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Article

A direct proof of stability of nonnegative weak solution for fractional p-Laplacian problem with concave nonlinearity

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Abstract: The present paper provides a direct proof of stability of nontrivial nonnegative weak solution for fractional p-Laplacian problem under concave nonlinearity condition. The main results of this work are extend the previously known results for the fractional Laplacian problem.

Keywords: Stability, weak solution, fractional p-Laplacian.

MSC: 35B35,35D30,35R11.

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1. Introduction

In this paper, we establish a direct proof of stability of nontrivial nonnegative weak solution for the following fractional p-Laplacian problem

$$\left.\begin{array}{lr} (-\Delta)_p^s u(x) = f(u) & x \in \Omega, \\ u > 0 & x \in \Omega, \\ u = 0 & x \in R^n/\Omega, \end{array}\right\} \tag{1}$$

where $\Omega \subset \mathbb{R}^n, n \geq 1$, is a bounded domain, $s \in (0,1), 1 < p < \infty$ and $(-\Delta_p)^s$ is the nonlocal nonlinear fractional p-Laplacian operator of order s defined by

$$(-\Delta)_p^s u(x) = C_{n,s,p}\ P.V. \int\limits_{R^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}} dy, \tag{2}$$

where $C_{n,s,p}$ is a constant depending on n, s, p and P.V. denotes the commonly used abbreviations for "the main value sense".

Under a certian concave nonlinearity condition on the function f, we establish that every nontrivial nonnegative weak solution of problem (1) is stable. The aim of this paper is to extend the results in [1] to the fractional p-Laplacian case.

The standard fractional Sobolev space is the natural space where one looks for a solution to the problem (1). We refer the reader to [2,3] for further results and references. Let Ω be an open set in $\mathbb{R}^n$, $s \in (0,1)$, and $p \in [0, \infty)$. We define the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ as follows

$$W^{s,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) : \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy < \infty\}, \tag{3}$$

equipped with the norm

$$\|u\|_{W^{s,p}\mathbb{R}^n)} = [\int\limits_{\mathbb{R}^n} |u|^p dx + \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy]^{1/p}. \tag{4}$$

Let

$$W_0^{s,p}\mathbb{R}^n) = \{u \in W^{s,p}(\mathbb{R}^n) : u = 0 \text{ a.e. } u \in \mathbb{R}^n \backslash \Omega\}, \tag{5}$$

be a closed linear subspace of $W^{s,p}\mathbb{R}^n)$, and its norm is given by

$$\|u\|_{W_0^{s,p}(\mathbb{R}^n)} = [\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy]^{1/p}, \tag{6}$$

which is equivalent to the norm given by (4). The fractional space $W_0^{s,p}(\mathbb{R}^n)$ is a suitable candidate to find "weak solutions" to the problem (1).

In the past few years, a great attention has been given to the subjects involving fractional p-Laplacian and nonlocal operators since it is not only a natural extension of the p-Laplacian equations but also presents many new phenomena described by nonlinear integral structures such as phase transition, population dynamics, optimization, finance, water waves, continuum mechanics, minimal surfaces, game theory and other sciences.

Numerous physical phenomena, including flow through porous media, reaction-diffusion issues, petroleum extraction, non-Newtonian fluids, etc., lead to the so-called Fractional p-Laplacian boundary value problems. In light of this, a lot of mathematicians have been interested in studying such problems and their generalizations recently.

Due to the many possible uses in reaction-diffusion problems, fluid mechanics, Newtonian fluids, glaciology, population dynamics, etc. see [4] and references therein, an extensive number of authors take an interest in the study of the stability and instability of nonnegative solutions of linear [5] , semilinear (see [6–9]), semiposiotne (see [10–12]), nonlinear (see [13–17]) and fractional [18,19] systems. Also, the existence of positive weak solutions for the prevuoius sytems have been studied (see [20–22]).

Most of the previous proofs of stability of nontrivial nonnegative weak solution for linear and nonlinear systems are computational and relatively long compared to the proof presented in this paper. Once the stability of of nontrivial nonnegative weak solution for problem (1) is established, we arrive at the main point of this article.

Definition 1. A bounded solution $u \in W_0^{s,p}(\mathbb{R}^n)$ of (1) is stable if

$$\int\limits_{R^n} \left|(-\Delta_p)^{\frac{s}{2}}\varphi\right|^2 dx \geq \int\limits_{\Omega} f'(u)\varphi^2 dx \quad \text{for all } \varphi \in W_0^{s,p}(\mathbb{R}^n).$$

It makes sense that a system is stable if it has the ability to recover from perturbations; a small change will not prevent the system from returning to equilibrium.

Lemma 2. *[23] For every $a, b \geq 0$ and $c, d > 0$, we have $(a-b)^2 \geq (c-d)[\frac{a^2}{c} - \frac{b^2}{d}]$. The equality holds if and only if $(a, b) = k(c, d)$ for some constant k.*

Proof. The result follows from the convexity of the mapping $t \to t^2$ (see [23]). □

Finally, let us explain the plan of the paper. In section 2, we introduce a direct proof to establish that every nontrivial nonnegative solution of problem (1) is stable under certian conditions. We introduce some applications in section 3.

2. Main Results

In this section, by a simple short method, we introduce a direct proof of stability of nontrivial nonnegative weak solution for problem (1) under a concave nonlinearity condition. Now, we state the main theorem in this paper.

Theorem 3. *If $f''(u) < 0$ and $f(0) \geq 0$, then every positive weak solution of (1) is stable.*

Proof. Let u be any nontrivial nonnegative weak solution of (1. Also, let $l(u) = uf'(u) - f(u)\ \forall u \in \mathbb{R}^+$. Then $l(0) \leq 0$ and $l'(u) < 0$, and so $l(u) < 0$. Testing the equation (1) with v^2/u gives

$$\begin{aligned}
0 &= \int\limits_{R^n} (-\Delta_p)^{\frac{s}{2}} u (-\Delta_p)^{\frac{s}{2}} (\frac{v^2}{u}) dx - \int\limits_{\Omega} f(u)(\frac{v^2}{u}) dx \\
&= \frac{C_{n,s,p}}{2} \int\limits_{R^n}\int\limits_{R^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y)) \left|\frac{v^2(x)}{u(x)} - \frac{v^2(y)}{u(y)}\right|^{p-2} [\frac{v^2(x)}{u(x)} - \frac{v^2(y)}{u(y)}]}{|x-y|^{n+sp}} dxdy \\
&\quad - \int\limits_{\Omega} f(u)(\frac{v^2}{u}) dx \\
&\leq \frac{C_{n,s,p}}{2} \int\limits_{R^n}\int\limits_{R^n} \frac{[|v(x)-v(y)|^{p-2}(v(x)-v(y))]^2}{|x-y|^{n+sp}} dxdy - \int\limits_{\Omega} f'(u) v^2 dx \\
&= \int\limits_{R^n} \left|(-\Delta_p)^{\frac{s}{2}} v\right|^2 dx - \int\limits_{R^n} f'(u) v^2 dx.
\end{aligned}$$

Then u is stable and the result follows. □

Remark 1. This result generalises the main result in [1], where the case $p = 2$ have been studied by the author.

3. Applications

Here, we introduce the following two examples to highlight the main results.

Example 1. Consider the following fractional version of the Allen-Cahn equation:

$$\left.\begin{array}{rr} (-\Delta_p)^s u(x) = u - u^3 & x \in \Omega, \\ u > 0 & x \in \Omega, \\ u = 0 & x \in R^n/\Omega. \end{array}\right\} \tag{7}$$

The fractional Allen-Cahn equation (7) arises in the study of interfaces, in the theory of superconductors and super-fluids, or in cosmology (see [24]). Hence according to Theorem 2, every nontrivial nonnegative weak solution u of (7) is stable.

Example 2. Consider the following nonlinear diffusion Fractional p-Laplacian system

$$\left.\begin{array}{rr} (-\Delta_p)^s u(x) = 1 - e^u & x \in \Omega, \\ u > 0 & x \in \Omega, \\ u = 0 & x \in R^n/\Omega. \end{array}\right\} \tag{8}$$

Hence according to Theorem 2, every nontrivial nonnegative weak solution u of (8) is stable.

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