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Coincidence point results for relational-theoretic contraction mappings in metric spaces with applications

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Abstract: In this article, we extend the classic Banach contraction principle to a complete metric space equipped with a binary relation. We accomplish this by generalizing several key notions from metric fixed point theory, such as completeness, closedness, continuity, g-continuity, and compatibility, to the relation-theoretic setting. We then use these generalized concepts to prove results on the existence and uniqueness of coincidence points, defined by two mappings acting on a metric space with a binary relation. As a consequence of our main results, we obtain several established metrical coincidence point theorems. We further provide illustrative examples that demonstrate the main results.

Keywords: Coincidence point; binary relations; R-completeness; R-continuity; R-connected sets; d-self-closedness.

MSC: 47H10; 54H25.

1. Introduction

In 1922, Banach [1] formulated the Banach contraction principle, a foundational result that is now widely used to prove the existence and uniqueness of solutions for a variety of mathematical and physical problems. Since its introduction, the Banach contraction mapping principle has been generalized and refined in numerous ways, leading to a wealth of articles dedicated to its improvement [2–22].

In 2001, Turinici [23] introduced the concept of order-theoretic fixed point results, which provided a novel approach to proving fixed point theorems. Later, in 2004, Ran and Reurings [24] developed an order-theoretic version of the Banach contraction principle and demonstrated its application to matrix equations. Samet and Turinici [25] further advanced this line of research by establishing fixed point results for nonlinear contractions based on the symmetric closure of an amorphous binary relation.

Alam and Imdad [26] developed a relation-theoretic fixed point theorem that extends the Banach contraction principle to arbitrary binary relations, thereby unifying various order-theoretic results that have been previously established.

In [27], Alam and Imdad introduced a novel variant of the Banach contraction principle for complete metric spaces endowed with a binary relation. Unlike previous fixed point theorems, their approach does not require the standard metrical concepts of contraction, completeness, and continuity, instead relying on relation-theoretic analogues of these notions.

Sawangsup et al. [6] developed a new class of fixed point theorems called F-contraction theorems that improve on Wardowski's original result [28]. They introduced a new binary relation on the space that does not have to be transitive or a partial order, and showed that under this relation, the F-contraction mapping is well-defined and has a fixed point.

Motivated by the above discussion, we present some new existence and uniqueness results for coincidence points in metric spaces equipped with an arbitrary binary relation. To prove our results, we introduce the concepts of R-completeness, R-closedness, R-continuity, (g,R)-continuity, R-compatibility, and R-connectedness. As corollaries of our results, we obtain several well-known theorems from the literature on metric fixed point theory. Finally, we provide some examples that illustrate our main results and highlight their originality.

Throughout this paper, we let R denote a non-empty binary relation on a set, but for brevity we refer to R simply as a binary relation. Furthermore, we use $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$ to represent the sets of natural numbers, rational numbers, and real numbers, respectively, and denote the sets of natural numbers, rational numbers and real numbers wherein $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2. Preliminaries

We start this section by presenting some basic relevant definitions, propositions, lemmas and theorems.

Definition 1. [3] Let X be a nonempty set. A subset R of X^2 is called a binary relation on X.

Notice that for each pair $x, y \in X$, one of the following holds:

(i) $(x,y) \in R$; means that "x is R-related to y" or "x relates to y under R". Sometimes, we write xRy instead of $(x,y) \in R$.

(ii) $(x,y) \notin R$; means that "x is not R-related to y" or "x doesn't relate to y under R".

Trivially, X^2 and $\varnothing$ being subsets of X^2 are binary relations on X, which are respectively called the universal relation (or full relation) and empty relation.

Definition 2. [26] Let R be a binary relation on a nonempty set X and $x, y \in X$. We say that x and y are R-comparative if either $(x,y) \in R$ or $(y,x) \in R$. We denote it by $[x,y] \in R$.

Proposition 1. *[27] If (X,d) is a metric space, R is a binary relation on X, f and g are two self-mappings on X and $\lambda \in [\,0,1)$, then the following contractive conditions are equivalent:*

(a) $d(fx,fy) \leq \lambda d(gx,gy)$, $\forall x,y \in X$ with $(gx,gy) \in R$,

(b) $d(fx,fy) \leq \lambda d(gx,gy)$, $\forall x,y \in X$ with $[gx,gy] \in R$.

Proof. The implication (b)⇒(a) is trivial. On the other hand, suppose that (a) holds. Take $x, y \in X$ with $[gx,gy] \in R$. If $(gx,gy) \in R$, then (b) is directly follow from (a). Otherwise, in case $(gy,gx) \in R$, using symmetry of d and (a), we obtain

$$d(fx,fy) = d(fy,fx) \leq \lambda d(gy,gx) = \lambda d(gx,gy) \tag{1}$$

Implies that $(a) \implies (b)$. $\square$

Proposition 2. *If (X,d) is a metric space, R is a binary relation on X, f and g are two self-mappings on X and $\lambda \in [\,0,1)$, then the following contractivity conditions are equivalent:*

(a) $d(fx,fy) \leq \lambda\,(M(gx,gy))$, $\forall x,y \in X$ with $(gx,gy) \in R$,

where

$$M(gx,gy) = max\left\{\begin{array}{c} d(gx,gy), \frac{d(gx,\, fx)d(gy,\, fy)}{d(gx,gy)}, \frac{d(gx,\, fx)d(gy,\, fy)}{d(gx,gy)+d(gx,\, fy)+d(gy,\, fx)} \\ \frac{d(gx,fx)d(gx,fy)+d(gy,fx)d(gy,fy)}{d(gy,fx)+d(gx,fy)} \end{array}\right\} \tag{2}$$

(b) $d(fx,fy) \leq \lambda\,(M(gx,gy))$, $\forall x,y \in X$ with $[gx,gy] \in R$,

where

$$M(gx,gy) = max\left\{ \begin{array}{c} d(gx,gy), \frac{d(gx,fx)d(gy,fy)}{d(gx,gy)}, \frac{d(gx,fx)d(gy,fy)}{d(gx,gy)+d(gx,fy)+d(gy,fx)} \\ \frac{d(gx,fx)d(gx,fy)+d(gy,fx)d(gy,fy)}{d(gy,fx)+d(gx,fy)} \end{array} \right\} \tag{3}$$

Proof. The implications follow from the proof of Proposition 3. □

Definition 3. [4] A binary relation R on a nonempty set X is called reflexive if $(x,x) \in R\ \forall x \in X$, symmetric if whenever $(x,y) \in R$ then $(y,x) \in R$, antisymmetric if whenever $(x,y) \in R$ and $(y,x) \in R$ then $x = y$, transitive if whenever $(x,y) \in R$ and $(y,z) \in R$ then $(x,z) \in R$,
(i) complete or connected or dichotomous if $[x,y] \in R\ \forall x,y \in X$,
(ii) weakly complete or weakly connected or trichotomous if $[x,y] \in R$ or $x = y\ \forall x,y \in X$.

Definition 4. [23,29,30] A binary relation R defined on a nonempty set X is called
(a) amorphous if R has no specific properties at all,
(b) strict order or sharp order if R is irreflexive and transitive,
(c) near-order if R is antisymmetric and transitive,
(d) pseudo-order if R is reflexive and antisymmetric,
(e) quasi-order or preorder if R is reflexive and transitive,
(f) partial order if R is reflexive, antisymmetric and transitive,
(g) simple order if R is weakly complete strict order,
(h) weak order if R is complete preorder,
(i) total order or linear order or chain if R is complete partial order,
(j) tolerance if R is reflexive and symmetric,
(k) equivalence if R is reflexive, symmetric and transitive.
Note that universal relation X^2 on a nonempty set X remains a complete equivalence relation.

Definition 5. [3] Let X be a nonempty set and R a binary relation on X.
(1) The inverse or transpose or dual relation of R, denoted by R^{-1} and is defined by

$$R^{-1} = \left\{(x,y) \in X^2 : (x,y) \in R\right\}. \tag{4}$$

(2) The symmetric closure of R, denoted by R^s, is defined to be the set $R \cup R^{-1}$ by

$$R^s = R \cup R^{-1}. \tag{5}$$

Indeed, R^s is the smallest symmetric relation on X containing R.

Proposition 3. *[26] For a binary relation R on a nonempty set X,*

$$(x,y) \in R^s \Leftrightarrow [x,y] \in R. \tag{6}$$

Definition 6. [26] Let X be a nonempty set, $E \subseteq X$ and R a binary relation on X. Then, the restriction of R to E, denoted by $R|_E$, is defined to be the set $R \cap E^2$ by

$$R|_E = R \cap E^2. \tag{7}$$

Indeed, $R|_E$ is a relation on E induced by R.

Definition 7. [31] Let X be a nonempty set and R a binary relation on X. A sequence $\{x_n\} \subset X$ is called R-preserving if

$$(x_n, x_{n+1}) \in R\ \forall\ n \in \mathbb{N}_0. \tag{8}$$

Definition 8. [26] Let X be a nonempty set and f a self-mapping on X. A binary relation R on X is called f-closed if for all $x, y \in X$,

$$(x,y) \in R \Rightarrow (fx, fy) \in R. \tag{9}$$

Definition 9. Let X be a nonempty set and f and g two self-mappings on X. A binary relation R on X is called (f,g)-closed if for all $x, y \in X$,

$$(gx, gy) \in R \Rightarrow (fx, fy) \in R. \tag{10}$$

Notice that under the restriction $g = I$, the identity mapping on X, Definition 2 reduces to Definition 8.

Proposition 4. *[27] Let X be a nonempty set, R a binary relation on X and f and g two self-mappings on X. If R is (f,g)-closed, then R^s is also (f,g)-closed.*

Definition 10. Let (X,d) be a metric space and R a binary relation on X. We say that (X,d) is R -complete if every R-preserving Cauchy sequence in X converges.

Remark 1. Every complete metric space is R-complete, for any binary relation R. Particularly, under the universal relation the notion of R-completeness coincides with usual completeness.

Definition 11. [31] Let (X,d) be a metric space and R a binary relation on X. A subset E of X is called R-closed if every R-preserving convergent sequence in E converges to a point of E.

Remark 2. Every closed subset of a metric space is R-closed, for any binary relation R. Particularly, under the universal relation the notion of R-closedness coincides with usual closedness.

Proposition 5. *An R-complete subspace of a metric space is R-closed.*

Proof. Let (X,d) be a metric space. Suppose that Y is an R-complete subspace of X. Take an R-preserving sequence $\{x_n\} \subset Y$ such that $x_n \xrightarrow{d} x \in X$. Since each convergent sequence is Cauchy, $\{x_n\}$ is an R-preserving Cauchy sequence in Y. Hence, R-completeness of Y implies that the limit of $\{x_n\}$ must lie in Y , that is, $x \in Y$. Therefore, Y is R-closed. □

Proposition 6. *An R-closed subspace of an R-complete metric space is R-complete.*

Proof. Let (X,d) be an R-complete metric space. Suppose that Y is R-closed subspace of X. Let $\{x_n\}$ be an R-preserving Cauchy sequence in Y. As X is R-complete, there exists $x \in X$ such that $x_n \xrightarrow{d} x$ and so $\{x_n\}$ is an R-preserving sequence converging to x. Hence, R-closeness of Y implies that $x \in Y$. Therefore, Y is R-complete. □

Definition 12. [27] Let (X,d) be a metric space, R a binary relation on X and $x \in X$. A mapping $f : X \to X$ is called R-continuous at x if for any R-preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, we have $f(x_n) \xrightarrow{d} f(x)$. f is called R-continuous if it is R-continuous at each point of X.

Note that every continuous mapping is R-continuous, for any binary relation R. Particularly, under the universal relation the notion of R-continuity coincides with usual continuity.

Definition 13. [27] Let (X,d) be a metric space, R a binary relation on X, g a self mapping on X and $x \in X$. A mapping $f : X \to X$ is called (g,R)-continuous at x if for any sequence $\{x_n\}$ such that $\{gx_n\}$ is R-preserving and $g(x_n) \xrightarrow{d} g(x)$, we have $f(x_n) \xrightarrow{d} f(x)$. Moreover, f is called (g,R)-continuous if it is (g,R)-continuous at each point of X.

Notice that under the restriction $g = I$, the identity mapping on X, Definition 13 reduces to Definition 12.

Remark 3. Every g-continuous mapping is (g, R)-continuous, for any binary relation R. Particularly, under the universal relation the notion of (g, R)-continuity coincides with usual g-continuity.

Definition 14. Let (X, d) be a metric space, R be a binary relation on X and f and g two self-mappings on X. We say that f and g are R-compatible if for any sequence $\{x_n\} \subset X$ such that $\{\mathrm{fx}_n\}$ and $\{gx_n\}$ are R-preserving and $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(x_n)$, we have

$$\lim_{n\to\infty} d(\mathrm{gf}x_n, fgx_n) = 0. \tag{11}$$

Remark 4. In a metric space (X, d) endowed with a binary relation R, commutativity $\Rightarrow$ weak commutativity $\Rightarrow$ compatibility $\Rightarrow$ R-compatibility $\Rightarrow$ weak compatibility.

Definition 15. [26] Let (X, d) be a metric space. A binary relation R on X is called d-self-closed if for any R-preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in R\ \forall\, k \in \mathbb{N}_0$.

Definition 16. [27] Let (X, d) be a metric space and g a self-mapping on X. A binary relation R on X is called (g, d)-self-closed if for any R-preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[gx_{n_k}, gx] \in R\ \forall\, k \in \mathbb{N}_0$.

Notice that under the restriction $g = I$, the identity mapping on X, Definition 16 reduces to Definition 15.

Definition 17. [2] Let X be a nonempty set and R a binary relation on X. A subset E of X is called R-directed if for each pair $x, y \in E$, there exists $z \in X$ such that $(x, z) \in R$ and $(y, z) \in R$.

Definition 18. [31] Let X be a nonempty set and R a binary relation on X. For $x, y \in X$, a path of length k (where k is a natural number) in R from x to y is a finite sequence $\{z_0, z_1, z_2, \ldots, z_k\} \subset X$ satisfying the following conditions:

(i) $z_0 = x$ and $z_k = y$,

(ii) $(z_i, z_{i+1}) \in R$ for each $i (0 \le i \le k-1)$.

Notice that a path of length k involves $k+1$ elements of X, although they are not necessarily distinct.

Definition 19. Let X be a nonempty set and R a binary relation on X. A subset E of X is called R -connected if for each pair $x, y \in E$, there exists a path in R from x to y.

Given a binary relation R and two self-mappings f and g defined on a nonempty set X, we use the following notations:

(i) $C(f, g) = \{x \in X : gx = fx\}$, i.e., the set of all coincidence points of f and g,

(ii) $\overline{C}(f, g) = \{\overline{x} \in X : \overline{x} = gx = fx, x \in X\}$, i.e., the set of all points of coincidence of f and g,

(iii) $X(f, R) = \{x \in X : (x, fx) \in R\}$,

(iv) $X(f, g, R) = \{x \in X : (gx, fx) \in R\}$.

Theorem 1. *[26] Let (X, d) be a complete metric space, R a binary relation on X and f a self-mapping on X. Suppose that the following conditions hold:*

(i) R is f-closed,

(ii) either f is continuous or R is d-self-closed,

(iii) $X(f, R)$ is nonempty,

(iv) there exists $\lambda \in [\,0, 1)$ such that

$$d(fx, fy) \le \lambda d(x, y),\ \forall x, y \in X \ \textit{with}\ (x, y) \in R \tag{12}$$

Then f has a fixed point. Moreover, if

(v) X is R^s -connected,

then f has a unique fixed point.

Lemma 1. *[5] Let X be a nonempty set and g a self-mapping on X. Then there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-one.*

Lemma 2. *[32] Let X be a nonempty set and f and g two self-mappings on X. If f and g are weakly compatible, then every point of coincidence of f and g is also a coincidence point of f and g.*

Definition 20. [6,33] Let X be a nonempty set and f and g two self-mappings on X. Then

(i) an element $x \in X$ is called a coincidence point of f and g if

$$g(x) = f(x), \tag{13}$$

(ii) if $x \in X$ is a coincidence point of f and g and $x \in X$ such that $x = g(x) = f(x)$, then x is called a point of coincidence of f and g,

(iii) if $x \in X$ is a coincidence point of f and g such that $x = g(x) = f(x)$, then x is called a common fixed point of f and g,

(iv) f and g are called commuting if

$$g\,(\mathrm{fx}) = f\,(\mathrm{gx})\ \forall x, y \in X \tag{14}$$

(v) f and g are called weakly compatible (or partially commuting or coincidentally commuting) if f and g commute at their coincidence points, i.e., for any $x \in X$,

$$gx = fx \Rightarrow \mathrm{gf}(x) = fg(x). \tag{15}$$

Definition 21. [34,35] Let (X, d) be a metric space and f and g two self-mappings on X. Then

(i) f and g are called weakly commuting if for all $x \in X$,

$$d(gfx, fgx) \leq d(gx, fx) \tag{16}$$

(ii) f and g are called compatible if $\lim_{n\to\infty} d\,(\mathrm{gf}x_n, fgx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} g\,(x_n) = \lim_{n\to\infty} f\,(x_n) \tag{17}$$

(iii) f is called g-continuous at some $x \in X$ if for all sequences $\{x_n\} \subset X$,

$$g\,(x_n) \xrightarrow{d} g(x) \Rightarrow f\,(x_n) \xrightarrow{d} f(x). \tag{18}$$

Moreover, f is called g-continuous if it is g-continuous at each point of X.

3. Main Results

Theorem 2. *Let X and Y be nonempty set equipped with a binary relation R and a metric d such that (X, d) is an R-complete subspace of X. Let $f, g : X \to X$ be two self-mappings. Suppose that the following conditions hold:*

(a) $f(X) \subseteq g(X) \cap Y$,

(b) R is (f, g)-closed,

(c) $X(f, g, R)$ is nonempty,

(d) there exists $\lambda \in [\,0, 1)$ such that

$d(fx, fy) \leq \lambda\,(M(gx, gy))\,,\ \forall x, y \in X$ with $(gx, gy) \in R$,

where

$$M(gx, gy) = max\left\{ \begin{array}{c} d(gx, gy), \frac{d(gx,\, fx)d(gy,\, fy)}{d(gx,gy)}, \frac{d(gx,\, fx)d(gy,\, fy)}{d(gx,gy)+d(gx,\, fy)+d(gy,\, fx)} \\ \frac{d(gx,fx)d(gx,fy)+d(gy,fx)d(gy,fy)}{d(gy,fx)+d(gx,fy)} \end{array} \right\} \tag{19}$$

Moreover, if

(e) f and g are R-compatible,

(f) g is R-continuous,
(g) either f is R-continuous or R is (f,d)-self-closed,
Then f and g have a coincidence point.

Proof. Observe that hypothesis (a) is equivalent to saying that $f(X) \subseteq g(X)$ and $f(X) \subseteq Y$. In light of assumption (c), we choose x_0 be an arbitrary element of $X(f,g,R)$, then $(gx_0, fx_0) \in R$. If $g(x_0) = f(x_0)$, then, the proof is complete. Otherwise, if $g(x_0) \neq f(x_0)$, then from $f(X) \subseteq g(X)$, we can choose $x_1 \in X$ such that $g(x_1) = f(x_0)$.

Again from $f(X) \subseteq g(X)$, we can choose $x_2 \in X$ such that $g(x_2) = f(x_1)$. Continuing this process, we construct a sequence $\{x_n\} \subset X$ such that

$$g(x_{n+1}) = f(x_n) \quad \text{for all} \quad n \in \mathbb{N}_0. \tag{20}$$

We now claim that $\{\mathrm{gx}_n\}$ is R-preserving sequence, i.e.,

$$(gx_n, gx_{n+1}) \in R \quad \text{for all} \quad n \in \mathbb{N}_0. \tag{21}$$

To show this, by induction. For $n = 0$ in (19) and the fact that $x_0 \in X(f,g,R)$, we have
$(gx_0, gx_1) \in R$
Which show that (19) holds.
Continuing this process for $n = 1$ and the fact that $x_1 \in X(f,g,R)$, we have

$$(gx_1, gx_2) \in R$$

Suppose that (4) holds for $n = k > 1$, i.e.,

$$(gx_k, gx_{k+1}) \in R. \tag{22}$$

Since, R is (f,g)-closed, we get

$$(fx_k, fx_{k+1}) \in R. \tag{23}$$

Also, for $n = k+1$, we get that

$$(gx_{k+1}, gx_{k+2}) \in R. \tag{24}$$

Hence, (4) holds by induction, for all $n \in \mathbb{N}_0$.
Similarly, the sequence $\{\mathrm{fx}_n\}$ is also R-preserving sequence, i.e.,

$$(fx_n, fx_{n+1}) \in R forall n \in \mathbb{N}_0. \tag{25}$$

Using (3),(4) and assumption (d), we obtain
$d(gx_n, gx_{n+1}) = d(fx_{n-1}, fx_n) \leq \lambda(M(gx_{n-1}, gx_n)), \forall x, y \in X$
where
$M(gx_{n-1}, gx_n) =$

$$\max\left\{\begin{array}{c} d(gx_{n-1}, gx_n), \frac{d(gx_{n-1}, fx_{n-1})d(gx_n, fx_n)}{d(gx_{n-1}, gx_n)}, \frac{d(gx_{n-1}, fx_{n-1})d(gx_n, fx_n)}{d(gx_{n-1}, gx_n)+d(gx_{n-1}, fx_n)+d(gx_n, fx_{n-1})}, \\ \frac{d(gx_{n-1}, fx_{n-1})d(gx_{n-1}, fx_n)+d(gx_n, fx_{n-1})d(gx_n, fx_n)}{d(gx_n, fx_{n-1})+d(gx_{n-1}, fx_n)} \end{array}\right\} \tag{26}$$

$= max\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1}), d(gx_{n-1}, gx_n), d(gx_{n-1}, gx_n)\}$

$$M(gx_{n-1}, gx_n) \leq max\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1})\} \tag{27}$$

If for some $n \geq 1$, we have $d(gx_{n-1}, gx_n) \leq d(gx_n, gx_{n+1})$.
From (8), we get

$$d\left(gx_n, gx_{n+1}\right) \leq \lambda d\left(gx_{n-1}, gx_n\right) < d\left(gx_n, gx_{n+1}\right) \tag{28}$$

a contradiction. Thus, for all $n \geq 1$, we have

$$\max\left\{d\left(gx_{n-1}, gx_n\right), d\left(gx_n, gx_{n+1}\right)\right\} = d\left(gx_{n-1}, gx_n\right) \tag{29}$$

Using (27) and (29), we get

$$d\left(gx_n, gx_{n+1}\right) \leq \lambda d\left(gx_{n-1}, gx_n\right) foralln \geq 1 \tag{30}$$

By induction, we have

$$d\left(gx_n, gx_{n+1}\right) \leq \lambda d\left(gx_{n-1}, gx_n\right) \leq \lambda^2 d\left(gx_{n-2}, gx_{n-1}\right) \leq \ldots \leq \lambda^n d\left(gx_0, gx_1\right) \tag{31}$$

for all $n \in \mathbb{N}$. So that

$$d\left(gx_n, gx_{n+1}\right) \leq \lambda^n d\left(gx_0, gx_1\right), \ \forall n \in \mathbb{N}. \tag{32}$$

For $n < m$ and (32), we obtain

$$d\left(gx_n, gx_m\right) \leq d\left(gx_n, gx_{n+1}\right) + d\left(gx_{n+1}, gx_{n+2}\right) + \ldots + d\left(gx_{m-1}, gx_m\right)$$

$\leq \left(\lambda^n + \lambda^{n+1} + \ldots + \lambda^{m-1}\right) d\left(gx_0, gx_1\right)$
$= \frac{\lambda^n - \lambda^m}{1-\lambda} d\left(gx_0, gx_1\right)$

$$\leq \frac{\lambda^n}{1-\lambda} d\left(gx_0, gx_1\right) \to 0, \tag{33}$$

as $m, n \to \infty$. Thus, $\{gx_n\}$ is a Cauchy sequence.

Since by (20), we have $\{gx_n\} \subset f(X) \subseteq Y$ so that $\{gx_n\}$ is R-preserving Cauchy sequence in Y. As Y is R-complete, there exists $u \in Y$ such that

$$\lim_{n\to\infty} g\left(x_n\right) = u. \tag{34}$$

Using (20) and (34), we get

$$\lim_{n\to\infty} f\left(x_n\right) = u. \tag{35}$$

Now, assume that (e), (f) and (h) hold. Using (22), (34) and assumption (f) (R-continuity of g), we get

$$\lim_{n\to\infty} g\left(gx_n\right) = g\left(\lim_{n\to\infty} gx_n\right) = g(u). \tag{36}$$

Again, using (25), (35) and assumption (f) (R-continuity of g), we have

$$\lim_{n\to\infty} g\left(fx_n\right) = g\left(\lim_{n\to\infty} fx_n\right) = g(u). \tag{37}$$

Since $\{fx_n\}$ and $\{gx_n\}$ are R-preserving from (21) and (25) and also, $\lim_{n\to\infty} g\left(x_n\right) = \lim_{n\to\infty} f\left(x_n\right) = u$ from (34) and (35), on using assumption (e) (R-compatibility of f and g), we get

$$\lim_{n\to\infty} d\left(gfx_n, fgx_n\right) = 0 \tag{38}$$

We now show that u is a coincidence point of f and g. By assumption (e), (f) and (g). We have the two cases:

Case 1 Suppose that f is R-continuous. Using (21), (34) and R-continuity of f, we get

$$\lim_{n\to\infty} f\left(gx_n\right) = f\left(\lim_{n\to\infty} gx_n\right) = f(u). \tag{39}$$

By (37), (38), (39) and continuity of d, we get

$$d(gu, fu) = d\left(\lim_{n\to\infty} gfx_n, \lim_{n\to\infty} fgx_n\right)$$

$$= \lim_{n\to\infty} d\left(\mathrm{gfx}_n, \mathrm{fgx}_n\right) = 0. \tag{40}$$

Hence, $d(gu, fu) = 0$, that is $gu = fu$. Thus, f and g have a coincidence point.

Case 2 Suppose that R is (f, d)-self-closed. Since $\{\mathrm{gx}_n\}$ is R-preserving from (21) and $g(x_n) \xrightarrow{d} u$ from (3.16), by using (g, d)-self-closedness of R, there exists a subsequence $\{gx_{n_r}\}$ of $\{\mathrm{gx}_n\}$ such that

$$[\mathrm{gg}x_{n_r}, gu] \in R \tag{41}$$

for all $r \in \mathbb{N}_0$.

Since $g(x_{n_r}) \xrightarrow{d} u$, so equations (34)-(38) hold for also $\{x_{n_r}\}$ instead of $\{x_n\}$. Using (41), assumption (d) and Proposition 2, we get

$$d\left(\mathrm{fg}x_{n_r}, fu\right) \leq \lambda d\left(\mathrm{gg}x_{n_r}, gu\right) \tag{42}$$

for all $r \in \mathbb{N}_0$.

By triangular inequality, (36),(37), (38) and (42), we get

$$\begin{aligned} d(gu, fu) &\leq d\left(gu, gfx_{n_r}\right) + d\left(\mathrm{gf}x_{n_r}, gfx_{n_r}\right) + d\left(\mathrm{fg}x_{n_r}, fu\right) \\ &\leq d\left(gu, gfx_{n_r}\right) + d\left(\mathrm{gf}x_{n_r}, gfx_{n_r}\right) + d\left(\mathrm{fg}x_{n_r}, fu\right) \to 0 \end{aligned} \tag{43}$$

as $r \to \infty$ so that $gu = fu$.

Thus, $u \in X$ is a coincidence point of f and g. □

Theorem 3. *Let X and Y be nonempty set equipped with a binary relation R and a metric d such that (X, d) is an R-complete subspace of X. Let $f, g : X \to X$ be two self-mappings. Suppose that the following conditions hold:*

(a) $f(X) \subseteq g(X) \cap Y$,

(b) R is (f, g)-closed,

(c) $X(f, g, R)$ is nonempty,

(d) there exists $\lambda \in [0, 1)$ such that

$d(fx, fy) \leq \lambda\left(M(gx, gy)\right), \forall x, y \in X$ with $(gx, gy) \in R$,

where

$$M(gx, gy) = max\left\{ \begin{array}{c} d(gx, gy), \frac{d(gx,\, fx)d(gy,\, fy)}{d(gx,gy)}, \frac{d(gx,\, fx)d(gy,\, fy)}{d(gx,gy)+d(gx,\, fy)+d(gy,\, fx)}, \\ \frac{d(gx,fx)d(gx,fy)+d(gy,fx)d(gy,fy)}{d(gy,fx)+d(gx,fy)} \end{array} \right\}$$

Moreover, if

(e) f and g are R-compatible,

(f) g is R-continuous,

(g) either f is R-continuous or R is (f, d)-self-closed,

Also, assume that

(h) $Y \subseteq g(X)$,

(i) either f is (g, R)-continuous or f and g are continuous or $R|_Y$ is d-self closed. Then f and g have a coincidence point.

Proof. The other proof follows from the proof of Theorem 2

Also, assume that (h) and (i) hold. Owing to assumption (h) ($Y \subseteq g(X)$), we can find some $z \in X$ such that $u = g(z)$. With (34) and (35), we have

$$\lim_{n\to\infty} g\left(x_n\right) = g(z). \tag{44}$$

$$\lim_{n\to\infty} f\left(x_n\right) = g(z). \tag{45}$$

We can now show that z is a coincidence point of f and g. By assumption (i), we have the following three cases:

Case 1 Suppose that f is (g, R)-continuous, then using (21) and 44, we get

$$\lim_{n\to\infty} f(x_n) = f(z). \tag{46}$$

Consider, (45) and 46, we get
$g(z) = f(z)$.
Hence, f and g have a coincidence point.

Case 2 Suppose that f and g are continuous. Owing to Lemma 1, there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-one. Define $T : g(E) \to g(X)$ by

$$T(\mathrm{ga}) = f(a), \quad \forall g(a) \in g(E) \tag{47}$$

where $a \in E$.

Since $g : E \to X$ is one-one and $f(X) \subseteq g(X)$, T is well defined. Again since f and g are continuous, it follows that T is continuous. Using the fact $g(X) = g(E)$, assumptions (a) and (h) reduce to respectively $f(X) \subseteq g(X) \cap Y$ and $Y \subseteq g(E)$, which follows that, without loss of generality, we are able to construct $\{x_n\}_{n=1}^{\infty} \subset E$ satisfying (20) and to choose $z \in E$.

Consider, (44), (45), (47) and continuity of T, we get

$$f(z) = T(\mathrm{gz}) = T\left(\lim_{n\to\infty} gx_n\right) = \lim_{n\to\infty} T(gx_n) = \lim_{n\to\infty} f(x_n) = g(z). \tag{48}$$

Implies, $f(z) = g(z)$.
Thus, $z \in E$ is a coincidence point of f and g.

Case 3 Suppose that $R|_Y$ is d-self-closed. Since $\{\mathrm{gx}_n\}$ is $R|_Y$-preserving from (21) and $g(x_n) \xrightarrow{d} g(z) \in Y$ from (44), using d-self-closedness of $R|_Y$, there exists a subsequence $\{gx_{n_r}\}$ of $\{\mathrm{gx}_n\}$ such that

$$[gx_{n_r}, gz] \in R|_Y \tag{49}$$

for all $r \in \mathbb{N}_0$.

On using 44, 49, assumption (d) and Proposition 2, we get

$$d(fx_{n_r}, fz) \leq \lambda(M(gx_{n_r}, gz))$$

where

$$M(gx_{n_r}, gz) = \max\left\{\begin{array}{l} d(gx_{n_r}, gz), \dfrac{d(gx_{n_r}, fx_{n_r})\, d(gz, fz)}{d(gx_{n_r}, gz)}, \\ \dfrac{d(gx_{n_r}, fx_{n_r})\, d(gz, fz)}{d(gx_{n_r}, gz) + d(gx_{n_r}, fz) + d(gz, fx_{n_r})}, \\ \dfrac{d(gx_{n_r}, fx_{n_r})\, d(gx_{n_r}, fz) + d(gz, fx_{n_r})\, d(gz, fz)}{d(gz, fx_{n_r}) + d(gx_{n_r}, fz)} \end{array}\right\} \tag{50}$$

$$d(fx_{n_r}, fz) \leq \lambda M(gx_{n_r}, gz) \longrightarrow 0 \tag{51}$$

as $n \longrightarrow \infty$.
So that

$$\lim_{n\to\infty} f(x_{n_r}) = f(z). \tag{52}$$

Using 45 and 52, we get $g(z) = f(z)$.
Thus, f and g have a coincidence point.

Now, as a consequence, we particularize Theorem 2 by assuming the R-completeness of whole space X. □

Corollary 4. *Let X be a nonempty set equipped with a binary relation R and a metric d such that (X, d) is an R-complete metric space. Let $f, g : X \to X$ be two self-mappings. Suppose that the following conditions hold:*

(a) $f(X) \subseteq g(X)$,
(b) R is (f,g)-closed,
(c) $X(f,g,R)$ is nonempty,
(d) there exists $\lambda \in [\,0,1)$ such that
$d(fx,fy) \leq \lambda\,(M(gx,gy))\,,\ \forall x,y \in X$ *with* $(gx,gy) \in R$,
where

$$M(gx,gy) = max\left\{\begin{array}{l} d(gx,gy), \frac{d(gx,\,fx)d(gy,\,fy)}{d(gx,gy)}, \frac{d(gx,\,fx)d(gy,\,fy)}{d(gx,gy)+d(gx,\,fy)+d(gy,\,fx)}, \\ \frac{d(gx,fx)d(gx,fy)+d(gy,fx)d(gy,fy)}{d(gy,fx)+d(gx,fy)} \end{array}\right\} \tag{53}$$

Moreover, if
(e) f and g are R-compatible,
(f) g is R-continuous,
(g) either f is R-continuous or R is (f,d)-self-closed,
Then f and g have a coincidence point.

Proof. The result proof follows easily on setting $Y = X$ in Theorem 2. □

Remark 5. If g is onto in Corollary 3.3, then we can drop assumption (a) as in this case it trivially holds.

Corollary 5. *Let X be a nonempty set equipped with a binary relation R and a metric d such that (X,d) is an R-complete metric space. Let $f,g : X \to X$ be two self-mappings. Suppose that the following conditions hold:*
(a) $f(X) \subseteq g(X)$,
(b) R is (f,g)-closed,
(c) $X(f,g,R)$ is nonempty,
(d) there exists $\lambda \in [\,0,1)$ such that
$d(fx,fy) \leq \lambda\,(M(gx,gy))\,,\ \forall x,y \in X$ *with* $(gx,gy) \in R$,
where

$$M(gx,gy) = \max\left\{\begin{array}{l} d(gx,gy), \\ \dfrac{d(gx,fx)d(gy,fy)}{d(gx,gy)}, \\ \dfrac{d(gx,fx)d(gy,fy)}{d(gx,gy)+d(gx,fy)+d(gy,fx)}, \\ \dfrac{d(gx,fx)d(gx,fy)+d(gy,fx)d(gy,fy)}{d(gy,fx)+d(gx,fy)} \end{array}\right\} \tag{54}$$

Moreover, if
(e) f and g are R-compatible,
(f) g is R-continuous,
(g) either f is R-continuous or R is (f,d)-self-closed,
Also, assume that
(h) there exists an R-closed subspace Y of X such that $f(X) \subseteq Y \subseteq g(X)$,
(i) either f is (g,R)-continuous or f and g are continuous or $R|_Y$ is d-self closed. Then f and g have a coincidence point.

Proof. The result corresponds to the proof in Theorem 3 easily on the setting $Y = X$ with part (h) and (i) follows, using Proposition 6. □

Remark 6. We can remove assumption (h) as it trivially holds for $Y = g(X) = X$ using Proposition 2.18. Whenever, f is onto, owing to assumption (a), g must be onto and hence again same conclusion is immediate.

In the following Theorem we show the existence and uniqueness of a coincidence point.

Theorem 6. *In addition to the hypotheses of Theorem 2, suppose that the following condition holds:*

(j) $f(X)$ is $R|^{s}_{g(X)}$-connected.

Then f and g have a unique point of coincidence.

Proof. Following the arguments of the proof Theorem 2, $\overline{C}(f,g) \neq \varnothing$. Suppose $\overline{x}, \overline{y} \in \overline{C}(f,g)$, then, there exist $x, y \in X$ such that

$$\overline{x} = g(x) = f(x) \text{and} \overline{y} = g(y) = f(y) \tag{55}$$

To show that $\overline{x} = \overline{y}$. Since $f(x), f(y) \in f(X) \subseteq g(X)$, by assumption (j), there exists a path say $\{gz_0, gz_1, gz_2, \dots, gz_k\}$ of some finite length k in $R|^{s}_{g(X)}$ from $f(x)$ to $f(y)$ where $z_0, z_1, z_2, \dots, z_k \in X$. With (55), we may choose $z_0 = x$ and $z_k = y$. Thus, we have

$$[\mathrm{gz}_i, \mathrm{gz}_{i+1}] \in R|_{g(X)} for each i (0 \leq i \leq k-1). \tag{56}$$

Define the constant sequences $z_n^0 = x$ and $z_n^k = y$, then using (55), we have

$g\left(z_{n+1}^0\right) = f\left(z_n^0\right) = \overline{x}$ and

$$g\left(z_{n+1}^k\right) = f\left(z_n^k\right) = \overline{y} for all n \in \mathbb{N}_0. \tag{57}$$

Let $z_0^1 = z_1,\ z_0^2 = z_2,\ \dots, z_0^{k-1} = z_{k-1}$. Since $f(X) \subseteq g(X)$, on the lines similar to that of Theorem 2, we can define sequences $\{z_n^1\}, \{z_n^2\}, \dots, \left\{z_n^{k-1}\right\}$ in X such that

$$g\left(z_{n+1}^1\right) = f\left(z_n^1\right), g\left(z_{n+1}^2\right) = f\left(z_n^2\right), \dots, g\left(z_{n+1}^{k-1}\right) = f\left(z_n^{k-1}\right) for all n \in \mathbb{N}_0. \tag{58}$$

Hence, we have

$$g\left(z_{n+1}^i\right) = f\left(z_n^i\right) for all n \in \mathbb{N}_0 and for each i (0 \leq i \leq k) \tag{59}$$

Now, we claim that

$$\left[gz_n^i, gz_n^{i+1}\right] \in R, for all n \in \mathbb{N}_0 \text{and for each} i (0 \leq i \leq k-1) \tag{60}$$

Inductively, it follows from (56) that (60) holds for $n = 0$. Suppose that (60) holds for $n = r > 0$, i.e.,

$$\left[gz_r^i, gz_r^{i+1}\right] \in R for each i (0 \leq i \leq k-1) \tag{61}$$

Since, R is (f,g)-closed, using Proposition 2.13, we obtain

$$\left[fz_r^i, fz_r^{i+1}\right] \in R for each i (0 \leq i \leq k-1), \tag{62}$$

which on using (60), gives rise

$$\left[gz_{r+1}^i, gz_{r+1}^{i+1}\right] \in R for each i (0 \leq i \leq k-1). \tag{63}$$

It follows that (60) holds for $n = r+1$.

Thus, by induction, (60) holds for all $n \in \mathbb{N}_0$. Now for all $n \in \mathbb{N}_0$ and for each $i (0 \leq i \leq k-1)$, denote $\beta_n^i =: d\left(gz_n^i, gz_n^{i+1}\right)$. Then, we claim that

$$\lim_{n\to\infty} \beta_n^i = 0 \tag{64}$$

On using (59), (60), assumption (d) and Proposition 2, for each $i (0 \leq i \leq k-1)$ and for all $n \in \mathbb{N}_0$, we obtain

$\beta_{n+1}^i = d\left(gz_{n+1}^i, gz_{n+1}^{i+1}\right)$

$= d\left(fz_n^i, fz_n^{i+1}\right)$

$\leq \lambda\left(M\left(gz_n^i, gz_n^{i+1}\right)\right)$

where

$M\left(gz_n^i, gz_n^{i+1}\right) =$

$$\max\left\{\begin{array}{c} d\left(gz_n^i, gz_n^{i+1}\right), \frac{d\left(gz_n^i, fz_n^i\right)d\left(gz_n^{i+1}, fz_n^{i+1}\right)}{d\left(gz_n^i, gz_n^{i+1}\right)}, \frac{d\left(gz_n^i, fz_n^i\right)d\left(gz_n^{i+1}, fz_n^{i+1}\right)}{d\left(gz_n^i, gz_n^{i+1}\right)+d\left(gz_n^i, fz_n^{i+1}\right)+d\left(gz_n^{i+1}, fz_n^i\right)}, \\ \frac{d\left(gz_n^i, fz_n^i\right)d\left(gz_n^i, fz_n^{i+1}\right)+d\left(gz_n^{i+1}, fz_n^i\right)d\left(gz_n^{i+1}, fz_n^{i+1}\right)}{d\left(gz_n^{i+1}, fz_n^i\right)+d\left(gz_n^i, fz_n^{i+1}\right)} \end{array}\right\}$$

$$\leq \lambda d\left(gz_n^i, gz_n^{i+1}\right) \tag{65}$$

$$= \lambda \beta_n^i. \tag{66}$$

By induction, we have

$$\beta_{n+1}^i \leq \lambda \beta_n^i \leq \lambda^2 \beta_{n-1}^i \leq \ldots \leq \lambda^{n+1} \beta_0^i \tag{67}$$

so that

$$\beta_{n+1}^i \leq \lambda^{n+1} \beta_0^i. \tag{68}$$

Implies

$$\lim_{n\to\infty} \beta_n^i = 0 for each i (0 \leq i \leq k-1). \tag{69}$$

Thus, we have (64) for each $i(0 \leq i \leq k-1)$.

By triangular inequality and (64), we obtain

$$d\left(\overline{x}, \overline{y}\right) \leq \beta_n^0 + \beta_n^1 + \ldots + \beta_n^{k-1} \to 0 as n \to \infty \tag{70}$$

Hence, $\overline{x} = \overline{y}$. □

Theorem 7. *In addition to the hypotheses of Theorem 6, suppose that the following condition holds:*

(k) one of f and g is one-one.

Then f and g have a unique coincidence point.

Proof. Following the argument of Theorem 2, $C(f,g) \neq \varnothing$. Take $x, y \in C(f,g)$, then, by Theorem 3.7, we have

$$g(x) = f(x) = f(y) = g(y). \tag{71}$$

Since f or g is one-one, we have $x = y$. □

Theorem 8. *In addition to the hypotheses in condition (h) and (i) of Theorem 6, suppose that the following condition holds:*

(l) f and g are weakly compatible.

Then f and g have a unique common fixed point.

Proof. Owing to Remark 4 as well as assumption (l), the mappings f and g are weakly compatible. Take $x \in C(f,g)$ and denote $g(x) = f(x) = \overline{x}$. Then by Lemma 2, $\overline{x} \in C(f,g)$. It follows from Theorem 6 with $y = \overline{x}$ that $g(x) = g(\overline{x})$, i.e., $\overline{x} = g(\overline{x})$, which yields that

$\overline{x} = g(\overline{x}) = f(\overline{x})$.

Hence, $\overline{x}$ is a common fixed point of f and g. To prove uniqueness, assume that $x^*$is another common fixed point of f and g. Then again from Theorem 6, we have

$x^* = g(x^*) = g(\overline{x}) = \overline{x}$.

Hence, $x^* = \overline{x}$.

We now present some examples that demonstrate the practical implications of our main theorems. □

Example 1. Consider $X = R$ endowed with usual metric and also define a binary relation

$R = \left\{(x,y) \in \mathbb{R}^2 : |x| - |y| \geq 0\right\}$. Then (X,d) is an R-complete metric space. Consider the mappings $f, g : X \to X$ defined by $f(x) = \frac{x^2}{3}$ and $g(x) = \frac{x^2}{2}$ for all $x \in X$.

Obviously, R is (f,g)-closed. Now, for $x, y \in X$ with $(gx, gy) \in R$, we have

$$d(fx, fy) = \left|\frac{x^2}{3} - \frac{y^2}{3}\right| = \frac{2}{3}\left|\frac{x^2}{2} - \frac{y^2}{2}\right| < \frac{3}{4}\left(d(gx, gy)\right)$$

Thus, f and g satisfy hypothesis (d) of Theorem 2 with $\lambda = \frac{3}{4}$. By a routine calculation, one can verify all the hypothesis in (e), (f) and (g) of Theorem 2. Hence, all the hypothesis of Theorem 2 are satisfied for $Y = X$, which guarantees that f and g have a coincidence point in X. Moreover, observe that (j) holds and henceforth in light of Theorem 6, f and g have a unique point of coincidence ($x = 0$), which remains also a unique common fixed point of Theorem 8.

Example 2. Consider $X = R$ equipped with usual metric and also define a binary relation $R = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \in \mathbb{Q}\}$. Consider the mappings $f, g : X \to X$ defined by $f(x) = 1$ and $g(x) = x^2 - 3$ for all $x \in X$. Clearly, R is (f, g)-closed.

Now, for $x, y \in X$ with $(gx, gy) \in R$, we have

$$d(fx, fy) = |1 - 1| = 0 \leq \left|x^2 - y^2\right| = \lambda d(gx, gy)$$

Thus, f and g satisfy assumption (d) of Theorem 2 for any arbitrary $\lambda \in [\,0, 1)$. Also, the mappings f and g are not R-compatible and hence (e), (f) and (g) does not hold. But the subspace $Y := g(x) = [\,-3, \infty)$ is R-complete and f and g are continuous, i.e., all the hypothesis mentioned in (h) and (i) are satisfied. Hence, in light of Theorem 2, f and g have a coincidence point in X. Further, in this example (j) holds and henceforth, owing to Theorem 6, f and g have a unique point of coincidence ($x = 1$).

It is important to note that neither the mapping f nor the mapping g is one-to-one. This means that property (k), which guarantees the uniqueness of coincidence points, cannot be applied to Theorem 8. Indeed, in this example, there are two different coincidence points: $x = 2$ and $x = -2$. Furthermore, the mappings f and g are not weakly compatible, meaning that property (l) cannot be used to ensure the uniqueness of a common fixed point. In fact, in this case, there is no common fixed point for f and g.

4. Applications

In this section, we obtain some results as consequences of our main results.

4.1. Coincidence points in ordered metric spaces via comparable mappings.

We begin by considering results involving comparable mappings, as introduced in works such as Turinici [36], Nieto and Rodriguez Lopez [40], and Alam and Imdad [28].

Definition 22. [6]. Let $(X, \preccurlyeq)$ be an ordered set and f and g two self-mappings on X. We say that f is g-comparable if for any $x, y \in X$,

$$g(x) \prec\succ g(y) \Rightarrow f(x) \prec\succ f(y) \tag{72}$$

Remark 7. It is clear that f is g-comparable iff $\prec\succ$ is (f, g)-closed.

Definition 23. [41]. Let $(X, \preccurlyeq)$ be an ordered set and $\{x_n\} \subset X$.

(i) the sequence $\{x_n\}$ is said to be termwise bounded if there is an element $z \in X$ such that each term of $\{x_n\}$ is comparable with z, i.e.,

$$x_n \prec\succ z \quad \forall\, n \in \mathbb{N}_0 \tag{73}$$

so that z is a c-bound of $\{x_n\}$ and

(ii) the sequence $\{x_n\}$ is said to termwise monotone if consecutive terms of $\{x_n\}$ are comparable, i.e.,

$$labeleq4.3(x_n \prec\succ x_n \qquad \forall\, n \in \mathbb{N}_0. \tag{74}$$

Remark 8. Clearly, $\{x_n\}$ is termwise monotone iff it is $\prec\succ$-preserving.

Definition 24. [41] Given a mapping $g : X \to X$, we say that an ordered metric space $(X, d, \preccurlyeq)$ has g-TCC (termwise monotone-convergence-c-bound) property if every termwise monotone sequence $\{x_n\}$ in X such that $x_n \xrightarrow{d} x$ has a subsequence, whose g-image is termwise bounded by g-image of limit (of the sequence) as a c-bound, i.e., $g\left(x_{n_k}\right) \prec\succ g(x)$ for all $k \in \mathbb{N}_0$.

Notice that under the restriction $g = I$, the identity mapping on X, Definition 4.5 transforms to the notion of TCC property.

Remark 9. Clearly, $(X, d, \preccurlyeq)$ has TCC property (resp. g-TCC property) iff $\prec\succ$ is d-self-closed (resp. (g, d)-self-closed).

Corollary 9. *[41] Let (X, d) be an ordered metric space and Y a complete subspace of X. Let f and g be two self-mappings on X. Suppose that the following conditions hold:*

(a) $f(X) \subseteq g(X) \cap Y$,
(b) f is g-comparable,
(c) there exists $x_0 \in X$ such that $g\left(x_0\right) \prec\succ f\left(x_0\right)$,
(d) there exists $\lambda \in [\,0,1)$ such that

$$d(fx, fy) \leq \lambda d(gx, gy),\ \forall x, y \in X \ \text{ with } \ gx \prec\succ gy, \tag{75}$$

Moreover, if

(e) f and g are compatible,
(f) g is continuous,
(g) either f is continuous or $(Y, d, \preccurlyeq)$ has g-TCC property,
Also, assume that
(h) $Y \subseteq g(X)$,
(i) either f is g-continuous or f and g are continuous or $(Y, d, \preccurlyeq)$ has TCC property.
Then f and g have a coincidence point.

Corollary 10. *Let (X, d) be an ordered metric space and Y a complete subspace of X. Let f and g be two self-mappings on X. Suppose that the following conditions hold:*

(a) $f(X) \subseteq g(X) \cap Y$,
(b) f is g-comparable,
(c) there exists $x_0 \in X$ such that $g\left(x_0\right) \prec\succ f\left(x_0\right)$,
(d) there exists $\lambda \in [\,0,1)$ such that
$d(fx, fy) \leq \lambda\left(M(gx, gy)\right),\ \forall x, y \in X$ with $gx \prec\succ gy$,
where

$$M(gx, gy) = \max \left\{ \begin{array}{l} d(gx, gy), \\ \dfrac{d(gx, fx) d(gy, fy)}{d(gx, gy)}, \\ \dfrac{d(gx, fx) d(gy, fy)}{d(gx, gy) + d(gx, fy) + d(gy, fx)}, \\ \dfrac{d(gx, fx) d(gx, fy) + d(gy, fx) d(gy, fy)}{d(gy, fx) + d(gx, fy)} \end{array} \right\} \tag{76}$$

Moreover, if

(e) f and g are compatible,
(f) g is continuous,
(g) either f is continuous or $(Y, d, \preccurlyeq)$ has g-TCC property,
Also, assume that
(h) $Y \subseteq g(X)$,
(i) either f is g-continuous or f and g are continuous or $(Y, d, \preccurlyeq)$ has TCC property.
Then f and g have a coincidence point.

4.2. Coincidence theorems under symmetric closure of a binary relation

We now consider results that involve the symmetric closure of a binary relation, which have been studied in works such as Samet and Turinici [25] and Berzig [38]. In this context, R is a binary relation on a nonempty set X, and S denotes the symmetric closure of R, that is, $S := R^s$.

Definition 25. [38]. Let f and g be two self-mappings on X. We say that f is g-comparative if for any $x, y \in X$,

$$(gx, gy) \in S \Rightarrow (fx, fy) \in S. \tag{77}$$

Remark 10. It is clear that f is g-comparative iff S is (f, g)-closed.

Definition 26. [2]. $(X, d,\ S)$ is regular if the following condition holds: if the sequence $\{x_n\}$ in X and the point $x \in X$ are such that

$$(x_n, x_{n+1}) \in S for all n and \lim_{n\to\infty} (x_{n_k}, x) = 0, \tag{78}$$

then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in S$ for all k.

Remark 11. Clearly, $(X, d,\ S)$ is regular iff S is d-self-closed.

Corollary 11. *[38] Let (X, d) be a metric space, R a binary relation on X and f and g two self-mappings on X. Suppose that the following conditions hold:*

(a) $f(X) \subseteq g(X)$,
(b) f is g-comparative,
(c) there exists $x_0 \in X$ such that $(gx_0, gx_0) \in S$,
(d) there exists $\lambda \in [\ 0, 1)$ such that

$$d(fx, fy) \leq \lambda d(gx, gy),\ \forall x, y \in X \ \text{with}\ (gx, gy) \in S, \tag{79}$$

(e) (X, d) is complete and $g(X)$ is closed,
(f) (X, d, S) is regular.
Then f and g have a coincidence point.

Corollary 12. *Let (X, d) be a metric space, R a binary relation on X and f and g two self-mappings on X. Suppose that the following conditions hold:*

(a) $f(X) \subseteq g(X)$,
(b) f is g-comparative,
(c) there exists $x_0 \in X$ such that $(gx_0, gx_0) \in S$,
(d) there exists $\lambda \in [\ 0, 1)$ such that
$d(fx, fy) \leq \lambda\,(M(gx, gy)),\ \forall x, y \in X$ with $(gx, gy) \in S$,
where

$$M(gx, gy) = \max \left\{ \begin{array}{l} d(gx, gy), \\ \dfrac{d(gx, fx)d(gy, fy)}{d(gx, gy)}, \\ \dfrac{d(gx, fx)d(gy, fy)}{d(gx, gy) + d(gx, fy) + d(gy, fx)}, \\ \dfrac{d(gx, fx)d(gx, fy) + d(gy, fx)d(gy, fy)}{d(gy, fx) + d(gx, fy)} \end{array} \right\} \tag{80}$$

(e) (X, d) is complete and $g(X)$ is closed,
(f) (X, d, S) is regular.
Then f and g have a coincidence point.

Conclusion

In this article, we extend fundamental metrical notions such as completeness, closedness, continuity, g-continuity, and compatibility to the relation-theoretic context. Utilizing these generalized concepts, we establish new results regarding the existence and uniqueness of coincidence points for mappings on metric spaces defined by arbitrary binary relations. Notably, when the relation is universal, our findings encompass classic coincidence point theorems. Additionally, our results lead to several well-known metrical coincidence point theorems. Finally, we provide illustrative examples to validate the effectiveness of our results.

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