Open Journal of Mathematical Analysis

PSR Press

Article

A direct proof of stability of nonnegative weak solution for fractional p-Laplacian problem with concave nonlinearity

Salah A. Khafagy[1,*]

[1] Department of Mathematics, Faculty of Science, Al-Azhar University,Nasr City (11884), Cairo, Egypt

* Correspondence: salahabdelnaby.211@azhar.edu.eg

Received: 20 April 2024; Accepted: 8 May 2024; Published: 30 June 2024.

Abstract: The present paper provides a direct proof of stability of nontrivial nonnegative weak solution for fractional p-Laplacian problem under concave nonlinearity condition. The main results of this work are extend the previously known results for the fractional Laplacian problem.

Keywords: Stability; weak solution; fractional p-Laplacian.

MSC: 35B35,35D30,35R11.

1. Introduction

In this paper, we establish a direct proof of stability of nontrivial nonnegative weak solution for the following fractional p-Laplacian problem

$$\left.\begin{array}{cc} (-\Delta)_p^s u(x) = f(u) & x \in \Omega, \\ u > 0 & x \in \Omega, \\ u = 0 & x \in R^n/\Omega, \end{array}\right\} \tag{1}$$

where $\Omega \subset \mathbb{R}^n, n \geq 1$, is a bounded domain, $s \in (0,1), 1 < p < \infty$ and $(-\Delta_p)^s$ is the nonlocal nonlinear fractional p-Laplacian operator of order s defined by

$$(-\Delta)_p^s u(x) = C_{n,s,p}\ P.V. \int_{R^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}} dy, \tag{2}$$

where $C_{n,s,p}$ is a constant depending on n, s, p and P.V. denotes the commonly used abbreviations for "the main value sense".

Under a certian concave nonlinearity condition on the function f, we establish that every nontrivial nonnegative weak solution of problem (1) is stable. The aim of this paper is to extend the results in [1] to the fractional p-Laplacian case.

The standard fractional Sobolev space is the natural space where one looks for a solution to the problem (1). We refer the reader to [2,3] for further results and references. Let Ω be an open set in $\mathbb{R}^n$, $s \in (0,1)$, and $p \in [0,\infty)$. We define the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ as follows

$$W^{s,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) : \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy < \infty\}, \tag{3}$$

equipped with the norm

$$\|u\|_{W^{s,p}\mathbb{R}^n)} = [\int_{\mathbb{R}^n} |u|^p dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy]^{1/p}. \tag{4}$$

Let

$$W_0^{s,p}\mathbb{R}^n) = \{u \in W^{s,p}(\mathbb{R}^n) : u = 0 \text{ a.e. } u \in \mathbb{R}^n \backslash \Omega\}, \tag{5}$$

be a closed linear subspace of $W^{s,p}\mathbb{R}^n)$, and its norm is given by

$$\|u\|_{W_0^{s,p}(\mathbb{R}^n)} = [\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}dxdy]^{1/p}, \tag{6}$$

which is equivalent to the norm given by (4). The fractional space $W_0^{s,p}(\mathbb{R}^n)$ is a suitable candidate to find "weak solutions" to the problem (1).

In the past few years, a great attention has been given to the subjects involving fractional p-Laplacian and nonlocal operators since it is not only a natural extension of the p-Laplacian equations but also presents many new phenomena described by nonlinear integral structures such as phase transition, population dynamics, optimization, finance, water waves, continuum mechanics, minimal surfaces, game theory and other sciences.

Numerous physical phenomena, including flow through porous media, reaction-diffusion issues, petroleum extraction, non-Newtonian fluids, etc., lead to the so-called Fractional p-Laplacian boundary value problems. In light of this, a lot of mathematicians have been interested in studying such problems and their generalizations recently.

Due to the many possible uses in reaction-diffusion problems, fluid mechanics, Newtonian fluids, glaciology, population dynamics, etc. see [4] and references therein, an extensive number of authors take an interest in the study of the stability and instability of nonnegative solutions of linear [5] , semilinear (see [6–9]), semiposiotne (see [10–12]), nonlinear (see [13–17]) and fractional [18,19] systems. Also, the existence of positive weak solutions for the prevuoius sytems have been studied (see [20–22]).

Most of the previous proofs of stability of nontrivial nonnegative weak solution for linear and nonlinear systems are computational and relatively long compared to the proof presented in this paper. Once the stability of of nontrivial nonnegative weak solution for problem (1) is established, we arrive at the main point of this article.

Definition 1. A bounded solution $u \in W_0^{s,p}(\mathbb{R}^n)$ of (1) is stable if

$$\int_{R^n} \left|(-\Delta_p)^{\frac{s}{2}}\varphi\right|^2 dx \geq \int_{\Omega} f'(u)\varphi^2 dx \quad \text{for all } \varphi \in W_0^{s,p}(\mathbb{R}^n).$$

It makes sense that a system is stable if it has the ability to recover from perturbations; a small change will not prevent the system from returning to equilibrium.

Lemma 2. *[23] For every $a, b \geq 0$ and $c, d > 0$, we have $(a-b)^2 \geq (c-d)[\frac{a^2}{c} - \frac{b^2}{d}]$. The equality holds if and only if $(a,b) = k(c,d)$ for some constant k.*

Proof. The result follows from the convexity of the mapping $t \to t^2$ (see [23]). □

Finally, let us explain the plan of the paper. In section 2, we introduce a direct proof to establish that every nontrivial nonnegative solution of problem (1) is stable under certian conditions. We introduce some applications in section 3.

2. Main Results

In this section, by a simple short method, we introduce a direct proof of stability of nontrivial nonnegative weak solution for problem (1) under a concave nonlinearity condition. Now, we state the main theorem in this paper.

Theorem 3. *If $f''(u) < 0$ and $f(0) \geq 0$, then every positive weak solution of (1) is stable.*

Proof. Let u be any nontrivial nonnegative weak solution of (1. Also, let $l(u) = uf'(u) - f(u)\ \forall u \in \mathbb{R}^+$. Then $l(0) \leq 0$ and $l'(u) < 0$, and so $l(u) < 0$. Testing the equation (1) with v^2/u gives

$$\begin{aligned}
0 &= \int_{R^n} (-\Delta_p)^{\frac{s}{2}} u (-\Delta_p)^{\frac{s}{2}} (\frac{v^2}{u}) dx - \int_{\Omega} f(u)(\frac{v^2}{u}) dx \\
&= \frac{C_{n,s,p}}{2} \int_{R^n} \int_{R^n} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) \left| \frac{v^2(x)}{u(x)} - \frac{v^2(y)}{u(y)} \right|^{p-2} [\frac{v^2(x)}{u(x)} - \frac{v^2(y)}{u(y)}]}{|x - y|^{n+sp}} dxdy \\
&\quad - \int_{\Omega} f(u)(\frac{v^2}{u}) dx \\
&\leq \frac{C_{n,s,p}}{2} \int_{R^n} \int_{R^n} \frac{[|v(x) - v(y)|^{p-2} (v(x) - v(y)]^2}{|x - y|^{n+sp}} dxdy - \int_{\Omega} f'(u) v^2 dx \\
&= \int_{R^n} \left| (-\Delta_p)^{\frac{s}{2}} v \right|^2 dx - \int_{R^n} f'(u) v^2 dx.
\end{aligned}$$

Then u is stable and the result follows. □

Remark 1. This result generalises the main result in [1], where the case $p = 2$ have been studied by the author.

3. Applications

Here, we introduce the following two examples to highlight the main results.

Example 1. Consider the following fractional version of the Allen-Cahn equation:

$$\left.\begin{array}{ll} (-\Delta_p)^s u(x) = u - u^3 & x \in \Omega, \\ u > 0 & x \in \Omega, \\ u = 0 & x \in R^n / \Omega. \end{array}\right\} \tag{7}$$

The fractional Allen-Cahn equation (7) arises in the study of interfaces, in the theory of superconductors and super-fluids, or in cosmology (see [24]). Hence according to Theorem 2, every nontrivial nonnegative weak solution u of (7) is stable.

Example 2. Consider the following nonlinear diffusion Fractional p-Laplacian system

$$\left.\begin{array}{ll} (-\Delta_p)^s u(x) = 1 - e^u & x \in \Omega, \\ u > 0 & x \in \Omega, \\ u = 0 & x \in R^n / \Omega. \end{array}\right\} \tag{8}$$

Hence according to Theorem 2, every nontrivial nonnegative weak solution u of (8) is stable.

Acknowledgments: We would like to express our sincere gratitude to the reviewers and the editor for their good cooperations on our manuscript. Also, the authors wish to thank Professor H. M. Serag (Mathematics Department, Faculty of Science, AL-Azhar University) for his unwavering support during the process of this work.

Conflicts of Interest: "The author declares no conflict of interest."

References

[1] Rasouli, S. (2021). A Note On The Stability Of Nonnegative Solutions To Classes Of Fractional Laplacian Problems With Concave Nonlinearity. *Applied Mathematics E-Notes*, **21**, 194-197.

[2] Demengel, F. & Demengel, G. (2012). Functional spaces for the theory of elliptic partial differential equations, Universitext, Springer, London, Translated from the 2007 French original by Reinie Erne.

[3] Di Nezza, E., Palatucci, G. & Valdinoci, E. (2012). Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **136**(5), 521–573.

[4] Atkinson, C. & Ali, K. (1992). Some boundary value problems for the Bingham model. *J. Non-Newton. Fluid Mech.*, **41**, 339-363.

[5] Afrouzi G. & Sadeeghi, Z. (2008). Stability results for a class of elliptic problems. *International Journal of Nonlinear Science*, **6**(2), 114–117.

[6] Karatson L. & Simon, P. (2001). On the stability properties of nonnegative solutions of semilinear problems with convex or concave nonlinearity. *Journal of Computational and Applied Mathematics* **131**, 497-501.

[7] Korman, P. & Shi, J. (2000). Instability and exact multiplicity of solutions of semilinear equations. *Electron J Diff Eqns, Conf.* **05**, 311-322.

[8] Maya, C. & Shivaji, R. (1998). Instability of nonnegative solutions for a class of semilinear elliptic boundary value problems. *J. Comput. App.l Math*, **88**, 125-128.

[9] Voros, I. (2004). Stability properties of nonnegative solutions of semilinear symmetric cooperative systems. *Electronic Jour. Diff. Eqns.*, **105**, 1-6.

[10] Ali, I., Castro, A. & Shivaji, R. (1993). Uniqueness and stability of nonnegative solutions for semipositone problems in a ball. *Proc. Amer. Math. Soc.*, **117**(3), 775–782.

[11] Brown, K. & Shivaji, R. (1991). Instability of nonnegative solutions for a class of semipositone problems. *Proc. Amer. Math. Soc.*, **112**, 121-124.

[12] Tertikas, A. (1992). Stability and instability of positive solutions of semilinear problems. *Proc. Amer. Math. Soc.*, **114**, 1035-1040.

[13] Afrouzi, G. & Rasouli, S. (2006). Stability properties of nonnegative solutions to a non-autonomous p-Laplacian equation. *Chaos. Solitons and fractals*, **29**, 1095-1099.

[14] Khafagy, S. (2015). On the stabiblity of positive weak solution for weighted p-Laplacian nonlinear system. *New Zealand Journal of Mathematics*, **45**, 39-43.

[15] Khafagy, S. & Serag, H. (2020). On the existence of positive weak solution for nonlinear system with singular weights. *Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences)*, 55(4), 259–267.

[16] Khafagy, S. & Serag, H. (2022). Stability of positive weak solution for Generalized weighted p-Fisher Kolmogoroff Nonlinear Stationary-Stat Problem. *Eur. J. Math. Anal.*, **2**, 1-8.

[17] Khafagy, S. & Serag, H. (2018) . Stability results of positive weak solution for singular p-Laplacian nonlinear system. *J. Appl. Math. & Informatics* **36** (3-4), 173-179.

[18] Khafagy, S. & Ezzat E. (2024). Existence and stability of positive weak solutions for a class of chemically reacting systems. *Eur. J. Math. Appl.*, 4(2), 1-12.

[19] Khafagy, S., Rasouli, S. & Serag, H. (2023). Existence results for fractional fisher-kolmogoroff steady state prob- lem. *Eur. J. Math. Appl.*, 3(20), 1–7.

[20] Khafagy, S. & Sadeghi, Z., 2024. Existence of positive weak solution for a weighted system of autocatalytic reaction steady state type. Eur. J. Math. Appl., 4(11), 1-7.

[21] Khafagy, S. & Serag, H. (2020). On the stability of positive weak solution for (p; q)-Laplacian nonlinear system. *Applied Mathematics E-Notes*, **20**, 108-114.

[22] Khafagy, S., El-Zahrani, E. & Serag, H. (2022). Existence and uniqueness of weak solution for nonlinear weighted (p, q)-laplacian system with application on an optimal control problem. *Jordan Journal of Mathematics and Statistics (JJMS)*, 15(4A), 983–998.

[23] Amghibech, S. (2008). On the discrete version of Picone's identity. Discrete Appl. Math. 156(1), 1-10.

[24] Allen, S.M. & Cahn, J.W. (1979). A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening, *Acta Metall*, **27**, 1085-1095.

© 2024 by the authors; licensee PSRP, Lahore, Pakistan. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC-BY) license (http://creativecommons.org/licenses/by/4.0/).