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Uniqueness of weak solution for nonlocal (p,q)-Kirchhoff system

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Abstract: The paper aims to investigate the existence and uniqueness of weak solution, using the Browder Theorem method, for the nonlocal (p,q)-Kirchhoff system:

$$\begin{cases} -K_1\big(\int_\Omega |\nabla\phi|^p\big)\Delta_p\phi + \lambda a(x)|\phi|^{p-2}\phi = f_1(x,\phi,\psi), & x\in\Omega \\ -K_2\big(\int_\Omega |\nabla\psi|^q\big)\Delta_q\psi + \lambda b(x)|\psi|^{q-2}v = f_2(x,\phi,\psi), & x\in\Omega \\ \phi=\psi=0, & x\in\partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, with K_1, K_2 be continuous functions and f_1, f_2 be Carathéodory functions.

Keywords: nonlocal elliptic, weak solution, p-Kirchhoff problem.

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1. Introduction

This paper investigates the existence and uniqueness of weak solutions to the nonlocal (p,q)-Kirchhoff system, using the Browder Theorem method. The system is given by:

$$\begin{cases} -K_1\left(\int_\Omega |\nabla\phi|^p\right)\Delta_p\phi + \lambda a(x)|\phi|^{p-2}\phi = f_1(x,\phi,\psi), & x\in\Omega, \\ -K_2\left(\int_\Omega |\nabla\psi|^q\right)\Delta_q\psi + \lambda b(x)|\psi|^{q-2}\psi = f_2(x,\phi,\psi), & x\in\Omega, \\ \phi=\psi=0, & x\in\partial\Omega, \end{cases} \tag{1}$$

where $\Delta_r z \equiv \operatorname{div}(|\nabla z|^{r-2}\nabla z)$ denotes the r-Laplacian for $r=p,q$, with $1<r<N$, λ is a positive parameter, and $0<\alpha\le a(x)\le\beta<\infty$, $0<\gamma\le b(x)\le\delta<\infty$. The domain Ω is a bounded subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Additionally, the functions K_1, K_2, f_1, and f_2 satisfy the following conditions:

(L1) The functions K_1 and K_2 are continuous and increasing, such that

$$0<k_i\le K_i(t)\le k_{i,\infty}, \quad \forall t\in[0,\infty), \quad i=1,2. \tag{2}$$

(L2) The functions f_1 and $f_2:\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions, decreasing with respect to the second and third variables, respectively. Specifically, if $w_2\le w_1$ and $s_2\le s_1$, then

$$f_1(x,w_1,s)\le f_1(x,w_2,s) \quad \text{and} \quad f_2(x,w,s_1)\le f_2(x,w,s_2), \tag{3}$$

for almost every $x\in\Omega$ and all $w_1,w_2,s_1,s_2\in\mathbb{R}$.

(L3) There exist $\bar{f}_1 \in L^{p'}(\Omega)$ and $\bar{f}_2 \in L^{q'}(\Omega)$ such that:

$$\begin{cases} |f_1(x,w,s)| \leq c_1 \left[\bar{f}_1(x) + |w|^{p-1} + |s|^{q/p'}\right], \\ |f_2(x,w,s)| \leq c_2 \left[\bar{f}_2(x) + |w|^{p/q'} + |s|^{q-1}\right], \end{cases} \tag{4}$$

where $c_1, c_2 > 0$, $p' = \frac{p}{p-1}$, and $q' = \frac{q}{q-1}$.

Previous studies have focused on nonlocal Kirchhoff-type elliptic systems, such as:

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2\right)\Delta u = h(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{5}$$

where M is a continuous function on $\mathbb{R}^+$, and $h(x,u)$ is continuous on $\bar{\Omega} \times \mathbb{R}$ (see [1–3]). The stationary version of the Kirchhoff equation associated with problem (5) is:

$$u_{tt} - M\left(\int_\Omega |\nabla_x u|^2\right)\Delta_x u = h(x,t),$$

where $M(t) = \alpha t + \beta$ with $\alpha, \beta > 0$. In [4], the authors established the existence of a positive weak solution for the nonlocal p-Kirchhoff-type system:

$$\begin{cases} -\left[K\left(\int_\Omega |\nabla u|^p\right)\right]^{p-1}\Delta_p u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{6}$$

using critical point theory, where $f \in \mathrm{CAR}(\bar{\Omega} \times \mathbb{R}^+)$ and K is a continuous increasing function satisfying:

$$K(t) \geq k_0 > 0 \quad \forall t \in \mathbb{R}^+. \tag{7}$$

In [5], the authors proved existence and multiplicity results for solutions of (6) using the Genus theory introduced by Krasnoselskii. The existence and uniqueness of weak solutions for the p-Laplacian system using the Browder theorem were studied in [6], while [7] generalized these results to the case of weighted p-Laplacians.

Boulaaras et al. [8] discussed the existence of weak solutions for the sublinear Kirchhoff elliptic system using the sub-super solutions method:

$$\begin{cases} -M_1\left(\int_\Omega |\nabla u|^2\right)\Delta u = \lambda_1 u^\alpha + \mu_1 v^\beta, & x \in \Omega, \\ -M_2\left(\int_\Omega |\nabla v|^2\right)\Delta v = \lambda_2 u^\delta + \mu_2 v^\gamma, & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{8}$$

where M_1, M_2 are continuous increasing functions, and λ_1, λ_2, μ_1, μ_2 are positive parameters, with $\alpha + \delta < 1$ and $\beta + \gamma < 1$.

The system (1) is classified as a nonlocal problem due to the integrals in the first two equations, which prevent the equations from being pointwise identities. It is analogous to the stationary version of the Kirchhoff equation:

$$\rho \frac{\partial^2 u}{\partial t^2} - \left[\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right]\frac{\partial^2 u}{\partial x^2} = 0, \tag{9}$$

originally presented by Kirchhoff in 1883 (see [9]), where ρ is the mass density, P_0 is the initial tension, E is the Young modulus, h is the cross-sectional area, and L is the length of the string.

The study of Kirchhoff and p-Kirchhoff type problems has been the subject of much attention due to their theoretical and practical significance. Notable works include [10–14], where topological and variational techniques were employed to prove the existence of weak solutions.

The Browder Theorem method has been successfully applied to prove the existence of positive weak solutions for various nonlinear systems (see [6,15–18]).

This work extends previous studies by considering a system with the p-Laplacian operator, which is particularly relevant in physical scenarios such as fluid mechanics (see [19]).

The structure of this paper is as follows: Section 2 provides a brief overview of relevant concepts, definitions, and theorems, along with the space setting for our problem. In Section 3, we present the proof of the main results.

2. Space Setting and Preliminaries

In this section, we outline important concepts, definitions, and theorems related to the operators used in this work, which are discussed in detail in [20].

Definition 1. Let K be a real Banach space, and let $T : K \to K^*$ be an operator. For all $\phi, \phi_1, \phi_2, \phi_n \in K$, the operator T is:

(a) **Bounded**: if it maps bounded sets to bounded sets, i.e.,

$$\forall r > 0, \exists M > 0 : \|\phi\| \le r \implies \|T(\phi)\| \le M, \quad \text{where } M \text{ depends on } r.$$

(b) **Coercive**: if

$$\lim_{\|\phi\| \to \infty} \frac{\langle T(\phi), \phi \rangle}{\|\phi\|} = \infty.$$

(c) **Monotone**: if

$$\langle T(\phi_1) - T(\phi_2), \phi_1 - \phi_2 \rangle \ge 0.$$

(d) **Strictly monotone**: if

$$\langle T(\phi_1) - T(\phi_2), \phi_1 - \phi_2 \rangle > 0 \quad \text{for } \phi_1 \neq \phi_2.$$

(e) **Strongly monotone**: if there exists $c > 0$ such that

$$\langle T(\phi_1) - T(\phi_2), \phi_1 - \phi_2 \rangle \ge c\|\phi_1 - \phi_2\|^2.$$

(f) **Continuous**: if $\phi_n \to \phi$ implies $T(\phi_n) \to T(\phi)$.
(g) **Strongly continuous**: if $\phi_n \xrightarrow{w} \phi$ implies $T(\phi_n) \to T(\phi)$.
(h) **Demicontinuous**: if $\phi_n \to \phi$ implies $T(\phi_n) \xrightarrow{w} T(\phi)$.

Remark 1.
- Every continuous operator is demicontinuous.
- Every strictly monotone operator is monotone.
- Every strongly monotone operator is coercive if T is linear on a Hilbert space K.

Theorem 1. *(Browder Theorem [21]) Let $T : K \to K^*$ be an operator on a reflexive real Banach space K. Moreover, if the operator T is: bounded, demicontinuous, monotone and coercive on the space K. Hence, the equation $T(u) = f$ has at least one solution $u \in K$ for each $f \in K^*$. If furthermore, T is strictly monotone operator, then the equation $T(u) = f$ has precisely one solution $u \in K$ for every $f \in K^*$.*

Next, we recall some background facts concerning the Sobolev spaces:

- $W^{1,p}(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left[\int_\Omega |u|^p + \int_\Omega |\nabla u|^p \right]^{\frac{1}{p}}. \tag{10}$$

- $W_0^{1,p}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$ with the norm

$$\|u\|_{W_0^{1,p}(\Omega)} = \left[\int_\Omega |\nabla u|^p \right]^{\frac{1}{p}}, \tag{11}$$

for $1 < p < \infty$, which are well defined reflexive Banach Spaces. Also, $\|u\|_{W^{1,p}(\Omega)}$ and $\|u\|_{W_0^{1,p}(\Omega)}$ are equivalent norms.

For simplicity, we consider $W_1 = W_0^{1,p}(\Omega)$ and $W_2 = W_0^{1,q}(\Omega)$. The space setting of our problem is the Banach space $W = W_1 \times W_2$ and the norm of $z = (\phi, \psi) \in W$ is defined as $\|z\|_W = \|\phi\|_{W_1} + \|\psi\|_{W_2}$, where $\|\phi\|_{W_1} = \big(\int_\Omega |\nabla\phi|^p \big)^{\frac{1}{p}}$ and $\|\psi\|_{W_2} = \big(\int_\Omega |\nabla\psi|^q \big)^{\frac{1}{q}}$. From the continuity of the embedding

$$W_1 \times W_2 \hookrightarrow\hookrightarrow L^p(\Omega) \times L^q(\Omega)$$

there exist positive constants C_p and C_q such that

$$\|\phi\|_{L^p(\Omega)} \le C_p \|\phi\|_{W_1}, \quad \|\psi\|_{L^q(\Omega)} \le C_q \|\psi\|_{W_2} \quad \forall (\phi, \psi) \in W. \tag{12}$$

Readers can find more details about the space setting in [22] and its references. Throughout this paper, the notation $\langle \cdot, \cdot \rangle$ represents the duality pairing between W and W^*.

3. Existence and Uniqueness Results

In this section, we prove that system (1) has a unique solution via the Browder theorem method.

Definition 2. We say $(\phi, v) \in W$ to be a weak solution for system (1) if

$$\begin{aligned} K_1\big(\|\phi\|_{W_1}^p\big) \int_\Omega |\nabla\phi|^{p-2}\nabla\phi \cdot \nabla\rho_1 + \lambda \int_\Omega a(x)|\phi|^{p-2}\phi\rho_1 &= \int_\Omega f_1(x, \phi, \psi)\rho_1 \quad \forall \rho_1 \in W_1, \\ K_2\big(\|\psi\|_{W_2}^q\big) \int_\Omega |\nabla\psi|^{q-2}\nabla\psi \cdot \nabla\rho_2 + \lambda \int_\Omega b(x)|\psi|^{q-2}\psi\rho_2 &= \int_\Omega f_2(x, \phi, \psi)\rho_2 \quad \forall \rho_2 \in W_2, \end{aligned}$$

where $\|\phi\|_{W_1}$ is the usual norm in W_1. The following theorem summarizes our main results for problem (1).

Theorem 2. *Let $(L1) - (L3)$ are satisfied, then system (1) has a unique solution.*

Proof. Suppose $\lambda \in \mathbb{R}^+$ and define the operator $T : W \to W^*$ as

$$T(\phi, \psi) := J(\phi, \psi) + \lambda S(\phi, \psi) - R(x, \phi, \psi),$$

where the operators $J, S : W \to W^*$ are given by

$$\langle J(\phi, \psi), (\rho_1, \rho_2) \rangle := \langle J_1(\phi), \rho_1 \rangle + \langle J_2(\psi), \rho_2 \rangle,$$

where,

$$\begin{aligned} \langle J_1(\phi), \rho_1 \rangle &= K_1\big(\|\phi\|_{W_1}^p\big) \int_\Omega |\nabla\phi|^{p-2}\nabla\phi \cdot \nabla\rho_1, \\ \langle J_2(\psi), \rho_2 \rangle &= K_2\big(\|\psi\|_{W_2}^q\big) \int_\Omega |\nabla\psi|^{q-2}\nabla\psi \cdot \nabla\rho_2, \end{aligned}$$

and,

$$\langle S(\phi, \psi), (\rho_1, \rho_2) \rangle := \langle S_1(\phi), \rho_1 \rangle + \langle S_2(\psi), \rho_2 \rangle,$$

where,

$$\langle S_1(\phi), \rho_1 \rangle = \int_\Omega a(x)|\phi|^{p-2}\phi\rho_1, \quad \langle S_2(\psi), \rho_2 \rangle = \int_\Omega b(x)|\psi|^{q-2}\psi\rho_2,$$

also, the operator $R : \Omega \times W \to W^*$ is given by

$$\langle R(x, \phi, \psi), (\rho_1, \rho_2) \rangle := \langle R_1(x, \phi, \psi), \rho_1 \rangle + \langle R_2(x, \phi, \psi), \rho_2 \rangle,$$

where,

$$\langle R_1(x, \phi, \psi), \rho_1 \rangle = \int_\Omega f_1(x, \phi, \psi)\rho_1, \quad \langle R_2(x, \phi, \psi), \rho_2 \rangle = \int_\Omega f_2(x, \phi, \psi)\rho_2,$$

$\forall(\rho_1,\rho_2) \in W$. We say $(\phi,\psi) \in W$ to be a weak solution for system (1) if

$$\langle T(\phi,\psi),(\rho_1,\rho_2)\rangle = \langle J(\phi,\psi),(\rho_1,\rho_2)\rangle + \lambda\,\langle S(\phi,\psi),(\rho_1,\rho_2)\rangle - \langle R(x,\phi,\psi),(\rho_1,\rho_2)\rangle = 0,$$

holds for any $(\rho_1,\rho_2) \in W$. Finding $(\phi,\psi) \in W$ that satisfies the operator equation $T(\phi,\psi) = 0$ is the equivalent of finding a weak solution for system (1).

We split our proof into several steps, in order to apply Browder Theorem:

Step 1. We prove the operators J, S and R are well defined. By Hölder's inequality, for the operator J, we have

$$\begin{aligned}|\langle J_1(\phi),\rho_1\rangle| &\le \left|K_1(\|\phi\|^p_{W_1})\right| \int_\Omega |\nabla\phi|^{p-1}\,|\nabla\rho_1| \\ &\le k_{1,\infty}\left(\int_\Omega |\nabla\phi|^p\right)^{\frac{1}{p'}}\left(\int_\Omega |\nabla\rho_1|^p\right)^{\frac{1}{p}} < \infty.\end{aligned}$$

Similarly, for the operator J_2. Therefore, since their sum is well defined, then the operator J is well defined. For the operator S, we have

$$|\langle S_1(\phi),\rho_1\rangle| \le \int_\Omega a(x)|\phi|^{p-1}\,|\rho_1| \le \beta\Big(\int_\Omega |\phi|^p\Big)^{\frac{1}{p'}}\left(\int_\Omega |\rho_1|^p\right)^{\frac{1}{p}} < \infty.$$

Similarly,

$$|\langle S_2(\psi),\rho_2\rangle| \le \int_\Omega b(x)|\psi|^{q-1}|\rho_2| \le \delta\left(\int_\Omega |\psi|^q\right)^{\frac{1}{q'}}\left(\int_\Omega |\rho_2|^q\right)^{\frac{1}{q}} < \infty.$$

So, both S_1 and S_2 are well defined, then the operator S is well defined.
Also, the operator R can be written as the sum of R_1 and R_2. For the operator R_1, we get

$$\begin{aligned}|\langle R_1(x,\phi,\psi),\rho_1\rangle| &\le c_1\left(\int_\Omega \left(\bar f_1(x) + |\phi|^{p-1} + |\psi|^{q/p'}\right)|\rho_1|\right) \\ &\le c_1\left[\left(\int_\Omega |\bar f_1(x)|^{p'}\right)^{\frac{1}{p'}} + \left(\int_\Omega |\phi|^p\right)^{\frac{1}{p'}} + \left(\int_\Omega |\psi|^q\right)^{\frac{1}{p'}}\right]\left(\int_\Omega |\rho_1|^p\right)^{\frac{1}{p}} \\ &= c_1\left[||\bar f_1||_{L^{p'}(\Omega)} + ||\phi||^{p/p'}_{L^p(\Omega)} + ||\psi||^{q/p'}_{L^q(\Omega)}\right]||\rho_1||_{L^p(\Omega)} < \infty.\end{aligned}$$

Similarly, for the operator R_2, and hence R is well defined.

Step 2. The operators J, S and R are bounded. Indeed, $\forall\phi,\psi$ such that $||\phi||_{W_1} \le H$, $||\psi||_{W_2} \le L$, for the operator J, we have

$$\begin{aligned}||J_1(\phi)||_{W^*} &= \sup_{||\rho_1||_{W_1}\le 1} |\langle J_1(\phi),\rho_1\rangle| \\ &\le k_{1,\infty}\sup_{||\rho_1||_{W_1}\le 1}\int_\Omega |\nabla\phi|^{p-1}|\nabla\rho_1| \\ &\le k_{1,\infty}\sup_{||\rho_1||_{W_1}\le 1}\left(\int_\Omega |\nabla\phi|^p\right)^{\frac{1}{p'}}\left(\int_\Omega |\nabla\rho_1|^p\right)^{\frac{1}{p}} \\ &\le k_{1,\infty}H^{p/p'}.\end{aligned}$$

Similarly,

$$\begin{aligned}||J_2(\psi)||_{W^*} &\le k_{2,\infty}\sup_{||\rho_2||_{W_2}\le 1}\left(\int_\Omega |\nabla\psi|^q\right)^{\frac{1}{q'}}\left(\int_\Omega |\nabla\rho_2|^q\right)^{\frac{1}{q}} \\ &\le k_{2,\infty}L^{q/q'}.\end{aligned}$$

Hence J is bounded.

Also, for the operator S, we have

$$\begin{aligned} ||S_1(\phi)||_{W^*} &= \sup_{||\rho_1||_{W_1}\le 1} |\langle S_1(\phi),\rho_1\rangle| \le \beta \sup_{||\rho_1||_{W_1}\le 1} \left(\int_\Omega |\phi|^p\right)^{\frac{1}{p'}} \left(\int_\Omega |\rho_1|^p\right)^{\frac{1}{p}} \\ &\le \beta {C_p}^p H^{p/p'}. \end{aligned}$$

Similarly,

$$\begin{aligned} ||S_2(\psi)||_{W^*} &\le \delta \sup_{||\rho_2||_{W_2}\le 1} \left(\int_\Omega |\psi|^q\right)^{\frac{1}{q'}} \left(\int_\Omega |\rho_2|^q\right)^{\frac{1}{q}} \\ &\le \delta {C_q}^q L^{q/q'}. \end{aligned}$$

Then S is bounded.

Finally, for the operator R_1, we have

$$\begin{aligned} ||R_1(x,\phi,\psi)||_{W^*} &= \sup_{||\rho_1||_{W_1}\le 1} |\langle R_1(x,\phi,\psi),\rho_1\rangle| \\ &\le c_1 \sup_{||\rho_1||_{W_1}\le 1} \int_\Omega \left(\bar{f}_1(x) + |\phi|^{p-1} + |\psi|^{q/p'}\right)|\rho_1| \\ &\le c_1 \sup_{||\rho_1||_{W_1}\le 1} \left[||\bar{f}_1||_{L^{p'}(\Omega)} + ||\phi||^{p/p'}_{L^p(\Omega)} + ||\psi||^{q/p'}_{L^q(\Omega)}\right] ||\rho_1||_{L^p(\Omega)} \\ &\le c_1 C_p (||\bar{f}_1||_{L^{p'}(\Omega)} + {C_p}^{p/p'}\, \|\phi\|^{p/p'}_{W_1} + {C_q}^{q/p'}\, ||\psi||^{q/p'}_{W_2}) \\ &\le c_1 C_p (\,||\bar{f}_1||_{L^{p'}(\Omega)} + {C_p}^{p/p'} M^{p/p'} + {C_q}^{q/p'} N^{q/p'}). \end{aligned}$$

Similarly,

$$\begin{aligned} ||R_2(x,\phi,\psi)||_{W^*} &= \sup_{||\rho_2||_{W_2}\le 1} |\langle R_2(x,\phi,\psi),\rho_2\rangle| \\ &\le c_2 C_q (\,||\bar{f}_2||_{L^{q'}(\Omega)} + {C_q}^{q/q'} N^{q/q'} + {C_p}^{p/q'} M^{p/q'}). \end{aligned}$$

Hence R is bounded.

Step 3. The operators J, S and R are continuous. Let

$$\begin{cases} \phi_n \longrightarrow \phi \quad \text{in } W_1 \implies \|\phi_n - \phi\|_{W_1} \longrightarrow 0 \implies \|\nabla\phi_n - \nabla\phi\|_{L^p(\Omega)} \longrightarrow 0, \\ \psi_n \longrightarrow \psi \quad \text{in } W_2 \implies \|\psi_n - \psi\|_{W_2} \longrightarrow 0 \implies \|\nabla\psi_n - \nabla\psi\|_{L^q(\Omega)} \longrightarrow 0. \end{cases}$$

Applying Dominated Convergence Theorem, for the operator J, we have

$$\begin{aligned} \|J_1(\phi_n) - J_1(\phi)\|_{W^*} &= \sup_{||\rho_1||_{W_1}\le 1} |\langle J_1(\phi_n) - J_1(\phi),\rho_1\rangle| \\ &\le k_{1,\infty} \left(\int_\Omega \left[|\nabla\phi_n|^{p-2}\nabla\phi_n - |\nabla\phi|^{p-2}\nabla\phi\right]^{p'}\right)^{\frac{1}{p'}} \to 0 \quad \text{for} \quad n\to\infty. \end{aligned}$$

Similarly,

$$\begin{aligned} \|J_2(\psi_n) - J_2(\psi)\|_{W^*} &= \sup_{||\rho_2||_{W_2}\le 1} |\langle J_2(\psi_n) - J_2(\psi),\rho_2\rangle| \\ &\le k_{2,\infty} \left(\int_\Omega \left[|\nabla\psi_n|^{q-2}\nabla\psi_n - |\nabla\psi|^{q-2}\nabla\psi\right]^{q'}\right)^{\frac{1}{q'}} \to 0 \quad \text{for} \quad n\to\infty. \end{aligned}$$

Hence J is continuous.

Also, for the operator S, we have

$$\|S_1(\phi_n) - S_1(\phi)\|_{W^*} = \sup_{\|\rho_1\|_{W_1} \le 1} |\langle S_1(\phi_n) - S_1(\phi), \rho_1\rangle|$$
$$\le \beta C_p \left(\int_\Omega \left[|\phi_n|^{p-2}\phi_n - |\phi|^{p-2}\phi\right]^{p'}\right)^{\frac{1}{p'}} \to 0 \quad \text{for } n \to \infty.$$

Similarly,

$$\|S_2(\psi_n) - S_2(\psi)\|_{W^*} \le \delta C_q \left(\int_\Omega \left[|\psi_n|^{q-2}\psi_n - |\psi|^{q-2}\psi\right]^{q'}\right)^{\frac{1}{q'}} \to 0 \quad \text{for } n \to \infty.$$

Hence S is continuous.

Finally, since f_1, f_2 be Carathéodory functions satisfy $(L3)$, then the Nemytskij operators R_1 acting from W into $L^{p'}(\Omega)$ and R_2 acting from W into $L^{q'}(\Omega)$ are continuous operators (see[20]). Hence R is continuous.

Step 4. We prove T is a monotone operator. Let $p \ge 2$, then we have (see [23])

$$|y_2|^p \ge |y_1|^p + p\,|y_1|^{p-2}\,y_1(y_2 - y_1) + \frac{|y_2 - y_1|^p}{2^{p-1} - 1} \quad \forall y_1, y_2 \in \mathbb{R}^N. \tag{13}$$

From $(L1)$ and using (13) for $p \ge 2$, we get

$$\langle J_1(\phi) - J_1(\rho_1), \phi - \rho_1\rangle = K_1(\|\phi\|^p_{W_1}) \int_\Omega \left[|\nabla\phi|^{p-2}\nabla\phi - |\nabla\rho_1|^{p-2}\nabla\rho_1\right](\nabla\phi - \nabla\rho_1)$$
$$\ge k_1\left[\int_\Omega |\nabla\phi|^{p-2}\nabla\phi(\nabla\phi - \nabla\rho_1) - \int_\Omega |\nabla\rho_1|^{p-2}\nabla\rho_1(\nabla\phi - \nabla\rho_1)\right]$$
$$\ge \frac{2k_1}{p(2^{p-1} - 1)} \int_\Omega |\nabla\phi - \nabla\rho_1|^p$$
$$= k_1\mu_p\,\|\phi - \rho_1\|^p_{W_1},$$

where $\mu_p = \frac{2}{p(2^{p-1}-1)}$. Similarly, for $q \ge 2$

$$\langle J_2(\psi) - J_2(\rho_2), v - \rho_2\rangle \ge \frac{2k_2}{q(2^{q-1} - 1)} \int_\Omega |\nabla\psi - \nabla\rho_2|^q$$
$$= k_2\mu_q\,\|\psi - \rho_2\|^q_{W_2},$$

where $\mu_q = \frac{2}{q(2^{q-1}-1)}$. Hence, for $p, q \ge 2$

$$\langle J(\phi, \psi) - J(\rho_1, \rho_2), (\phi, \psi) - (\rho_1, \rho_2)\rangle \ge k_1\mu_p\,\|\phi - \rho_1\|^p_{W_1} + k_2\mu_q\,\|\psi - \rho_2\|^q_{W_2}. \tag{14}$$

Also, we have

$$\langle S_1(\phi) - S_1(\rho_1), \phi - \rho_1\rangle = \int_\Omega a(x)\left[|\phi|^{p-2}\phi - |\rho_1|^{p-2}\rho_1\right](\phi - \rho_1)$$
$$\ge \frac{2}{p(2^{p-1} - 1)} \int_\Omega a(x)\,|\phi - \rho_1|^p$$
$$\ge \alpha\mu_p\,\|\phi - \rho_1\|^p_{L^p(\Omega)} \ge 0.$$

Similarly,

$$\langle S_2(\psi) - S_2(\rho_2), \psi - \rho_2\rangle \ge \gamma\mu_q\,\|\psi - \rho_2\|^q_{L^q(\Omega)} \ge 0.$$

Hence,

$$\langle S(\phi, \psi) - S(\rho_1, \rho_2), (\phi, \psi) - (\rho_1, \rho_2)\rangle \ge 0. \tag{15}$$

Also, from $(L2)$, we have

$$[f_1(x, \phi, \psi) - f_1(x, \rho_1, \psi)](\phi - \rho_1) \le 0,$$

consequently,

$$\langle R_1(x,\phi,\psi) - R_1(x,\rho_1,\psi), \phi - \rho_1\rangle = \int_\Omega [f_1(x,\phi,\psi) - f_1(x,\rho_1,\psi)](\phi - \rho_1) \leq 0,$$

similarly,

$$\langle R_2(x,\phi,\psi) - R_2(x,\phi,\rho_2), \psi - \rho_2\rangle \leq 0,$$

so,

$$\langle R(x,\phi,\psi) - R(x,\rho_1,\rho_2), (\phi,\psi) - (\rho_1,\rho_2)\rangle \leq 0. \tag{16}$$

Equations (14), (15) and (16), for $p, q \geq 2$ imply that

$$\begin{aligned}\langle T(\phi,\psi) - T(\rho_1,\rho_2), (\phi,\psi) - (\rho_1,\rho_2)\rangle &\geq k_1\mu_p \,\|\phi - \rho_1\|^p_{W_1} + k_2\mu_q \,\|\psi - \rho_2\|^q_{W_2} \\ &\geq c_{\min}[\|\phi - \rho_1\|^p_{W_1} + \|\psi - \rho_2\|^q_{W_2}],\end{aligned} \tag{17}$$

where $c_{\min} = \min\{k_1\mu_p, k_2\mu_q\}$. Hence, T is monotone.
Step 5. Now, we prove T is a coercive operator. Equation (17) gives us the following:

$$\langle T(\phi,\psi), (\phi,\psi)\rangle \geq \langle T(0,0), (\phi,\psi)\rangle + c_{\min}\left[\, \|\phi\|^p_{W_1} + \|\psi\|^q_{W_2}\,\right].$$

On the other side,

$$\begin{aligned}\langle T(0,0), (\phi,\psi)\rangle &= \langle J(0,0), (\phi,\psi)\rangle + \lambda\,\langle S(0,0), (\phi,\psi)\rangle - \langle R(x,0,0), (\phi,\psi)\rangle \\ &= -\int_\Omega f_1(x,0,0)\phi - \int_\Omega f_2(x,0,0)\psi \\ &\geq -c_1\int_\Omega \bar{f}_1(x)\phi - c_2\int_\Omega \bar{f}_2(x)\psi \\ &\geq -c_1\Big(\int_\Omega [\bar{f}_1(x)]^{p'}\Big)^{\frac{1}{p'}}\Big(\int_\Omega |\phi|^p\Big)^{\frac{1}{p}} - c_2\Big(\int_\Omega [\bar{f}_2(x)]^{q'}\Big)^{\frac{1}{q'}}\Big(\int_\Omega |\psi|^q\Big)^{\frac{1}{q}} \\ &\geq -c_1C_p\,||\bar{f}_1||_{L^{p'}(\Omega)}\,||\phi||_{W_1} - c_2C_q\,||\bar{f}_2||_{L^{q'}(\Omega)}\,||\psi||_{W_2},\end{aligned}$$

then,

$$\langle T(\phi,\psi), (\phi,\psi)\rangle \geq c_{\min}[\|\phi\|^p_{W_1} + \|\psi\|^q_{W_2}] - c_1C_p\,||\bar{f}_1||_{L^{p'}(\Omega)}\,||\phi||_{W_1} - c_2C_q\,||\bar{f}_2||_{L^{q'}(\Omega)}\,||\psi||_{W_2}.$$

So, one can have

$$\lim \frac{\langle T(\phi,\psi), (\phi,\psi)\rangle}{||(\phi,\psi)||_W} = \infty \quad \text{when} \quad ||(\phi,\psi)||_W \to \infty.$$

Hence, T is a coercive operator, consequently, there exists a weak solution for system (1).
Step 6. The uniqueness of weak solution for system (1) directly follows from (17). Let $(\phi_1,\psi_1), (\phi_2,\psi_2)$ be weak solutions for system (1) such that $(\phi_1,\psi_1) \neq (\phi_2,\psi_2)$. Now, from (14), we have

$$\begin{aligned}0 &= \langle T(\phi_1,\psi_1) - T(\phi_2,\psi_2), (\phi_1,\psi_1) - (\phi_2,\psi_2)\rangle \\ &\geq c_{\min}[\|\phi_1 - \phi_2\|^p_{W_1} + \|\psi_1 - \psi_2\|^q_{W_2}] \geq 0 \quad \text{for} \quad p,q \geq 2,\end{aligned}$$

therefore $(\phi_1,\psi_1) = (\phi_2,\psi_2)$. $\square$

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