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Contributions to hyperbolic 1-parameter inequalities

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Abstract: In this article we provide classes of hyperbolic chains of inequalities depending on a certain parameter n. New refinements as well as new results are offered. Some graphical analyses support the theoretical results.

Keywords: hyperbolic functions; sinc function; hyperbolic inequalities; inequality chains

MSC: 26D07, 33B10, 33B20, 26D15.

1. Introduction

In recent years, sharp inequalities involving trigonometric and hyperbolic functions have received a lot of attention. This is easily explained by their usefulness in all areas of mathematics. Such inequalities, as well as refinements of the so-called Jordan, Cusa-Huygens and Wilker inequalities, can be found in [1–9], and the references therein.

In this article, our interest in hyperbolic inequalities remains in establishing the counterpart of recent developments in trigonometric inequalities. In particular, we extend some results studied in [10]. In fact, many hyperbolic inequalities already exist. For example, Wilker-type inequalities for hyperbolic functions have been studied by Wu and Debnath [11]. In addition, the following chain of inequalities is known: For $x \in (0, \infty)$, we have

$$\frac{1}{2}\left[\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x}\right] > \frac{1}{3}\left[\frac{2\sinh x}{x} + \frac{\tanh x}{x}\right] >$$
$$\frac{1}{2}\left[\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x}\right] > \frac{1}{3}\left[\frac{2x}{\sinh x} + \frac{x}{\tanh x}\right] > 1. \tag{1}$$

See [9, Theorems 2.2 or 4.2]. The following inequality, the hyperbolic equivalent of a famous trigonometric inequality, was derived by Lazarevic: For $x \neq 0$, we have

$$\left(\frac{\sinh x}{x}\right)^3 > \cosh x.$$

In [4], a deep work was done on this topic. In particular, the theorems below represent sharp hyperbolic chains of inequalities.

Theorem 1. *For every integer $n \geq 1$ and $x \in (0, \infty)$, we have*

$$0 < \frac{\sinh x}{x} + \frac{\tanh x}{2x} - \frac{3}{2} < \left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} - 2 < \ldots$$
$$< f(n, x) = \left(\frac{\sinh x}{x}\right)^n + \frac{n\tanh x}{2x} - \frac{n+2}{2}. \tag{2}$$

Theorem 2. *For every integer $n \geq 1$ and $x \in (0, \infty)$, we have*

$$0 < \frac{x}{\sinh x} + \frac{x}{2\tanh x} - \frac{3}{2} < \left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} - 2 < \dots$$
$$< g(n,x) = \left(\frac{x}{\sinh x}\right)^n + \frac{nx}{2\tanh x} - \frac{n+2}{2}. \quad (3)$$

Theorem 3. *For every integer $n \geq 2$ and $x \in (0, \infty)$, we have*

$$0 < \left(\frac{\sinh x}{x}\right)^2 - \frac{2\cosh x}{3} - \frac{1}{3} < \left(\frac{\sinh x}{x}\right)^3 - \cosh x < \dots$$
$$< \left(\frac{\sinh x}{x}\right)^n - \frac{n\cosh x}{3} + \frac{n-3}{3}. \quad (4)$$

For this last result, the case $n = 3$ corresponds to the Lazarevic inequality. However, this is not true for $n = 1$. In fact, for $x \in (0, \infty)$, the following inequality is well known:

$$\frac{\sinh x}{x} - \frac{\cosh x}{3} - \frac{2}{3} < 0.$$

This illustrates the complex role of n in this kind of chain of inequalities. For example, Theorem 3 states that Lazarevic-type inequalities hold for any integer $n \geq 2$.

In fact, as proved in [4], classical hyperbolic inequalities are enclosed in chains of inequalities. We call the n-order hyperbolic Huygens-Wilker function the function defined as follows:

$$f(n,x) = \left(\frac{\sinh x}{x}\right)^n + \frac{n\tanh x}{2x} - \frac{n+2}{2}. \quad (5)$$

The inequality $f(n,x) > 0$ becomes the Huygens inequality and the Wilker inequality for $n = 1$ and $n = 2$, respectively.

In a similar way, for the second hyperbolic Huygens inequality and hyperbolic Wilker inequality, we obtain the comparison

$$0 < \frac{x}{\sinh x} + \frac{x}{2\tanh x} - \frac{3}{2} < \left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} - 2.$$

See [8]. In this sense, we call the n-order second Huygens-Wilker function the following function:

$$g(n,x) = \left(\frac{x}{\sinh x}\right)^n + \frac{nx}{2\tanh x} - \frac{n+2}{2}. \quad (6)$$

Concerning the Lazarevic inequality, it can be expressed as follows: For $x \in (0, \infty)$, we have

$$\left(\frac{\sinh x}{x}\right)^3 - \cosh x > 0.$$

Based on it, we call the n-order Lazarevic function the following function:

$$h(n,x) = \left(\frac{\sinh x}{x}\right)^n - \frac{n\cosh x}{3} + \frac{n-3}{3}.$$

These three special n-order functions will be at the center of our findings.

This article is organized as follows: In Section 2, we propose to refine and extend the above inequality chains. We provide additional inequalities depending on n, which are often reduced to the well-known correspondences for $n = 2$ or 3. We also provide some graphics to support the analysis. Section 3 contains the detailed proofs of the considered intermediate lemmas and corollaries. In Section 4, we discuss and propose some possible developments from these results.

2. Main results

2.1. n-order hyperbolic Huygens-Wilker inequalities

In this part, we aim to extend the chain of inequalities described in Equation (1) to that given by Equation (2). For this purpose, we establish the result below, which extends Theorem 2 demonstrated in [4].

Theorem 4. *First of all, let us recall the function in Equation (5), i.e.,*

$$f(n,x) = \left(\frac{\sinh x}{x}\right)^n + \frac{n\tanh x}{2x} - \frac{n+2}{2}.$$

For every integer $n \geq 2$ and for $x \in (0,\infty)$, the following chain of inequalities holds:

$$\ldots > \frac{f(n,x)}{n(22+5n)} > \frac{f(n-1,x)}{(n-1)(5n+17)} > \frac{f(n-2,x)}{(n-2)(5n+12)} > \ldots.$$

Proof of Theorem 4. Let us consider the following intermediary function:

$$\phi(n,x) = \frac{f(n,x)}{n(5n+22)} = \frac{1}{n(5n+22)}\left[\left(\frac{\sinh x}{x}\right)^n + \frac{n\tanh x}{2x} - \frac{n+2}{2}\right].$$

The derivative of $\phi(n,x)$ with respect to n is given by

$$\begin{aligned}
\frac{\partial\phi(n,x)}{\partial n} &= \frac{\left(\frac{\sinh x}{x}\right)^n \ln\left(\frac{\sinh x}{x}\right) + \frac{\tanh x}{2x} - \frac{1}{2}}{n\,(5n+22)} \\
&- \frac{\left(\frac{\sinh x}{x}\right)^n + \frac{n\tanh x}{2x} - \frac{n}{2} - 1}{n^2\,(5n+22)} - \frac{5\left[\left(\frac{\sinh x}{x}\right)^n + \frac{n\tanh x}{2x} - \frac{n}{2} - 1\right]}{n\,(5n+22)^2} \\
&= \frac{\left(\frac{\sinh x}{x}\right)^n\left[n\,(5n+22)\ln\left(\frac{\sinh x}{x}\right) - 10n - 22\right]}{n^2\,(5n+22)^2} - \frac{5\tanh x}{2x\,(5n+22)^2} + \frac{5n^2+20n+44}{2n^2\,(5n+22)^2} \\
&= \frac{\left(\frac{\sinh x}{x}\right)^n\left[n\,(5n+22)\ln\left(\frac{\sinh x}{x}\right) - 10n - 22\right] - 5n^2\frac{\tanh x}{2x} + \frac{5n^2}{2} + 10n + 22}{n^2(5n+22)^2}.
\end{aligned}$$

To proceed further, we need the technical lemma below.

Lemma 5.

- *The function*

$$-5n^2\frac{\tanh x}{2x} + \frac{5n^2+20n+44}{2}$$

  is always non negative.
- *The function*

$$n\,(5n+22)\ln\left(\frac{\sinh x}{x}\right) - 10n - 22$$

  has a unique positive root x_n for every integer $n \geq 2$. Moreover, the sequence (x_n) is decreasing as n grows:

$$x_2 = 2.111605780, \quad x_3 = 1.753855952, \quad x_4 = 1.542294291, \ldots$$

Note: In order to make the reading easier, the proof of this lemma and all the following lemmas and corollaries are postponed in Section 3.

In particular, this lemma implies that, for $x \geq x_n$,

$$n\,(5n+22)\ln\left(\frac{\sinh x}{x}\right) - 10n - 22 > 0.$$

We then deduce that $\frac{\partial\phi(n,x)}{\partial n} > 0$.

For the case $x \in (0, x_n)$ and $n \geq 2$, a different approach is needed. The result below is a key lemma.

Lemma 6. *For every integer $n \geq 2$ and for $x \in (0, x_n)$, where x_n is the root of*

$$n\,(5n+22)\ln\left(\frac{\sinh x}{x}\right) - 10n - 22,$$

so $x_n < 2.1116$, the following inequalities hold:

1. $$\left(\frac{\sinh x}{x}\right)^n > 1 + \frac{nx^2}{6} + \frac{n(5n-2)x^4}{360}.$$
2. $$1 - \frac{x^2}{3} + \frac{2x^4}{15} - \frac{17x^6}{315} < \frac{\tanh x}{x} < 1 - \frac{x^2}{3} + \frac{2x^4}{15} - \frac{17x^6}{315} + \frac{62x^8}{2835}.$$
3. $$\frac{x^2}{6} - \frac{x^4}{180} + \frac{x^6}{2835} - \frac{x^8}{37800} < \ln\left(\frac{\sinh x}{x}\right) < \frac{x^2}{6} - \frac{x^4}{180} + \frac{x^6}{2835}.$$

In the light of this result, let us now consider the numerator of $\frac{\partial\phi(n,x)}{\partial n}$. We want to minimize it for $x \in (0, x_n)$. After some tedious development, we find that

$$\begin{aligned}
&\left(\frac{\sinh x}{x}\right)^n\left[n\,(5n+22)\ln\left(\frac{\sinh x}{x}\right) - 10n - 22\right] - 5n^2\frac{\tanh x}{2x} + \frac{5n^2}{2} + 10n + 22\\
&> \left[1 + \frac{nx^2}{6} + \left(\frac{1}{120}n + \frac{1}{72}n\,(n-1)\right)x^4\right]\times\\
&\left[-10n - 22 + n\,(5n+22)\left(\frac{x^2}{6} - \frac{x^4}{180} + \frac{x^6}{2835} - \frac{x^8}{37800}\right)\right]\\
&-\frac{5n^2}{2}\left(1 - \frac{x^2}{3} + \frac{2x^4}{15} - \frac{17x^6}{315} + \frac{62x^8}{2835}\right) + \frac{5n^2}{2} + 10n + 22 = -\frac{1}{40824000}x^6 nA(n,x),
\end{aligned}$$

where

$$\begin{aligned}
A(n,x) &= -472500n^3 + 15750x^2n^3 - 1000n^3x^4 + 75x^6n^3 - 3100x^4n^2 + 5720x^4n + 300x^6n^2\\
&- 132x^6n - 3916800n - 316800 - 1701000n^2 + 23760x^2 + 2156880nx^2 + 51000n^2x^2\\
&= \left(15750x^2 - 1000x^4 + 75x^6 - 472500\right)n^3 + \left(300x^6 - 1701000 + 51000x^2 - 3100x^4\right)n^2\\
&+ \left(-3916800 + 2156880x^2 + 5720x^4 - 132x^6\right)n - 316800 + 23760x^2.
\end{aligned}$$

Moreover, the partial derivative of $A(n,x)$ with respect to n is

$$\begin{aligned}
\frac{\partial A(n,x)}{\partial n} &= 3\left(15750x^2 - 1000x^4 + 75x^6 - 472500\right)n^2\\
&+ 2\left(300x^6 - 1701000 + 51000x^2 - 3100x^4\right)n - 3916800 + 2156880x^2 + 5720x^4 - 132x^6.
\end{aligned}$$

It has a non positive maximum at $n = 2$ and $x < 2.111605780$ since

$$A(2,x) = 2549880x^2 - 18680x^4 + 1968x^6 - 16390800 < 0.$$

This implies that $A(n,x) < 0$, and $\frac{\partial\phi(n,x)}{\partial n} > 0$. Theorem 4 is thus proved. □

A corollary derived from this theorem is presented below.

Corollary 7. *For $n = 2$ and for $x \in (0, \infty)$, the following inequalities hold:*

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > -\frac{14}{9} + \frac{64}{27}\frac{\sinh x}{x} + \frac{32}{27}\frac{\tanh x}{x} > \frac{2}{3}\left[\frac{2\sinh x}{x} + \frac{\tanh x}{x}\right],$$

and imply the inequalities in Equation (1).

Let us illustrate these inequalities in Figure 1. We consider the following functions:

$$c(x) = \left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} + \frac{14}{9} - \frac{64}{27}\frac{\sinh x}{x} - \frac{32}{27}\frac{\tanh x}{x}$$

and

$$d(x) = -\frac{14}{9} + \frac{64}{27}\frac{\sinh x}{x} + \frac{32}{27}\frac{\tanh x}{x} - \frac{2}{3}\left[\frac{2\sinh x}{x} + \frac{\tanh x}{x}\right].$$

This figure shows that $c(x) > 0$ and $d(x) > 0$ for $x \in (0, 4)$ (the same is, in fact, observed for any $x \in (0, \infty)$).

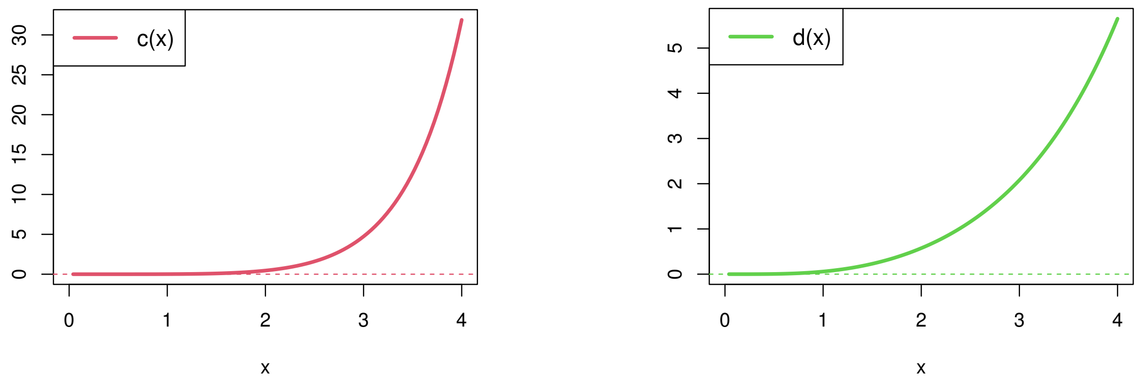

Figure 1. Curves of $c(x)$ (left) and $d(x)$ (right) for $x \in (0, 4)$

Another corollary is proposed below by distinguishing the values of a tuning parameter k.

Corollary 8. *For every integer $n \geq 2$ and for $x \in (0, \infty)$, the following inequalities hold:*

$$f(n, x) = \left(\frac{\sinh x}{x}\right)^n + \frac{n \tanh x}{2x} - \frac{n+2}{2} > kx^4, \quad k \leq \frac{n(5n+22)}{360},$$

and

$$f(n, x) = \left(\frac{\sinh x}{x}\right)^n + \frac{n \tanh x}{2x} - \frac{n+2}{2} < kx^4, \quad k > \frac{n(5n+22)}{360}, \quad x \leq x_{n,k},$$

where $x_{n,k}$ is the (unique) positive root of $\left(\frac{\sinh x}{x}\right)^n + \frac{n \tanh x}{2x} - \frac{n+2}{2} - kx^4 = 0$.

2.2. n-order hyperbolic second Huygens-Wilker inequalities

The result below proves that the classical inequalities in Equation (3) belong to a chain of inequalities that is sharper than the one given in Theorem 2 (established in [4]).

Theorem 9. *First of all, let us recall the function in Equation (6), i.e.,*

$$g(n, x) = \left(\frac{x}{\sinh x}\right)^n + \frac{nx}{2\tanh x} - \frac{n+2}{2}.$$

For every integer $n \geq 2$ and for $x \in (0, \infty)$, the following chain of inequalities holds:

$$\cdots < \frac{g(n,x)}{n(5n-2)} < \frac{g(n-1,x)}{(n-1)(5n-7)} < \frac{g(n-2,x)}{(n-2)(5n-12)} < \cdots.$$

In particular, for $n = 2$, we have

$$\frac{x^2}{16\,(\sinh x)^2} + \frac{x}{16\tanh x} < \frac{x}{3\sinh x} + \frac{x}{6\tanh x} - \frac{3}{8}.$$

We then extend the inequalities in Equation (3) as follows:

$$\begin{aligned}\frac{1}{8} &< \frac{x}{12\sinh x} + \frac{x}{24\tanh x} < \frac{x^2}{16\,(\sinh x)^2} + \frac{x}{16\tanh x} \\ &< \frac{x}{3\sinh x} + \frac{x}{6\tanh x} - \frac{3}{8}.\end{aligned}$$

Proof of Theorem 9. Let us set

$$\psi(n,x) = \frac{g(n,x)}{n(5n-2)} = \frac{1}{n(5n-2)}\left[\left(\frac{x}{\sinh x}\right)^n + \frac{nx}{2\tanh x} - \frac{n+2}{2}\right].$$

Based on it, we consider the following difference function:

$$\begin{aligned}\psi(n,x) - \psi(n-1,x) &= \frac{g(n,x)}{n(5n-2)} - \frac{g(n-1,x)}{(n-1)(5n-7)} \\ &= \frac{\left(\frac{x}{\sinh x}\right)^n + \frac{nx}{2\tanh x} - \frac{n}{2} - 1}{n\,(5n-2)} - \frac{\left(\frac{x}{\sinh x}\right)^{n-1} + \frac{(n-1)x}{2\tanh x} - \frac{n+1}{2}}{(n-1)\,(5n-7)} \\ &= \left(\frac{x}{\sinh x}\right)^{n-1}\left(\frac{x}{(\sinh x)\,n\,(5n-2)} - \frac{1}{(n-1)\,(5n-7)}\right) \\ &\quad - \frac{5x}{2\,(5n-2)\,(5n-7)\tanh x} + \frac{5n^2+15n-14}{2n\,(5n-2)\,(n-1)\,(5n-7)}.\end{aligned}$$

The lemma below is needed to continue the proof.

Lemma 10.

- *The function*

$$\frac{x}{(\sinh x)n\,(5n-2)} - \frac{1}{(n-1)\,(5n-7)}$$

  is always non positive.
- *The function*

$$-\frac{5x}{2\,(5n-2)\,(5n-7)\tanh x} + \frac{5n^2+15n-14}{2n\,(5n-2)\,(n-1)\,(5n-7)}$$

  has a unique positive root x_n for every integer $n \geq 2$. Moreover, the sequence (x_n) is decreasing as n grows:

$$x_2 = 3.594569985, \quad x_3 = 2.499381175, \quad x_4 = 2.028584013, \ldots$$

In particular, this lemma implies that

$$-\frac{5x}{2\,(5n-2)\,(5n-7)\tanh x} + \frac{5n^2+15n-14}{2n\,(5n-2)\,(n-1)\,(5n-7)} > 0$$

for $x \geq x_n$. We then deduce that

$$\psi(n,x) - \psi(n-1,x) > 0$$

for $x \geq x_n$. In order to examine the case $x \in (0, x_n)$ and $n \geq 2$, we need the result below.

Lemma 11. *For every integer $n \geq 2$ and for $x \in (0,\infty)$, the following inequalities hold:*

1.
$$1-\frac{nx^2}{6}+\left(\frac{1}{180}n+\frac{1}{72}n^2\right)x^4+\left(-\frac{1}{1080}n^2-\frac{1}{2835}n-\frac{1}{1296}n^3\right)x^6 < \left(\frac{x}{\sinh x}\right)^n < 1-\frac{nx^2}{6}+\left(\frac{1}{180}n+\frac{1}{72}n^2\right)x^4.$$

2.
$$\frac{x}{\tanh x} < 1+\frac{1}{3}x^2-\frac{1}{45}x^4+\frac{2}{945}x^6-\frac{1}{4725}x^8+\frac{2}{93555}x^{10}.$$

On the basis of this result, we consider the following natural difference function, with the aim of maximizing it:

$$\begin{aligned}
&\left(\frac{x}{\sinh x}\right)^{n-1}\left[\frac{x}{(\sinh x)\,n\,(5n-2)}-\frac{1}{(n-1)\,(5n-7)}\right] \\
&-\frac{5x}{2\,(5n-2)\,(5n-7)\tanh x}+\frac{5n^2+15n-14}{2n\,(5n-2)\,(n-1)\,(5n-7)} \\
&<\left[1-\left(\frac{n-1}{6}\right)x^2+\left(-\frac{n}{45}+\frac{1}{120}+\frac{n^2}{72}\right)x^4\right]\times \\
&\left[\frac{1-\frac{x^2}{6}+\frac{7x^4}{360}}{n\,(5n-2)}-\frac{1}{(n-1)\,(5n-7)}\right] \\
&-5\frac{1+\frac{x^2}{3}-\frac{x^4}{45}+\frac{2x^6}{945}-\frac{x^8}{4725}+\frac{2x^{10}}{93555}}{(5n-2)\,(5n-7)}+\frac{5n^2+15n-14}{2n\,(5n-2)\,(n-1)\,(5n-7)} \\
&=\frac{x^6}{29937600\,(5n-7)\,n\,(5n-2)}B(n,x),
\end{aligned}$$

where

$$\begin{aligned}
B(n,x) &= 40425x^2n^3-346500n^3+554400n^2-121275x^2n^2 \\
&-1600x^4n+21780n+130647x^2n-33957x^2-388080 \\
&=(-346500+40425y)\,n^3+(-121275y+554400)\,n^2 \\
&+\left(-1600y^2+130647y+21780\right)n-33957y-388080.
\end{aligned}$$

Its derivative with respect to n is obtained as

$$\begin{aligned}
\frac{\partial B(n,x)}{\partial n} &= 3\,(-346500+40425y)\,n^2+2\,(-121275y+554400)\,n \\
&-1600y^2+130647y+21780.
\end{aligned}$$

It has a non positive maximum at $n=2$ and $x<3.594569985$ or $y=x^2<12.92093338$, as well for $B(n,x)$ since

$$B(2,x)=-3200y^2+65637y-898920<0.$$

This means that $\psi(n,x)-\psi(n-1,x)=\frac{g(n,x)}{n(5n-2)}-\frac{g(n-1,x)}{(n-1)(5n-7)}<0$. Theorem 9 is then demonstrated. □

The corollary below is derived from this theorem.

Corollary 12. *For $n = 2$ and for $x \in (0, \infty)$, the following inequalities hold:*

$$\frac{1}{8} < \frac{x}{12\sinh x} + \frac{x}{24\tanh x} < \frac{x^2}{16(\sinh x)^2} + \frac{x}{16\tanh x}$$
$$< \frac{x}{3\sinh x} + \frac{x}{6\tanh x} - \frac{3}{8}.$$

Figure 2 provides a graphical analysis of some of these inequalities. Let us consider the following functions:

$$p(x) = \frac{x}{3\sinh x} + \frac{x}{6\tanh x} - \frac{3}{8} - \frac{x^2}{16(\sinh x)^2} - \frac{x}{16\tanh x}$$

and

$$q(x) = \frac{x^2}{16(\sinh x)^2} + \frac{x}{16\tanh x} - \frac{x}{12\sinh x} - \frac{x}{24\tanh x}.$$

This figure shows that $p(x) > 0$ and $q(x) > 0$ for $x \in (0, 6)$ (the same is observed for any $x \in (0, \infty)$).

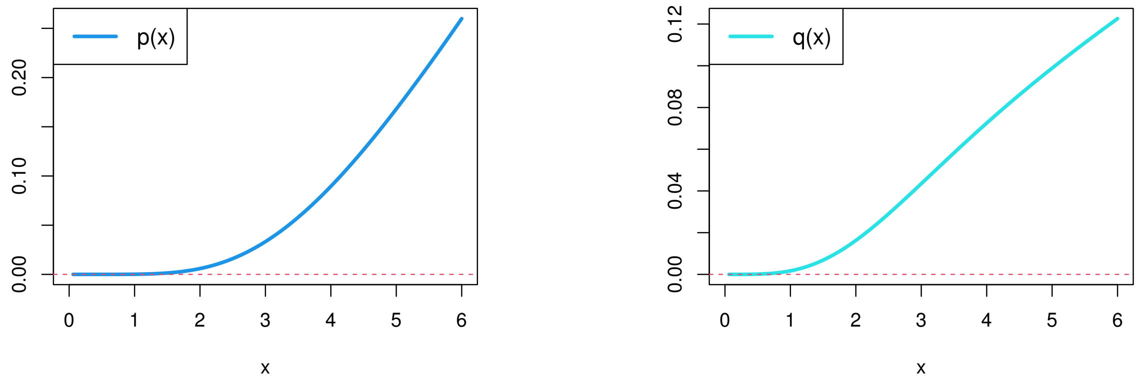

Figure 2. Curves of $p(x)$ (left) and $q(x)$ (right) for $x \in (0, 6)$

2.3. n-order Lazarevic inequalities

In this part, we examine the inequalities in Equation (4) and attempt to include them in a chain of inequalities that is sharper than that given by Theorem 3 (established in [4]).

Theorem 13. *Let us consider the function*

$$h(n, x) = \left(\frac{\sinh x}{x}\right)^n - \frac{n\cosh x}{3} + \frac{n-3}{3}.$$

Then for every integer $n \geq 3$ and for $x \in (0, \infty)$, the following inequalities hold:

$$\frac{h(n, x)}{n(n-7)} > \frac{h(n-1, x)}{(n-1)(5n-12)} \frac{h(n-2, x)}{(n-2)(5n-17)} > \ldots.$$

Proof of Theorem 13. Let us consider

$$v(n, x) = \frac{h(n, x)}{n(5n-7)} = \frac{1}{n(5n-7)}\left[\left(\frac{\sinh x}{x}\right)^n - \frac{n\cosh x}{3} + \frac{n-3}{3}\right].$$

The derivative of $\nu(n,x)$ with respect to n is

$$\begin{aligned}\frac{\partial \nu(n,x)}{\partial n} &= \left[\left(\frac{\sinh x}{x}\right)^n \ln\left(\frac{\sinh x}{x}\right) - \frac{1}{3}\cosh x + \frac{1}{3}\right] n^{-1}(5n-7)^{-1} \\
&- \left[\left(\frac{\sinh x}{x}\right)^n - \frac{n}{3}\cosh x + \frac{n}{3} - 1\right] n^{-2}(5n-7)^{-1} \\
&- 5\left[\left(\frac{\sinh x}{x}\right)^n - \frac{n}{3}\cosh x + \frac{n}{3} - 1\right] n^{-1}(5n-7)^{-2} \\
&= \frac{\left(\frac{\sinh x}{x}\right)^n \left[5\ln\left(\frac{\sinh x}{x}\right) n^2 - 7\ln\left(\frac{\sinh x}{x}\right) n - 10n + 7\right]}{n^2(5n-7)^2} \\
&+ \frac{5n^2\cosh x}{3n^2(5n-7)^2} - \frac{5n^2 - 30n + 21}{3n^2(5n-7)^2}.\end{aligned}$$

To go on the proof, we need the lemma below.

Lemma 14.

- *The function*
$$\frac{5\cosh x}{3(5n-7)^2} - \frac{5n^2-30n+21}{3n^2(5n-7)^2}$$
is always non negative.
- *The function*
$$\ln\left(\frac{\sinh x}{x}\right) - \frac{10n+7}{n(5n-7)}$$
has a unique positive root x_n for every integer $n \geq 3$. Moreover, the sequence (x_n) is decreasing as n grows:
$$x_3 = 2.619951007, \quad x_4 = 2.072479003, \quad x_5 = 1.772720990, \ldots$$

In particular, this lemma implies that

$$\ln\left(\frac{\sinh x}{x}\right) - \frac{10n+7}{n(5n-7)} > 0$$

for $x \geq x_n$. We then deduce that $\frac{\partial \nu(n,x)}{\partial n} > 0$ for $x \geq x_n$.

For the case $x \in (0, x_n)$ and $n \geq 3$, we need the lemma below.

Lemma 15. *For every integer $n \geq 2$ and for $x \in (0, x_n)$, the following inequalities hold:*

1. $$\left(\frac{\sinh x}{x}\right)^n < 1 + \frac{nx^2}{6} + \frac{n(5n-2)x^4}{360} + \frac{n(35n^2-42n+16)}{45360}x^6.$$
2. $$1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \frac{x^8}{40320} < \cosh x.$$
3. $$\frac{x^2}{6} - \frac{x^4}{180} + \frac{x^6}{2835} - \frac{x^8}{37800} < \ln\left(\frac{\sinh x}{x}\right) < \frac{x^2}{6} - \frac{x^4}{180} + \frac{x^6}{2835}.$$

Using this lemma, we want to minimize the following derivative function:

$$\begin{aligned}
\frac{\partial v(n,x)}{\partial n} &= \frac{\left(\frac{\sinh x}{x}\right)^n \left(5\ln\left(\frac{\sinh x}{x}\right)n^2 - 7\ln\left(\frac{\sinh x}{x}\right)n - 10n + 7\right)}{n^2(5n-7)^2} \\
&+ \frac{5n^2\cosh x}{3n^2(5n-7)^2} - \frac{5n^2-30n+21}{3n^2(5n-7)^2} \\
&> \left[1 + \frac{nx^2}{6} + \left(\frac{n}{120} + \frac{n(n-1)}{72}\right)x^4 + \left(\frac{n(n-1)}{720} + \frac{n}{5040} + \frac{n(n-1)(n-2)}{1296}\right)x^6\right] \times \\
&\frac{\left[n(5n-7)\left(\frac{x^2}{6} - \frac{x^4}{180} + \frac{x^6}{2835} - \frac{x^8}{37800}\right) - 10n + 7\right]}{n^2(5n-7)^2} \\
&+ \frac{5\left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \frac{x^8}{40320}\right)}{3(5n-7)^2} - \frac{5n^2-30n+21}{3n^2(5n-7)^2} \\
&= -\frac{x^6}{5143824000n(5n-7)^2}C(n,x),
\end{aligned}$$

where

$$\begin{aligned}
C(n,x) &= -412650n^3x^4 - 3307500x^2n^4 + 110250x^4n^4 + 575820x^4n^2 - 299880x^4n - 19845000n^3 \\
&- 36174600n + 55566000n^2 + 5812695nx^2 - 952560x^2 - 12152700n^2x^2 + 10584000x^2n^3 \\
&- 1365x^8n^3 + 1122x^8n^2 - 336x^8n - 7000x^6n^4 + 525x^8n^4 - 31970x^6n^2 + 9772x^6n + 27650x^6n^3 \\
&= \left(110250x^4 + 525x^8 - 3307500x^2 - 7000x^6\right)n^4 \\
&+ \left(-19845000 + 10584000x^2 - 1365x^8 + 27650x^6 - 412650x^4\right)n^3 \\
&+ \left(575820x^4 - 12152700x^2 + 55566000 + 1122x^8 - 31970x^6\right)n^2 \\
&+ \left(-299880x^4 - 36174600 + 5812695x^2 - 336x^8 + 9772x^6\right)n - 952560x^2.
\end{aligned}$$

The second derivative of $C(n,x)$ with respect to n is obtained as

$$\begin{aligned}
&12\left(110250x^4 + 525x^8 - 3307500x^2 - 7000x^6\right)n^2 \\
&+ 6\left(-19845000 + 10584000x^2 - 1365x^8 + 27650x^6 - 412650x^4\right)n \\
&+ 1151640x^4 - 24305400x^2 + 111132000 + 2244x^8 - 63940x^6.
\end{aligned}$$

It has a non positive maximum at $n = 3$ and $x < 2.619951007$, as well as for $C(n,x)$ since

$$C(3,x) = 2071440x^4 - 75028275x^2 - 144244800 + 14760x^8 - 78864x^6 < 0,$$

(because $C(3,x)$ has only 4.109378762 as positive root, recalling that $x < 2.619951007$) and

$$\frac{\partial C(3,x)}{\partial n} = 3920490x^4 + 26241x^8 - 138545505x^2 - 191498x^6 - 238593600 < 0,$$

(because $\frac{\partial C(3,x)}{\partial n}$ has only 4.191812255 as positive root and $x < 2.619951007$).

This means $\frac{\partial v(n,x)}{\partial n} > 0$. Theorem 13 is thus proved. □

We end this part by an immediate corollary.

Corollary 16. *For $n = 3$ and for $x \in (0, \infty)$, the following inequalities hold:*

$$\left(\frac{\sinh x}{x}\right)^3 - \cosh x > 4\left(\frac{\sinh x}{x}\right)^2 - \frac{8}{3}\cosh x - \frac{4}{3}.$$

Figure 3 provides graphical evidence of this inequality. We introduce the following function:

$$r(x) = \left(\frac{\sinh x}{x}\right)^3 - \cosh x - 4\left(\frac{\sinh x}{x}\right)^2 + \frac{8}{3}\cosh x + \frac{4}{3}.$$

This figure shows that $r(x) > 0$ for $x \in (0, 4)$ (the same is observed for any $x \in (0, \infty)$).

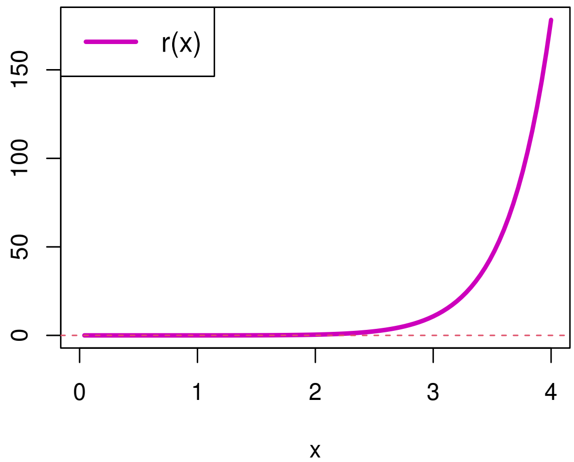

Figure 3. Curves of $r(x)$ for $x \in (0, 4)$

3. Proofs of the lemmas and corollaries

The proofs of all non-trivial lemmas and corollaries are given below, in the order in which they appear.

Proof of Lemma 5.

- Let us remark that

$$-5n^2\frac{\tanh x}{2x} + \frac{5n^2 + 20n + 44}{2} = \frac{(5n^2 + 20n + 44)\,x - 5n^2\tanh x}{2x}.$$

  The derivative of the numerator is

$$20n + 44 + 5n^2(\tanh x)^2.$$

  Since it is positive, we get the desired result.
- The derivative of $\ln\left(\frac{\sinh x}{x}\right)$ is

$$\left(\frac{\cosh x}{x} - \frac{\sinh x}{x^2}\right) x\,(\sinh x)^{-1}.$$

  Since it is positive, the function

$$n\,(5n + 22)\ln\left(\frac{\sinh x}{x}\right) - 10n - 22$$

  has only one real solution for a fixed n.

This ends the proof of this lemma. □

Proof of Lemma 6.

1. We have the following series representation of $\frac{\sinh x}{x}$:

$$\frac{\sinh x}{x} = 1 + \frac{x^2}{6} + \frac{x^4}{120} + \frac{x^6}{5040} + \ldots = \sum_{k\geq 0} \frac{x^{2k}}{(2k+1)!},$$

for $x \in (0, \infty)$. See [12] and [13]. On the other hand, the Taylor expansions of $\frac{\sinh x}{x}$ gives, for any $k \geq 1$,

$$\frac{\sinh x}{x} = 1 + \frac{x^2}{6} + \frac{x^4}{120} + \frac{x^6}{5040} + \ldots + \frac{x^{2k}}{(2k+1)!} + \frac{x^{2k+2}}{(2k+3)!} \frac{\sinh(\theta x)}{x},$$

for $\theta \in (0, 1)$.

We can remark that, for $x \in (0, x_n)$, we have

$$1 + \frac{x^2}{6} + \frac{x^4}{120} < \frac{\sinh x}{x}.$$

By an induction on n, we get

$$1 + \frac{nx^2}{6} + \frac{n(5n-2)x^4}{3600} < \left(\frac{\sinh x}{x}\right)^n.$$

2. Let us now consider the function

$$\frac{\tanh x}{x} - 1 + \frac{x^2}{3} - \frac{2x^4}{15} + \frac{17x^6}{315}.$$

Its derivative is

$$-\frac{1}{105} \frac{-105x + 105x(\tanh x)^2 + 105 \tanh x - 70x^3 + 56x^5 - 34x^7}{x^2}.$$

The numerator can be written as

$$-105x + 105x(\tanh x)^2 + 105 \tanh x - 70x^3 + 56x^5 - 34x^7.$$

Its derivative is

$$-210x(\tanh x)^3 + 210x \tanh x - 238x^6 - 210x^2 + 280x^4.$$

On the other hand, the following inequalities are provable:

$$x - \frac{x^3}{3} < \tanh x < x - \frac{x^3}{3} + \frac{2x^5}{15},$$

and they imply that

$$\begin{aligned} &- 105x + 105x(\tanh x)^2 + 105 \tanh x - 70x^3 + 56x^5 - 34x^7 \\ &< -210x\left(x - \frac{x^3}{3}\right)^3 + 210x\left(x - \frac{x^3}{3} + \frac{2x^5}{15}\right) - 238x^6 - 210x^2 + 280x^4 \\ &= \frac{70}{9}x^8(x-3)(x+3) < 0, \end{aligned}$$

since $x < x_n < x_2 = 2.1116\ldots$. As a result, the left inequality is proved.

Now, let us introduce the function

$$\tanh x - x + \frac{x^3}{3} - \frac{2x^5}{15} + \frac{17x^7}{315} - \frac{62x^9}{2835}.$$

Its derivative is

$$-(\tanh x)^2 + x^2 - \frac{2x^4}{3} + \frac{17}{45}x^6 - \frac{62}{315}x^8.$$

Owing to the previous left inequality, we deduce that

$$\begin{aligned} &-(\tanh x)^2 + x^2 - \frac{2x^4}{3} + \frac{17}{45}x^6 - \frac{62}{315}x^8 \\ &< -\left(x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17}{315}x^7\right)^2 + x^2 - \frac{2x^4}{3} + \frac{17}{45}x^6 - \frac{62}{315}x^8 \\ &= -\frac{1}{99225}x^{10}\left(5334 - 1428x^2 + 289x^4\right), \end{aligned}$$

which is non positive. This prove the right inequality.

3. Let us consider the function

$$\ln\left(\frac{\sinh x}{x}\right) - \frac{x^2}{6} + \frac{1}{180}x^4 - \frac{1}{2835}x^6.$$

Its derivative is

$$\frac{945x\cosh x - 945\sinh x - 315x^2\sinh x + 21x^4\sinh x - 2x^6\sinh x}{945x\sinh x}.$$

After simplifying by $\cosh x$, the numerator can be written as

$$\begin{aligned} &\left(-945 - 315x^2 + 21x^4 - 2x^6\right)\tanh x + 945x < \left(-315x^2 + 21x^4 - 2x^6\right)x \\ &= x^3\left(-315 + 21x^2 - 2x^4\right) < 0, \end{aligned}$$

because $-315 + 21x^2 - 2x^4$ has no root. Then this numerator is non positive for $x \in (0, x_n)$. The right inequality is demonstrated.

By the same way, let us introduce the function

$$\ln\left(\frac{\sinh x}{x}\right) - \frac{x^2}{6} + \frac{x^4}{180} - \frac{x^6}{2835} + \frac{x^8}{37800}.$$

Its derivative is

$$\frac{1}{4725}\frac{4725\cosh x - 4725\sinh x - 1575x^2\sinh x + 105x^4\sinh x - 10x^6\sinh x + x^8\sinh x}{x\sinh x}.$$

The derivative of the numerator is

$$\begin{aligned} &\left(8x^7 - 60x^5 + 420x^3 + 1575x\right)\sinh x + \left(x^8 - 10x^6 + 105x^4 - 1575x^2\right)\cosh x \\ &= \left[x^8 - 10x^6 + 105x^4 - 1575x^2 + \left(8x^7 - 60x^5 + 420x^3 + 1575x\right)\tanh x\right]\cosh x. \end{aligned}$$

The bracket terrm is upper than

$$\begin{aligned} &x^8 - 10x^6 + 105x^4 - 1575x^2 + \left(8x^7 - 60x^5 + 420x^3 + 1575x\right)\left(x - \frac{1}{3}x^3\right) \\ &= 29x^8 - 210x^6 - \frac{8}{3}x^{10}, \end{aligned}$$

which is non negative for $x \in (0, x_n)$ (because it has no roots). This proves the left inequality.

This ends the proof of this lemma. □

Proof of Corollary 7. For $n = 2$, Theorem 4 implies that

$$\frac{1}{64}\frac{(\sinh x)^2}{x^2} + \frac{1}{64}\frac{\tanh x}{x} - \frac{1}{32} > \frac{1}{27}\frac{\sinh x}{x} + \frac{1}{54}\frac{\tanh x}{x} - \frac{1}{18}.$$

Furthermore, the last inequality is sharper than Equation (1), i.e.,

$$\frac{1}{2}\left[\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x}\right] > \frac{1}{3}\left[\frac{2\sinh x}{x} + \frac{\tanh x}{x}\right].$$

The desired result is obtained. □

Proof of Corollary 8. For $n = 2$, owing to [11, Theorem 1], we obtain

$$\frac{(\sinh x)^2}{x^2} + \frac{\tanh x}{x} - 2 > kx^4, \quad k \leq \frac{8}{45}.$$

On the other hand, we have

$$\frac{(\sinh x)^2}{x^2} + \frac{\tanh x}{x} - 2 < kx^4, \quad k > \frac{8}{45}, \quad x \leq x_{2,k},$$

where $x_{2,k}$ is the positive root of $\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} - 2 - kx^4 = 0$.
The stated results are derived from Theorem 4. □

Proof of Lemma 10.

- It is enough to remark that
$$\frac{x}{\sinh x} < 1 < \frac{n(5n-2)}{(n-1)(5n-7)}.$$
- The derivative of
$$-5\frac{x}{\tanh x} + \frac{5n^2 + 15n - 14}{n(n-1)}$$
is
$$-5\frac{\sinh x \cosh x - x}{(\sinh x)^2}$$
and it is negative. That means the function has only one solution for fixed n.

The proof is completed. □

Proof of Lemma 11.

1. Let us consider the following function:
$$\left(\frac{x}{\sinh x}\right)^n - 1 + \frac{nx^2}{6} - \left(\frac{7}{360}n + \frac{1}{72}n(n-1)\right)x^4.$$
Its derivative is given by
$$\frac{1}{\sinh x} - \frac{x\cosh x}{(\sinh x)^2}\frac{\sinh x}{x} + \frac{nx}{3} - 4\left(\frac{7}{360}n + \frac{1}{72}n(n-1)\right)x^3.$$
Let us notice that the following inequality holds:
$$\frac{1}{\sinh x} - \frac{x\cosh x}{(\sinh x)^2} < -\frac{1}{3}x + \frac{7}{90}x^3 - \frac{31}{2520}x^5.$$

We thus deduce that the derivative is lower than

$$n\left(-\frac{x}{3}+\frac{7}{90}x^3-\frac{31}{2520}x^5\right)+\frac{nx}{3}-4\left(\frac{7}{360}n+\frac{1}{72}n\left(n-1\right)\right)x^3$$
$$=-\frac{31}{2520}nx^5-\frac{n\left(n-1\right)}{18}x^3<0.$$

Therefore, we assert that

$$\left(\frac{x}{\sinh x}\right)^n-1+\frac{nx^2}{6}-\left(\frac{7}{360}n+\frac{1}{72}n\left(n-1\right)\right)x^4<0,$$

implying the desired right inequality.

On the other hand, let us consider

$$\left(\frac{x}{\sinh x}\right)^n-1+\frac{nx^2}{6}-\left(\frac{1}{180}n+\frac{1}{72}n^2\right)x^4-\left(-\frac{1}{2835}n-\frac{1}{1080}n^2-\frac{1}{1296}n^3\right)x^6,$$

and use

$$\left(\frac{x}{\sinh x}\right)^{n+1}=\left(\frac{x}{\sinh x}\right)^n\left(\frac{x}{\sinh x}\right),$$

in order to prove by induction on n.

To obtain the left inequality, it is sufficient to prove the following inequality:

$$\frac{x}{\sinh x}\left[1-\frac{nx^2}{6}+\left(\frac{1}{180}n+\frac{1}{72}n^2\right)x^4+\left(-\frac{1}{2835}n-\frac{1}{1080}n^2-\frac{1}{1296}n^3\right)x^6\right]$$
$$>1-\left(\frac{41}{9072}n+\frac{31}{15120}+\frac{7}{2160}n^2+\frac{1}{1296}n^3\right)x^6$$
$$-\left(-\frac{n}{30}-\frac{7}{360}-\frac{1}{72}n^2\right)x^4-\left(\frac{n+1}{6}\right)x^2.$$

Let us recall that

$$\frac{x}{\sinh x}>1-\frac{1}{6}x^2+\frac{7}{360}x^4-\frac{31}{15120}x^6+\frac{127}{604800}x^8-\frac{73}{3421440}x^{10}.$$

We thus derive

$$\left(1-\frac{1}{6}x^2+\frac{7}{360}x^4-\frac{31}{15120}x^6+\frac{127}{604800}x^8-\frac{73}{3421440}x^{10}\right)\times$$
$$\left(1-\frac{nx^2}{6}+\left(\frac{1}{180}n+\frac{1}{72}n^2\right)x^4+\left(-\frac{1}{2835}n-\frac{1}{1080}n^2-\frac{1}{1296}n^3\right)x^6\right)$$
$$-1+\left(\frac{41}{9072}n+\frac{31}{15120}+\frac{7}{2160}n^2+\frac{1}{1296}n^3\right)x^6+\left(-\frac{n}{30}-\frac{7}{360}-\frac{1}{72}n^2\right)x^4$$
$$+\left(\frac{n+1}{6}\right)x^2=\frac{1}{5431878144000}x^8A(n,x),$$

where

$$A(n,x) = 2762242560n + 2305195200n^2 + 698544000n^3 - 289230480nx^2 - 252473760x^2n^2$$
$$- 81496800x^2n^3 + 29580912x^4n + 26153820x^4n^2 + 8593200x^4n^3 - 1046196x^6n$$
$$- 2665782x^6n^2 - 880110x^6n^3 + 40880x^8n + 107310x^8n^2 + 89425x^8n^3 + 1140622560$$
$$- 115894800x^2$$
$$= \left(-880110x^6 - 81496800x^2 + 89425x^8 + 698544000 + 8593200x^4\right) n^3$$
$$+ \left(2305195200 + 26153820x^4 + 107310x^8 - 2665782x^6 - 252473760x^2\right) n^2$$
$$+ \left(2762242560 - 1046196x^6 + 40880x^8 + 29580912x^4 - 289230480x^2\right) n$$
$$+ 1140622560 - 115894800x^2.$$

The derivative of $A(n,x)$ with respect to n is

$$\frac{\partial A(n,x)}{\partial n} = 3\left(-880110x^6 - 81496800x^2 + 89425x^8 + 698544000 + 8593200x^4\right) n^2$$
$$+ 2\left(2305195200 + 26153820x^4 + 107310x^8 - 2665782x^6 - 252473760x^2\right) n$$
$$+ 2762242560 - 1046196x^6 + 40880x^8 + 29580912x^4 - 289230480x^2.$$

This derivative is positive for $n \geq 2$, implying that $A(n,x) > 0$. The left inequality is proved.

2. Let us now consider the difference:

$$\frac{x}{\tanh x} - \left(1 + \frac{x^2}{3} - \frac{x^4}{45} + \frac{2}{945}x^6 - \frac{1}{4725}x^8 + \frac{2}{93555}x^{10}\right).$$

Its derivative is given by

$$\frac{1}{\tanh x} - \frac{x\left[1 - (\tanh x)^2\right]}{(\tanh x)^2} - \frac{2x}{3} + \frac{4}{45}x^3 - \frac{4}{315}x^5 + \frac{8}{4725}x^7 - \frac{4}{18711}x^9.$$

Moreover, we may easily express

$$\frac{1}{\tanh x} - \frac{x\left[1 - (\tanh x)^2\right]}{(\tanh x)^2} = \frac{\sinh(2x) - 2x}{\cosh(2x) - 1} = \coth(x) - \frac{x}{(\sinh x)^2}.$$

Let us recall that

$$\coth(x) < \frac{1}{x} + \frac{1}{3}x - \frac{1}{45}x^3 + \frac{2}{945}x^5 - \frac{1}{4725}x^7 + \frac{2}{93555}x^9$$

and

$$\frac{x}{(\sinh x)^2} > \frac{1}{x} - \frac{1}{3}x + \frac{1}{15}x^3 - \frac{2}{189}x^5 + \frac{1}{675}x^7 - \frac{2}{10395}x^9.$$

Thus, we have

$$\frac{1}{\tanh x} - \frac{x\left[1 - (\tanh x)^2\right]}{(\tanh x)^2}$$
$$< \frac{1}{x} + \frac{1}{3}x - \frac{1}{45}x^3 + \frac{2}{945}x^5 - \frac{1}{4725}x^7 + \frac{2}{93555}x^9$$
$$- \left(\frac{1}{x} - \frac{1}{3}x + \frac{1}{15}x^3 - \frac{2}{189}x^5 + \frac{1}{675}x^7 - \frac{2}{10395}x^9\right)$$
$$= \frac{2x}{3} - \frac{4}{45}x^3 + \frac{4}{315}x^5 - \frac{8}{4725}x^7 \frac{4}{18711}x^9 < 0.$$

We then derive

$$\frac{x}{\tanh x} - \left(1 + \frac{x^2}{3} - \frac{x^4}{45} + \frac{2}{945}x^6 - \frac{1}{4725}x^8 + \frac{2}{93555}x^{10}\right) < 0.$$

The desired inequality is obtained.

The lemma is proved. □

Proof of Corollary 12. By Theorem 9, we get

$$\frac{x^2}{16(\sinh x)^2} + \frac{x}{16\tanh x} < \frac{x}{3\sinh x} + \frac{x}{6\tanh x} - \frac{3}{8}.$$

The last inequality is sharper than the one in Equation (1); we have

$$\frac{1}{2}\left[\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x}\right] > \frac{1}{3}\left[\frac{2x}{\sinh x} + \frac{x}{\tanh x}\right].$$

This ends the proof. □

Proof of Lemma 14.

- It suffices to notice that

$$5\cosh x > 5 > 5 - \frac{3(10n-7)}{n^2} = \frac{5n^2 - 30n + 21}{3n^2\,(5n-7)^2}.$$

- The derivative of $\ln\left(\frac{\sinh x}{x}\right)$ is

$$\frac{\left(\frac{\cosh x}{x} - \frac{\sinh x}{x^2}\right)x}{\sinh x} = \coth x - \frac{1}{x}.$$

  Since it is positive, the function has only one solution for fixed n.

This ends the proof. □

Proof of Lemma 15. The proof is analogous to that given in Lemma 11 for the two first points. The third has already been proved in Lemma 6. □

4. Conclusions

Possibilities of going further and improving the above three theorems are discussed in this section.

4.1. About Theorem 4

In [11, Theorem 2], it is proved that, for any $x \neq 0$, $\lambda > 0$, $\mu > 0$, $q > 0$ and $p \geq \frac{2q\mu}{\lambda}$, the following inequality holds:

$$f_{wd} = \frac{\lambda}{\mu+\lambda}\left(\frac{\sinh x}{x}\right)^p + \frac{\mu}{\mu+\lambda}\left(\frac{\tanh x}{x}\right)^q > 1.$$

For the particular case $q = 1$, $p = n$, $\mu = n$, $\lambda = 2$, we find

$$f(n,x) = \left(\frac{\sinh x}{x}\right)^n + \frac{n}{2}\left(\frac{\tanh x}{x}\right) - \frac{n+2}{2} > 0.$$

Thus, we may naturally expect to discover other properties for f_{wd} similar to those provided for $f(n,x)$.

4.2. About Theorem 9

It seems that the following inequalities are true:

$$\frac{x^3}{(\sinh x)^3}+\frac{3x}{2\tanh x}-\frac{5}{2}<kx^4,\quad k\geq\frac{13}{120},$$

and

$$\frac{x^3}{(\sinh x)^3}+\frac{3x}{2\tanh x}-\frac{5}{2}>kx^4,\quad k<\frac{13}{120},\quad x\leq x_{3,k},$$

where $x_{3,k}$ is the positive root of $\frac{x^3}{(\sinh x)^3}+\frac{3x}{2\tanh x}-\frac{5}{2}-kx^4=0$.
Hence, by Theorem 9, we get

$$g(n,x)=\left(\frac{x}{\sinh x}\right)^n+\frac{nx}{2\tanh x}-\frac{n+2}{2}<kx^4,\quad k\geq\frac{n(5n-2)}{360},$$

and

$$g(n,x)=\left(\frac{x}{\sinh x}\right)^n+\frac{nx}{2\tanh x}-\frac{n+2}{2}>kx^4,\quad k<\frac{n(5n-2)}{360},\quad x\leq x_{n,k},$$

where $x_{n,k}$ is the positive root of $\left(\frac{x}{\sinh x}\right)^n+\frac{nx}{2\tanh x}-\frac{n+2}{2}-kx^4=0$.

4.3. About Theorem 13

It seems that the following inequalities hold:

$$h(3,x)=\left(\frac{\sinh x}{x}\right)^3-\cosh x>kx^4,\quad k\leq\frac{1}{15},$$

and

$$h(3,x)=\left(\frac{\sinh x}{x}\right)^3-\cosh x<kx^4,\quad k>\frac{1}{15},\quad x\leq x_{3,k},$$

where $x_{3,k}$ is the positive root of $\left(\frac{\sinh x}{x}\right)^3-\cosh x-kx^4=0$.
Thus, by Theorem 13, we have

$$h(n,x)=\left(\frac{\sinh x}{x}\right)^n-\frac{n\cosh x}{3}+\frac{n-3}{3}>kx^4,\quad k\leq\frac{n(5n-7)}{360},$$

and

$$h(n,x)=\left(\frac{\sinh x}{x}\right)^n-\frac{n\cosh x}{3}+\frac{n-3}{3}<kx^4,\quad k>\frac{n(5n-7)}{360},\quad x\leq x_{n,k},$$

where $x_{n,k}$ is the positive root of $\left(\frac{\sinh x}{x}\right)^n-\frac{n\cosh x}{3}+\frac{n-3}{3}-kx^4=0$.

The rigorous proofs and improvements of these results require further investigation. We leave this to future work.

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References

[1] Chen, C.-P., & Cheung, W.-S. (2012). Sharpness of Wilker and Huygens type inequalities. *Journal of Inequalities and Applications*, *2012*, 72. https://doi.org/10.1186/1029-242X-2012-72

[2] Chen, C.-P., & Malesevic, B. (2023). Sharp inequalities related to the Adamovic-Mitrinovic, Cusa, Wilker, and Huygens results. *Filomat*, *37*(19), 6319-6334. https://doi.org/10.2298/FIL2319319C

[3] Chouikha, A. R. (2020). New sharp inequalities related to classical trigonometric inequalities. *Journal of Inequalities and Special Functions*, *11*(4), 27-35. ISSN: 2217-4303. Retrieved from http://www.ilirias.com/jiasf

[4] Chouikha, A. R. (2023). On the 1-parameter trigonometric and hyperbolic inequalities chains. Retrieved from https://hal.science/hal-04435124

[5] Mortici, C. (2011). The natural approach of Wilker-Cusa-Huygens inequalities. *Mathematical Inequalities and Applications, 14*, 535-541.

[6] Neuman, E. (2012). On Wilker and Huygens type inequalities. *Mathematical Inequalities and Applications, 15*(2), 271-279.

[7] Neuman, E., & Sandor, J. (2010). On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker and Huygens inequalities. *Mathematical Inequalities and Applications, 13*(4), 715-723.

[8] Wu, S.-H., & Srivastava, H. M. (2008). A further refinement of Wilker's inequality. *Integral Transforms and Special Functions, 19*, 757-765.

[9] Zhu, L., & Malesevic, B. (2021). New Wilker-type and Huygens-type inequalities. *Hacettepe Journal of Mathematics and Statistics, 50*(1), 46-62.

[10] Chouikha, A. R., & Chesneau, C. (2023). Contributions to trigonometric 1-parameter inequalities. Retrieved from https://hal.science/hal-04500965

[11] Wu, S., & Debnath, L. (2012). Wilker-type inequalities for hyperbolic functions. *Applied Mathematics Letters, 25*(5), 837-842.

[12] Abramowitz, M., & Stegun, I. (1972). *Handbook of mathematical functions*. Washington, DC: National Bureau of Standards.

[13] Gradshteyn, I. S., & Ryzhik, I. M. (2015). *Table of integrals, series, and products* (8th ed.). San Diego, CA: Academic Press.