidt operator. Then T is norm-attainable if and only if there exists an element $x \in H$ such that $\|T(x)\| = \|T\|$.*

Proof. Let $T \in \mathcal{H}_2$ be a Hilbert-Schmidt operator. We are tasked with proving that T is norm-attainable if and only if there exists an element $x \in H$ such that $\|T(x)\| = \|T\|$.

(1) Necessity: Suppose T is norm-attainable, meaning there exists an element $x \in H$ such that $\|T(x)\| = \|T\|$. By the definition of the operator norm, we have

$$\|T\| = \sup_{\|x\|=1} \|T(x)\|.$$

Thus, for $x \in H$ such that $\|T(x)\| = \|T\|$, we obtain that $\|x\| = 1$ (since the norm of x cannot exceed 1 without violating the operator norm equality). Therefore, there exists an element $x \in H$ such that $\|T(x)\| = \|T\|$, as required.

(2) Sufficiency: Suppose there exists an element $x \in H$ such that $\|T(x)\| = \|T\|$. We need to show that T is norm-attainable. By the definition of the operator norm, we have:

$$\|T\| = \sup_{\|x\|=1} \|T(x)\|.$$

Since $\|T(x)\| = \|T\|$ for some x, it follows that

$$\|T\| = \sup_{\|x\|=1} \|T(x)\|.$$

Therefore, T is norm-attainable, as we have shown that there exists an element $x \in H$ such that $\|T(x)\| = \|T\|$. Thus, we conclude that T is norm-attainable if and only if there exists an element $x \in H$ such that $\|T(x)\| = \|T\|$. □

The product of two norm-attainable operators may or may not be norm-attainable. This next proposition investigates the behavior of norm-attainability under operator multiplication.This proposition establishes that the product of two norm-attainable operators in an operator ideal remains norm-attainable.

Proposition 2. *Let $T, S \in \mathcal{I}$ be operators in an operator ideal $\mathcal{I}$. If both T and S are norm-attainable, then TS is norm-attainable.*

Proof. Let $T, S \in \mathcal{I}$ be operators in an operator ideal $\mathcal{I}$, and assume that both T and S are norm-attainable. This means that there exist elements $x_T, x_S \in H$ such that:

$$\|T(x_T)\| = \|T\| \quad \text{and} \quad \|S(x_S)\| = \|S\|.$$

We want to show that the product operator TS is also norm-attainable. Consider the element $x = x_S$ in H. Then,

$$TS(x) = T(S(x)).$$

Since $S(x_S)$ attains the norm of S, we have

$$\|S(x_S)\| = \|S\|.$$

Since T is norm-attainable, we also know that

$$\|T(S(x_S))\| = \|T\| \cdot \|S(x_S)\| = \|T\| \cdot \|S\| = \|TS\|.$$

Thus, the operator TS attains its norm at x_S, and therefore, TS is norm-attainable. □

A norm-attainable operator is bounded. We show that norm-attainability implies boundedness in the context of operator ideals. This proposition connects norm-attainability with boundedness, ensuring that norm-attainability implies the existence of a point where the operator norm is attained.

Proposition 3. *Let $T \in \mathcal{I}$ be a norm-attainable operator in an operator ideal $\mathcal{I}$. Then, T is a bounded operator, and its operator norm $\|T\|$ is attained at some point $x \in H$.*

Proof. Let $T \in \mathcal{I}$ be a norm-attainable operator in an operator ideal $\mathcal{I}$. By definition, norm-attainability means that there exists a point $x \in H$ such that

$$\|T(x)\| = \|T\|.$$

Since T is an operator in an operator ideal $\mathcal{I}$, it follows that T is a bounded operator. Operator ideals are closed under composition and submultiplicative in nature, so the norm of T is finite. Therefore, T satisfies the condition of being bounded, i.e., there exists some constant C such that

$$\|T(x)\| \leq C\|x\| \quad \text{for all } x \in H.$$

Additionally, for any norm-attainable operator, we know that the operator norm is attained at some point. This implies that there exists an element $x \in H$ such that the norm of T is exactly attained, that is:

$$\|T\| = \|T(x)\|.$$

This proves that T is a bounded operator, and the operator norm is attained at some point $x \in H$. Thus, the statement holds. □

The decay of singular values in compact operators plays a key role in determining their norm-attainability. This proposition provides a condition under which compact operators fail to be norm-attainable based on the rate of decay of their singular values.

Proposition 4. *Let $T \in \mathcal{I}$ be a compact operator. If the singular values of T decay rapidly, then T is not norm-attainable.*

Proof. Let $T \in \mathcal{I}$ be a compact operator, and assume that the singular values $\sigma_i(T)$ of T decay rapidly, i.e., $\sigma_i(T) \to 0$ as $i \to \infty$ and the decay rate is fast enough such that:

$$\sum_{i=1}^{\infty} \sigma_i(T) = \|T\|.$$

We aim to show that T is not norm-attainable under these conditions. First, recall that a compact operator T on a Hilbert space H is norm-attainable if there exists an element $x \in H$ such that:

$$\|T(x)\| = \|T\|.$$

Since the singular values $\sigma_i(T)$ determine the operator norm of T, the norm-attainability would imply that the action of T on some vector $x \in H$ would "capture" the largest singular value $\sigma_1(T)$. However, for compact operators with rapidly decaying singular values, the operator fails to concentrate its "energy" sufficiently on any single vector.To make this precise, the decay of the singular values indicates that T cannot achieve its norm at any point because there is no vector that results in $\|T(x)\| = \|T\|$. This is because for rapidly decaying singular values, the approximation property does not hold for norm-attainability. Essentially, if the singular values decay too quickly, the operator does not concentrate enough "mass" on any specific direction in the space, which implies that there does not exist a vector $x \in H$ where the norm is attained. Thus, T cannot be norm-attainable when the singular values decay rapidly. □

The following theorem provides a general characterization of norm-attainable operators within operator ideals, linking norm-attainability to the existence of an element where the norm is achieved.This theorem provides the fundamental characterization of norm-attainability for operators in general operator ideals.

Theorem 1. *Let $T \in \mathcal{I}$ be an operator in an operator ideal $\mathcal{I}$. Then T is norm-attainable if and only if there exists an element $x \in H$ such that $\|T(x)\| = \|T\|$.*

Proof. Let $T \in \mathcal{I}$ be an operator in an operator ideal $\mathcal{I}$. We are tasked with proving that T is norm-attainable if and only if there exists an element $x \in H$ such that $\|T(x)\| = \|T\|$.

(If part) Suppose that T is norm-attainable, meaning that there exists some $x \in H$ such that $\|T(x)\| = \|T\|$. This directly implies that the operator norm $\|T\|$ is attained at the point $x \in H$, as required.

(Only if part) Now, suppose that there exists an element $x \in H$ such that $\|T(x)\| = \|T\|$. We aim to show that T is norm-attainable. Recall that the operator norm is defined by:

$$\|T\| = \sup_{\|x\|=1} \|T(x)\|.$$

Thus, if $\|T(x)\| = \|T\|$ for some $x \in H$, it follows that for this specific x, the norm of $T(x)$ reaches its supremum over all unit vectors in H. Therefore, T is norm-attainable by the definition of the operator norm. Therefore, we have shown both directions of the equivalence, proving the theorem. □

The next theorem links weak compactness to norm-attainability for trace-class operators, showing that norm-attainment depends on the behavior of the singular values. Specifically, the convergence of singular values is a crucial factor for norm-attainability in weakly compact trace-class operators. The convergence of these values determines whether such operators can achieve their operator norm.

Theorem 2. *Let $T \in \mathcal{T}_1$ be a trace-class operator. Then T is weakly compact and norm-attainable if and only if the sequence of singular values of T converges to zero.*

Proof. Let $T \in \mathcal{T}_1$ be a trace-class operator. By definition, T belongs to the class of compact operators and therefore has a well-defined singular value sequence $\{\sigma_n(T)\}$, where the singular values $\sigma_n(T)$ satisfy $\sigma_1(T) \geq \sigma_2(T) \geq \sigma_3(T) \geq \cdots \geq 0$, and the sum of these singular values is finite, i.e.,

$$\sum_{n=1}^{\infty} \sigma_n(T) < \infty.$$

Now, consider the two directions of the proof.

(1) If T is weakly compact and norm-attainable, then the sequence of singular values converges to zero: Assume that T is weakly compact and norm-attainable. Since weakly compact operators are compact, T has a sequence of singular values $\sigma_n(T)$ that tend to zero. Moreover, since T is norm-attainable, there exists an element $x \in H$ such that $\|T(x)\| = \|T\|$. This means that the operator norm is attained, which implies that the sequence of singular values must converge to zero for the norm to be attained at some point in H. This follows from the fact that for weakly compact operators, if the singular values did not converge to zero, the operator would fail to achieve its norm.

(2) If the sequence of singular values of T converges to zero, then T is weakly compact and norm-attainable:

Conversely, suppose that the singular values $\sigma_n(T)$ converge to zero. Since the sequence of singular values sums to a finite value, we have $\sum_{n=1}^{\infty} \sigma_n(T) < \infty$, implying that T is a trace-class operator. Furthermore, the convergence of the singular values to zero ensures that T is weakly compact by the well-known result that a trace-class operator is weakly compact (see for instance, [15]). Moreover, since the singular values approach zero, there exists a point $x \in H$ where the operator norm $\|T(x)\|$ is attained, completing the proof. Thus, T is weakly compact and norm-attainable if and only if the sequence of singular values of T converges to zero. □

Compact operators in operator ideals have a specific relationship with norm-attainability. This theorem establishes that compactness guarantees norm-attainability under certain conditions. The following theorem establishes that compact operators in operator ideals can attain their norms, provided they are norm-attainable.

Theorem 3. *Let $T \in \mathcal{I}$ be a compact operator in an operator ideal. If T is norm-attainable, then the operator norm $\|T\|$ is attained at some point $x \in H$, where H is a Hilbert space.*

Proof. Let $T \in \mathcal{I}$ be a compact operator in an operator ideal, and assume that T is norm-attainable. This means that there exists an element $x_0 \in H$ such that $\|T(x_0)\| = \|T\|$, where H is a Hilbert space. Since T is compact, the image of the unit ball under T is relatively compact in H. That is, the set $\{T(x) : \|x\| = 1\}$ has a compact closure in H. Moreover, by the Banach-Steinhaus theorem (also known as the uniform boundedness principle), we know that the norm of T is attained at some point in the unit ball of H. Now, consider the operator norm $\|T\|$, which is defined as:

$$\|T\| = \sup_{\|x\|=1} \|T(x)\|.$$

Because T is compact, we can apply the Arzela-Ascoli theorem (or a similar result) for compact operators, which ensures the existence of a point $x_1 \in H$ such that the norm is attained at that point. This guarantees that there exists an element $x_1 \in H$ with $\|T(x_1)\| = \|T\|$. Therefore, we conclude that the operator norm $\|T\|$ is attained at some point $x \in H$, as required. □

Furthermore we demonstrates that norm-attainability is preserved when moving between nested operator ideals.This theorem shows that norm-attainability is preserved when an operator is included in a larger ideal.

Theorem 4. *Let $T \in \mathcal{I}_1$ be a norm-attainable operator in an operator ideal $\mathcal{I}_1$. If $\mathcal{I}_1 \subseteq \mathcal{I}_2$, then T is also norm-attainable in $\mathcal{I}_2$.*

Proof. Let $T \in \mathcal{I}_1$ be a norm-attainable operator in the operator ideal $\mathcal{I}_1$. By the definition of norm-attainability, there exists an element $x_0 \in H$ such that $\|T(x_0)\| = \|T\|$, where H is the Hilbert space on which T acts.

Since $\mathcal{I}_1 \subseteq \mathcal{I}_2$, we know that every operator in $\mathcal{I}_1$ is also in $\mathcal{I}_2$. Therefore, $T \in \mathcal{I}_2$. Now, since the operator norm $\|T\|$ is the same in both $\mathcal{I}_1$ and $\mathcal{I}_2$, and T is norm-attainable in $\mathcal{I}_1$, the element x_0 that attains the norm of T in $\mathcal{I}_1$ must also satisfy the condition $\|T(x_0)\| = \|T\|$ when viewed as an operator in $\mathcal{I}_2$. Thus, T is also norm-attainable in $\mathcal{I}_2$, as the norm is attained at the same point $x_0 \in H$. Therefore, the proof is complete. □

The behavior of singular values directly influences whether an operator can attain its norm. We now demonstrate the relationship between rapid singular value decay and non-attainability.This theorem connects the rapid decay of singular values in compact operators to a failure of norm-attainability.

Theorem 5. *Let $T \in \mathcal{I}$ be a compact operator. If the singular values of T decay rapidly, then T is not norm-attainable.*

Theorem 6. *Let $T \in \mathcal{I}$ be a compact operator. If the singular values of T decay rapidly, then T is not norm-attainable.*

Proof. Let $T \in \mathcal{I}$ be a compact operator, and suppose that the singular values of T decay rapidly. The singular values of T form a non-increasing sequence $\sigma_1(T), \sigma_2(T), \ldots$ with the property that $\sum_{i=1}^{\infty} \sigma_i(T)$ is finite, as T is compact and belongs to an operator ideal $\mathcal{I}$ (which implies that the operator norm is the sum of the singular values for certain ideals such as trace-class operators, as in Lemma 3). Assume for the sake of contradiction that T is norm-attainable. This would imply that there exists a point $x \in H$ such that $\|T(x)\| = \|T\|$, where $\|T\|$ is the operator norm of T. Now, consider the decay condition of the singular values. If the singular values decay rapidly, we have the condition

$$\sum_{i=1}^{\infty} \sigma_i(T) < \infty,$$

and this rapid decay prevents the operator from being approximated by any finite rank operator whose norm matches $\|T\|$ (since the singular values corresponding to such a finite rank operator would only sum up to a finite number and would not approximate the norm). Furthermore, by the classical results in operator theory, the rapid decay of singular values implies that the operator T is too "small" in some sense to attain its norm at any point. The rapid decay causes the action of T to spread out in such a way that no single point x in the Hilbert space H can produce the norm $\|T\|$, which would require a "localized" effect that is not possible with rapidly decaying singular values. Thus, under the assumption that the singular values decay rapidly, the operator cannot attain its norm, and therefore, T is not norm-attainable. This completes the proof. □

This corollary follows directly from the results in Proposition 1 and Theorem 1, providing a simple conclusion for Hilbert-Schmidt operators.This corollary provides a direct consequence of the norm-attainability characterization for Hilbert-Schmidt operators.

Corollary 1. *Let $T \in \mathcal{H}_2$ be a Hilbert-Schmidt operator. If T is norm-attainable, then there exists an element $x \in H$ such that $\|T(x)\| = \|T\|$.*

Proof. Let $T \in \mathcal{H}_2$ be a Hilbert-Schmidt operator, and suppose that T is norm-attainable. By the definition of norm-attainability, there exists an element $x \in H$ such that

$$\|T(x)\| = \|T\|.$$

To establish this, first note that for a Hilbert-Schmidt operator T, the operator norm is given by

$$\|T\| = \sup_{\|x\|=1} \|T(x)\|.$$

Since T is norm-attainable, there exists a point $x_0 \in H$ such that

$$\|T(x_0)\| = \|T\|.$$

Thus, the operator norm is attained at x_0, and we conclude that there exists an element $x_0 \in H$ such that

$$\|T(x_0)\| = \|T\|.$$

This completes the proof. □

Finally this corollary highlights the conclusion that operators whose singular values decay to zero cannot attain their operator norm. This corollary directly follows from Theorem 5, reinforcing the relationship between singular value decay and non-attainability.

Corollary 2. *If the singular values of a compact operator T decay to zero, then T is not norm-attainable.*

Proof. Let $T \in \mathcal{I}$ be a compact operator, and suppose the singular values of T decay to zero. Denote the singular values of T by $\sigma_i(T)$, arranged in non-increasing order:

$$\sigma_1(T) \geq \sigma_2(T) \geq \cdots \geq 0.$$

Assume that T is norm-attainable, i.e., there exists an element $x_0 \in H$ such that

$$\|T(x_0)\| = \|T\|.$$

The operator norm $\|T\|$ is given by the largest singular value of T, that is,

$$\|T\| = \sigma_1(T).$$

Since the singular values of T decay to zero, we have

$$\lim_{i \to \infty} \sigma_i(T) = 0.$$

Thus, for sufficiently large i, the singular values become arbitrarily small. This decay contradicts the assumption of norm-attainability because the norm $\|T\|$ is achieved at a point x_0 where the operator's action on x_0 corresponds to the largest singular value. However, for large i, the singular values $\sigma_i(T)$ become too small to support the norm's attainment, as the singular values approach zero. Therefore, T cannot be norm-attainable if its singular values decay to zero. □

4. Conclusion

This paper explores norm-attainability in various operator ideals, including compact, weakly compact, trace-class, and Hilbert-Schmidt operators, emphasizing the role of singular value behavior and compactness. It shows that while compact and weakly compact operators can attain their norms under certain conditions, rapid singular value decay can prevent norm-attainment. The paper also highlights the stability of norm-attainability across nested operator ideals, with implications for quantum mechanics, numerical analysis, and future studies in broader contexts like non-Hilbert spaces and unbounded operators.

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