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Convergence analysis of tunable product sequences and series with two tuning parameters and two functions

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Abstract: The study of innovative sequences and series is important in several fields. In this article, we examine the convergence properties of a particular product series that offers adaptability through two parameters and two functions. Based on this analysis, we extend our investigation to a related series. Our main theorems are proved in detail and include several new intermediate results that can be used for other convergence analysis purposes. This is particularly the case for a generalized version of the Riemann sum formula. Several precise examples are presented and discussed, including one related to the gamma function.

Keywords: mathematical analysis; product sequences; Riemann sum formula; series; Cauchy root convergence rule.

MSC: 40A05.

1. Introduction

Because it can be used in a variety of scientific contexts, convergence analysis of tunable sequences (and series) is of interest in many fields. In particular, in mathematical analysis, it provides valuable information about the limits and properties of functions, thus facilitating the development of robust computational algorithms and numerical methods. In fields such as statistical modeling, where manipulation of functions and parameters plays a crucial role, convergence analysis of versatile sequences helps to improve algorithmic efficiency and predictive accuracy. In addition, applications in engineering, finance and physics benefit from the knowledge gained from convergence analysis, enabling the optimization of systems, the prediction of complex phenomena and the design of new approaches to computing. Old and new references on this vast topic include [1–12].

The main objective of this article is to study the behavior of specific product sequences and series governed by adaptive functions and parameters. It can be divided into three interrelated parts. In the first part, we study the convergence behavior of a special sequence of products denoted by $u_n(g,h;\alpha,\beta)$. More precisely, this sequence is defined as a product of terms involving two functions, g and h, each modulated by an integer variable, n, and two tuning parameters, α and β. It has the following general form:

$$''u_n(g,h;\alpha,\beta)=\prod_{k=1}^{n}\left\{1+\frac{1}{n^{\beta(1-\alpha)}}g\left[\frac{1}{n^{\alpha}}h\left(\frac{k}{n}\right)\right]\right\}^{1/n^{(1-\alpha)(1-\beta)}}''.$$

Under certain conditions on g, h, α, and β, we establish precise convergence results for $u_n(g,h;\alpha,\beta)$. The techniques used include appropriate decompositions, the triangular inequality, the Riemann sum formula, differentiation rules, logarithmic inequalities, and the Taylor inequality. In particular, an extended version of the Riemann sum formula is established and may be of independent interest. In this sense, we follow the spirit of the works in [2–4,6–9].

In the second part, based on the information obtained from the product sequence analysis, we extend our investigation to a corresponding specific series, denoted $\Xi(g,h;\alpha,\beta)$. Logically, this series contains the same functions g and h, as well as the tuning parameters α and β, although in a different mathematical formulation. It has the following general form:

$$''\Xi(g,h;\alpha,\beta) = \sum_{n=1}^{+\infty}\prod_{k=1}^{n}\left\{1+\frac{1}{n^{\beta(1-\alpha)}}g\left[\frac{1}{n^{\alpha}}h\left(\frac{k}{n}\right)\right]\right\}^{1/n^{\alpha\beta-\alpha-\beta}}''.$$

We determine some conditions on g, h, α, and β under which it converges. The Cauchy root convergence rule is a key tool here. In the third and last part, we give three examples to illustrate theoretically and graphically the convergence of these sequences and series. One example concerns the gamma function.

The remainder of the article is organized as follows: Section 2 discusses the existence of $u_n(g,h;\alpha,\beta)$ and introduces the two main theorems on the convergence conditions for $u_n(g,h;\alpha,\beta)$ and $\Xi(g,h;\alpha,\beta)$. The detailed proofs for each theorem are given in Sections 3 and 4, respectively. Concrete examples are investigated in Section 5. Finally, Section 6 concludes the article. There is also an appendix that provides a perspective for further investigation.

2. Results

In this section, we first discuss the mathematical validity of the considered sequence as a preliminary study. Then we examine its convergence as well as the corresponding series as the main results.

2.1. Preliminary study

The result below presents the basic assumptions and two complementary sets of conditions that give a mathematical sense to the main sequence.

Proposition 1. *Let g and h be two continuous functions, with h defined on $[0,1]$ and g defined on $[\min(0,I_h),\max(0,J_h)]$, where $I_h=\inf_{t\in[0,1]}h(t)$ and $J_h=\sup_{t\in[0,1]}h(t)$. For any $n\in\mathbb{N}\setminus\{0\}$, $\alpha\geq 0$ and $\beta\in\mathbb{R}$, let us set*

$$u_n(g,h;\alpha,\beta)=\prod_{k=1}^{n}\left\{1+\frac{1}{n^{\beta(1-\alpha)}}g\left[\frac{1}{n^{\alpha}}h\left(\frac{k}{n}\right)\right]\right\}^{1/n^{(1-\alpha)(1-\beta)}}.$$

Then $u_n(g,h;\alpha,\beta)$ is well-defined in the mathematical sense under one of the two following conditions:

Cond 1: *g is non-negative.*
Cond 2: *$\beta(1-\alpha)\geq 0$ and $n\geq K_g^{1/[\beta(1-\alpha)]}$, where $K_g=\sup_{t\in[\min(0,I_h),\max(0,J_h)]}|g(t)|$.*

Proof. Since the exponent $1/n^{(1-\alpha)(1-\beta)}$ is real number, the quantity $u_n(g,h;\alpha,\beta)$ is well-defined in the mathematical sense if the term under the curly bracket is non-negative, i.e., if

$$\inf_{k\in\{1,\ldots,n\}}\frac{1}{n^{\beta(1-\alpha)}}g\left[\frac{1}{n^{\alpha}}h\left(\frac{k}{n}\right)\right]\geq -1. \tag{1}$$

Under **Cond 1**, i.e., g is non-negative, Equation (1) is satisfied; we obviously have

$$\inf_{k\in\{1,\ldots,n\}}\frac{1}{n^{\beta(1-\alpha)}}g\left[\frac{1}{n^{\alpha}}h\left(\frac{k}{n}\right)\right]\geq 0\geq -1.$$

Let us now justify **Cond 2**. Since $\alpha\geq 0$, for any $k=1,\ldots,n$, we have $(1/n^{\alpha})h(k/n)\in[\min(0,I_h),\max(0,J_h)]$. Therefore, under **Cond 2**, we have

$$\left|\frac{1}{n^{\beta(1-\alpha)}}g\left[\frac{1}{n^{\alpha}}h\left(\frac{k}{n}\right)\right]\right|\leq\frac{1}{n^{\beta(1-\alpha)}}K_g\leq 1,$$

which implies Equation (1). This ends the proof. □

Obviously, **Cond 1** and **Cond 2** are satisfied for a wide range of parameters and functions. Moreover, for both of them, it is important to note that β can be negative.

We are now in a position to present our main results, which mainly concern some convergence properties of $u_n(g,h;\alpha,\beta)$ with respect to n.

2.2. Main results

The result below examines the convergence of $u_n(g,h;\alpha,\beta)$ with respect to n.

Theorem 2. *Let g and h be two continuous functions, with h defined on $[0,1]$ and g defined on $[\min(0,I_h),\max(0,J_h)]$, where $I_h=\inf_{t\in[0,1]}h(t)$ and $J_h=\sup_{t\in[0,1]}h(t)$. For any $n\in\mathbb{N}\setminus\{0\}$, $\alpha\geq 0$ and $\beta\in\mathbb{R}$, let us set*

$$u_n(g,h;\alpha,\beta)=\prod_{k=1}^{n}\left\{1+\frac{1}{n^{\beta(1-\alpha)}}g\left[\frac{1}{n^{\alpha}}h\left(\frac{k}{n}\right)\right]\right\}^{1/n^{(1-\alpha)(1-\beta)}}$$

*under **Cond 1** or **Cond 2** as described in Proposition 1, so that $u_n(g,h;\alpha,\beta)$ is well-defined. Then, the limit results below are satisfied.*

Case 1: *If $\alpha>0$, $\beta(1-\alpha)\geq 0$, $g(0)=0$, and g is twice differentiable with $\sup_{t\in[\min(0,I_h),\max(0,J_h)]}|g''(t)|<+\infty$, then we have*

$$\lim_{n\to+\infty}u_n(g,h;\alpha,\beta)=\exp\left[g'(0)\int_0^1 h(t)dt\right].$$

Case 2: *If $\alpha=0$ and $\beta>0$, then we have*

$$\lim_{n\to+\infty}u_n(g,h;\alpha,\beta)=\exp\left\{\int_0^1 g[h(t)]dt\right\}.$$

Case 3: *If $\alpha=0$ and $\beta=0$, then we have*

$$\lim_{n\to+\infty}u_n(g,h;\alpha,\beta)=\exp\left[\int_0^1\log\left\{1+g[h(t)]\right\}dt\right].$$

This proposition shows how the values of α and β, and the nature of g and h, affect $\lim_{n\to+\infty}u_n(g,h;\alpha,\beta)$; the limits obtained are really different in each case. The proof uses various convergence techniques and inequalities, including the Riemann sum formula, logarithmic inequalities and the Taylor inequality.

The result below completes Theorem 2 by investigating the convergence of a special series based on $u_n(g,h;\alpha,\beta)$. It is thus adaptable with two tuning parameters and two functions.

Theorem 3. *Let g and h be two continuous functions, with h defined on $[0,1]$ and g defined on $[\min(0,I_h),\max(0,J_h)]$, where $I_h=\inf_{t\in[0,1]}h(t)$ and $J_h=\sup_{t\in[0,1]}h(t)$. For any $n\in\mathbb{N}\setminus\{0\}$, $\alpha\geq 0$ and $\beta\in\mathbb{R}$, let us consider **Cond 1** and*

***Cond 3**: "**Cond 2** with $K_g<1$",*

*where **Cond 1** and **Cond 2** are described in Proposition 1, and set*

$$\Xi(g,h;\alpha,\beta)=\sum_{n=1}^{+\infty}\prod_{k=1}^{n}\left\{1+\frac{1}{n^{\beta(1-\alpha)}}g\left[\frac{1}{n^{\alpha}}h\left(\frac{k}{n}\right)\right]\right\}^{1/n^{\alpha\beta-\alpha-\beta}}$$

*under **Cond 1** or **Cond 3**. Then $\Xi(g,h;\alpha,\beta)$ is well-defined and the results below hold.*

Case I: *If $\alpha>0$, $\beta(1-\alpha)\geq 0$, $g(0)=0$, g is twice differentiable with $\sup_{t\in[\min(0,I_h),\max(0,J_h)]}|g''(t)|<+\infty$, and either $g'(0)<0$ and $\int_0^1 h(t)dt>0$, or $g'(0)>0$ and $\int_0^1 h(t)dt<0$, then $\Xi(g,h;\alpha,\beta)$ is convergent.*

Case II: *If $\alpha=0$, $\beta>0$, and $\int_0^1 g[h(t)]dt<0$, which has a sense under **Cond 3** only, then $\Xi(g,h;\alpha,\beta)$ is convergent.*

Case III: *If $\alpha=0$, $\beta=0$, and $\int_0^1\log\{1+g[h(t)]\}dt<0$, which has a sense under **Cond 3** only, then $\Xi(g,h;\alpha,\beta)$ is convergent.*

The proof is mainly based on Theorem 2 and the Cauchy root convergence rule. We can also state that $\Xi(g,h;\alpha,\beta)$ is divergent under one of the following case:

1. $\alpha > 0$, $\beta(1-\alpha) \geq 0$, $g(0) = 0$, g is twice differentiable with $\sup_{t\in[\min(0,I_h),\max(0,J_h)]} |g''(t)| < +\infty$, and either $g'(0) < 0$ and $\int_0^1 h(t)dt < 0$, or $g'(0) > 0$ and $\int_0^1 h(t)dt > 0$.
2. $\alpha = 0$, $\beta > 0$, and $\int_0^1 g[h(t)]dt > 0$, which is true under **Cond 1**.
3. $\alpha = 0$, $\beta = 0$, and $\int_0^1 \log\{1 + g[h(t)]\}\, dt > 0$, which is true under **Cond 1**.

In fact, Theorem 3 holds if we consider **Cond 2** instead of **Cond 3** (so without $K_g < 1$), but we need to replace $\Xi(g,h;\alpha,\beta)$ by

$$\Xi_*(g,h;\alpha,\beta,m) = \sum_{n=m}^{+\infty} \prod_{k=1}^{n} \left\{1 + \frac{1}{n^{\beta(1-\alpha)}} g\left[\frac{1}{n^{\alpha}} h\left(\frac{k}{n}\right)\right]\right\}^{1/n^{\alpha\beta-\alpha-\beta}},$$

where m is an integer greater than the smallest integer greater than $K_g^{1/[\beta(1-\alpha)]}$.

The rest of the article is devoted to the proofs of these theorems and examples.

3. Proof of Theorem 2

3.1. Intermediate results

The proof is based on some intermediate results which may be of independent interest

The proposition below can be described as an extended Riemann sum formula, using composition functions and a tuning parameter. We thus make a contribution to the field, with potential applications beyond the purposes of this study (see again [2–4,6–9]).

Proposition 4. *Let g and h be two continuous functions, with h defined on $[0,1]$ and g defined on $[\min(0,I_h),\max(0,J_h)]$, where $I_h = \inf_{t\in[0,1]} h(t)$ and $J_h = \sup_{t\in[0,1]} h(t)$. For any $\gamma \geq 0$, let us set*

$$v_n(g,h;\gamma) = \frac{1}{n^{1-\gamma}} \sum_{k=1}^{n} g\left[\frac{1}{n^{\gamma}} h\left(\frac{k}{n}\right)\right].$$

Then, the limit results below are satisfied.

- *If $\gamma > 0$, $g(0) = 0$, g is twice differentiable, and $\sup_{t\in[\min(0,I_h),\max(0,J_h)]} |g''(t)| < +\infty$, then we have*

$$\lim_{n\to+\infty} v_n(g,h;\gamma) = g'(0)\int_0^1 h(t)dt.$$

- *If $\gamma = 0$, then we have*

$$\lim_{n\to+\infty} v_n(g,h;\gamma) = \int_0^1 g[h(t)]dt.$$

Proof. Let us first consider the case $\gamma > 0$, which is the more technical one. It follows from a suitable decomposition and the triangular inequality that

$$\begin{aligned}
&\left|v_n(g,h;\gamma) - g'(0)\int_0^1 h(t)dt\right| \\
&= \left|\frac{1}{n^{1-\gamma}} \sum_{k=1}^{n} g\left[\frac{1}{n^{\gamma}} h\left(\frac{k}{n}\right)\right] - g'(0)\frac{1}{n}\sum_{k=1}^{n} h\left(\frac{k}{n}\right) + g'(0)\frac{1}{n}\sum_{k=1}^{n} h\left(\frac{k}{n}\right) - g'(0)\int_0^1 h(t)dt\right| \\
&\leq A_n + |g'(0)|B_n, \qquad (2)
\end{aligned}$$

where

$$A_n = \left|\frac{1}{n^{1-\gamma}} \sum_{k=1}^{n} g\left[\frac{1}{n^{\gamma}} h\left(\frac{k}{n}\right)\right] - g'(0)\frac{1}{n}\sum_{k=1}^{n} h\left(\frac{k}{n}\right)\right|$$

and

$$B_n = \left| \frac{1}{n} \sum_{k=1}^{n} h\left(\frac{k}{n}\right) - \int_0^1 h(t)dt \right|.$$

Let us investigate an upper bound for A_n. By using again the triangular inequality, we have

$$\begin{aligned} A_n &= \frac{1}{n^{1-\gamma}} \left| \sum_{k=1}^{n} \left\{ g\left[\frac{1}{n^\gamma} h\left(\frac{k}{n}\right)\right] - g'(0)\frac{1}{n^\gamma} h\left(\frac{k}{n}\right) \right\} \right| \\ &\leq \frac{1}{n^{1-\gamma}} \sum_{k=1}^{n} \left| g\left[\frac{1}{n^\gamma} h\left(\frac{k}{n}\right)\right] - g'(0)\frac{1}{n^\gamma} h\left(\frac{k}{n}\right) \right|. \end{aligned} \quad (3)$$

Since $g(0) = 0$ and $\sup_{t\in[\min(0,I_h),\max(0,J_h)]} |g''(t)| < \infty$, by setting $M = \sup_{t\in[\min(0,I_h),\max(0,J_h)]} |g''(t)|$, the Taylor inequality at the point 0 implies that, for any $t \in [\min(0, I_h), \max(0, J_h)]$,

$$|g(t) - tg'(0)| = |g(t) - g(0) - tg'(0)| \leq \frac{M}{2}t^2.$$

Therefore, by applying this inequality with $t = (1/n^\gamma)h(k/n) \in [\min(0, I_h), \max(0, J_h)]$ into Equation (3) and introducing $L_h = \sup_{t\in[0,1]} |h(t)|$, we have

$$A_n \leq \frac{M}{2} \left\{ \frac{1}{n^{1-\gamma}} \sum_{k=1}^{n} \left[\frac{1}{n^\gamma} h\left(\frac{k}{n}\right)\right]^2 \right\} \leq \frac{M}{2} L_h^2 \frac{1}{n^\gamma}.$$

Since $\gamma > 0$, we have

$$0 \leq \lim_{n\to+\infty} A_n \leq \frac{M}{2} L_h^2 \lim_{n\to+\infty} \frac{1}{n^\gamma} = 0,$$

so $\lim_{n\to+\infty} A_n = 0$.

Concerning B_n, the analysis is more direct. By an application of the Riemann sum formula, we have

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} h\left(\frac{k}{n}\right) = \int_0^1 h(t)dt,$$

which implies that $\lim_{n\to+\infty} B_n = 0$.

Therefore, based on Equation (2), we have

$$0 \leq \lim_{n\to+\infty} \left| v_n(g, h; \gamma) - g'(0) \int_0^1 h(t)dt \right| \leq \lim_{n\to+\infty} A_n + |g'(0)| \lim_{n\to+\infty} B_n = 0,$$

implying that

$$\lim_{n\to+\infty} v_n(g, h; \gamma) = g'(0) \int_0^1 h(t)dt.$$

For the case $\gamma = 0$, an immediate application of the Riemann sum formula to the composition function $g \circ h$ gives

$$\begin{aligned} \lim_{n\to+\infty} v_n(g, h; \gamma) &= \lim_{n\to+\infty} \frac{1}{n^{1-\gamma}} \sum_{k=1}^{n} g\left[\frac{1}{n^\gamma} h\left(\frac{k}{n}\right)\right] = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} (g \circ h)\left(\frac{k}{n}\right) \\ &= \int_0^1 (g \circ h)(t)dt = \int_0^1 g[h(t)]dt. \end{aligned}$$

This ends the proof. □

This result will be at the center of the proof of Theorem 2.

The proposition below examines a special "triple composition" version of the previous result, which yields manageable limits.

Proposition 5. *Let g and h be two continuous functions, with h defined on $[0,1]$ and g defined on $[\min(0, I_h), \max(0, J_h)]$, where $I_h = \inf_{t\in[0,1]} h(t)$ and $J_h = \sup_{t\in[0,1]} h(t)$. In addition, let T be a continuous transform (or function) defined on the image set $g([\min(0, I_h), \max(0, J_h)])$. For any $\gamma \geq 0$, let us consider*

$$w_n(g, h, T; \gamma) = \frac{1}{n^{1-\gamma}} \sum_{k=1}^{n} T\left\{g\left[\frac{1}{n^{\gamma}} h\left(\frac{k}{n}\right)\right]\right\}.$$

Then, the limit results below are satisfied.

- *If $\gamma > 0$, $g(0) = 0$, g is twice differentiable, $\sup_{t\in[\min(0,I_h),\max(0,J_h)]} |g''(t)| < +\infty$, $T(0) = 0$, T is twice differentiable, $T'(0) = 0$, and $\sup_{t\in g([\min(0,I_h),\max(0,J_h)])} |T''(t)| < \infty$, then we have*

$$\lim_{n\to+\infty} w_n(g, h, T; \gamma) = 0.$$

- *If $\gamma = 0$, then we have*

$$\lim_{n\to+\infty} w_n(g, h, T; \gamma) = \int_0^1 T\{g[h(t)]\} dt.$$

Proof. If $\gamma > 0$, it follows from Proposition 4 applied with the composition function $T \circ g$ instead of g that

$$\lim_{n\to+\infty} w_n(g, h, T; \gamma) = \lim_{n\to+\infty} \frac{1}{n^{1-\gamma}} \sum_{k=1}^{n} (T \circ g) \left[\frac{1}{n^{\gamma}} h\left(\frac{k}{n}\right)\right] = (T \circ g)'(0) \int_0^1 h(t)dt.$$

We have $(T \circ g)'(x) = g'(x)(T' \circ g)(x)$. Since $g(0) = 0$ and $T'(0) = 0$, we obtain

$$(T \circ g)'(0) = (T \circ g)'(x)|_{x=0} = g'(x)(T' \circ g)(x)|_{x=0} = g'(0)T'[g(0)] = g'(0)T'(0) = 0.$$

Hence, we establish that

$$\lim_{n\to+\infty} w_n(g, h, T; \gamma) = 0 \times \int_0^1 h(t)dt = 0.$$

If $\gamma = 0$, an immediate application of the Riemann sum formula to the composition function $T \circ g \circ h$ gives

$$\lim_{n\to+\infty} w_n(g, h, T; \gamma) = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} (T \circ g \circ h) \left(\frac{k}{n}\right) = \int_0^1 (T \circ g \circ h)(t)dt = \int_0^1 T\{g[h(t)]\} dt.$$

This concludes the proof. □

In the proof of Theorem 2, Proposition 5 will be applied only for the special transformation $T(x) = x^2$. We end this preliminary part with a lemma on bounds for a logarithmic function.

Lemma 6. *For any $a \in (-1, 0]$ and $x \geq a$, we have*

$$x - \frac{1}{2(1+a)^2} x^2 \leq \log(1+x) \leq x.$$

Proof. For any $x > -1$, including $x \geq a$ with $a \in (-1, 0]$, it is well known that $\log(1+x) \leq x$. Classically, we can consider the function $p(x) = \log(1+x) - x$. It has the derivative $p'(x) = 1/(1+x) - 1$ which is positive for $x \in (-1, 0)$, equal to 0 for $x = 0$, and negative for $x \in (0, +\infty)$, meaning that $x = 0$ is a maximum for $p(x)$. Therefore, for any $x > -1$, we have $p(x) \leq p(0) = 0$, implying that $\log(1+x) \leq x$.

For the lower bound, Let us distinguish the case $x \geq 0 \geq a$, and the case $x \in (a, 0]$. First, let us consider the case $x \geq 0$. For any $t \in [0, 1]$, we have $0 \leq (1-t)(1+t) = 1 - t^2 \leq 1$, which implies that $1 - t \leq 1/(1+t)$. By integrating the involved functions with respect to t over $[0, x]$, we get

$$x - \frac{x^2}{2} = t - \frac{t^2}{2}\bigg|_{t=0}^{t=x} = \int_0^x (1-t)dt \leq \int_0^x \frac{1}{1+t}dt = \log(1+t)|_{t=0}^{t=x} = \log(1+x).$$

This corresponds to the desired inequality with $a = 0$.

For the case $x \in (a, 0]$, let us consider the function

$$q(x) = \log(1+x) - x + \frac{1}{2(1+a)^2}x^2.$$

Then we have

$$q'(x) = \frac{1}{1+x} - 1 + \frac{1}{(1+a)^2}x, \quad q''(x) = -\frac{1}{(1+x)^2} + \frac{1}{(1+a)^2}.$$

Since $x \in (a, 0]$, we have $1 + x \geq 1 + a \geq 0$, so that $q''(x) \geq 0$, which means that $q'(x)$ is increasing. Therefore, for any $x \in (a, 0]$, we have $q'(x) \leq q'(0) = 0$, implying that $q(x)$ is decreasing. Hence, for any $x \in (a, 0]$, we have $q(x) \geq q(0) = 0$, which gives

$$\log(1+x) \geq x - \frac{1}{2(1+a)^2}x^2.$$

This ends the case $x \in (a, 0]$. Let us mention that this lower bound is also provable by using the Taylor inequality at the point 0. The desired bounds are obtained. □

3.2. Main proof

We are now ready to prove Theorem 2. After a natural transformation of $u_n(g, h; \alpha, \beta)$, we distinguish between the cases **Case 1** and **Case 2**.

3.2.1. Logarithmic transformation

Let us consider the logarithmic transformation of $u_n(g, h; \alpha, \beta)$ given by

$$\begin{aligned} z_n(g, h; \alpha, \beta) &= \log\left[u_n(g, h; \alpha, \beta)\right] \\ &= \log\left[\prod_{k=1}^{n}\left\{1 + \frac{1}{n^{\beta(1-\alpha)}}g\left[\frac{1}{n^\alpha}h\left(\frac{k}{n}\right)\right]\right\}^{1/n^{(1-\alpha)(1-\beta)}}\right] \\ &= \frac{1}{n^{(1-\alpha)(1-\beta)}}\sum_{k=1}^{n}\log\left\{1 + \frac{1}{n^{\beta(1-\alpha)}}g\left[\frac{1}{n^\alpha}h\left(\frac{k}{n}\right)\right]\right\}. \end{aligned} \tag{4}$$

This logarithmic series expression will be at the center of the proof.

3.2.2. For Case 1

Let us notice that

$$\frac{1}{n^{\beta(1-\alpha)}}g\left[\frac{1}{n^\alpha}h\left(\frac{k}{n}\right)\right] \geq a_n = \begin{cases} 0 & \text{under } \mathbf{C}ond1, \\ -K_g\dfrac{1}{n^{\beta(1-\alpha)}} & \text{under } \mathbf{C}ond2. \end{cases}$$

Clearly, we have $\lim_{n\to+\infty} a_n = 0$. To bound $z_n(g, h; \alpha, \beta)$, let us apply the inequalities in Lemma 6 with

$$x = \frac{1}{n^{\beta(1-\alpha)}}g\left[\frac{1}{n^\alpha}h\left(\frac{k}{n}\right)\right]$$

and $a = a_n \in (-1, 0]$. With this setting, the main logarithmic term can be bounded as follows:

$$\begin{aligned}
&\frac{1}{n^{\beta(1-\alpha)}} g\left[\frac{1}{n^{\alpha}} h\left(\frac{k}{n}\right)\right] - \frac{1}{2(1+a_n)^2}\left\{\frac{1}{n^{\beta(1-\alpha)}} g\left[\frac{1}{n^{\alpha}} h\left(\frac{k}{n}\right)\right]\right\}^2 \\
&\leq \log\left\{1 + \frac{1}{n^{\beta(1-\alpha)}} g\left[\frac{1}{n^{\alpha}} h\left(\frac{k}{n}\right)\right]\right\} \leq \frac{1}{n^{\beta(1-\alpha)}} g\left[\frac{1}{n^{\alpha}} h\left(\frac{k}{n}\right)\right].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\frac{1}{n^{(1-\alpha)(1-\beta)}} \sum_{k=1}^{n} \left\{\frac{1}{n^{\beta(1-\alpha)}} g\left[\frac{1}{n^{\alpha}} h\left(\frac{k}{n}\right)\right]\right\} \\
&- \frac{1}{2(1+a_n)^2} \frac{1}{n^{(1-\alpha)(1-\beta)}} \sum_{k=1}^{n} \left\{\frac{1}{n^{\beta(1-\alpha)}} g\left[\frac{1}{n^{\alpha}} h\left(\frac{k}{n}\right)\right]\right\}^2 \\
&\leq z_n(g, h; \alpha, \beta) \leq \frac{1}{n^{(1-\alpha)(1-\beta)}} \sum_{k=1}^{n} \left\{\frac{1}{n^{\beta(1-\alpha)}} g\left[\frac{1}{n^{\alpha}} h\left(\frac{k}{n}\right)\right]\right\},
\end{aligned}$$

which is equivalent to

$$C_n - \frac{1}{2(1+a_n)^2} D_n \leq z_n(g, h; \alpha, \beta) \leq C_n, \tag{5}$$

where

$$C_n = \frac{1}{n^{1-\alpha}} \sum_{k=1}^{n} g\left[\frac{1}{n^{\alpha}} h\left(\frac{k}{n}\right)\right]$$

and

$$D_n = \frac{1}{n^{(1-\alpha)(1+\beta)}} \sum_{k=1}^{n} \left\{g\left[\frac{1}{n^{\alpha}} h\left(\frac{k}{n}\right)\right]\right\}^2.$$

We clearly have

$$\lim_{n \to +\infty} \frac{1}{2(1+a_n)^2} = \frac{1}{2}.$$

So, let us focus on the limits of C_n and D_n.

Under the considered assumptions, it follows from the first item in Proposition 4 with $\gamma = \alpha > 0$ that

$$\lim_{n \to +\infty} C_n = \lim_{n \to +\infty} \frac{1}{n^{1-\alpha}} \sum_{k=1}^{n} g\left[\frac{1}{n^{\alpha}} h\left(\frac{k}{n}\right)\right] = g'(0) \int_0^1 h(t) dt.$$

Furthermore, it follows from the first item in Proposition 5 with $\gamma = \alpha > 0$ and $T(x) = x^2$, and $\beta(1-\alpha) \geq 0$, that

$$\begin{aligned}
&\lim_{n \to +\infty} D_n = \lim_{n \to +\infty} \frac{1}{n^{(1-\alpha)(1+\beta)}} \sum_{k=1}^{n} \left\{g\left[\frac{1}{n^{\alpha}} h\left(\frac{k}{n}\right)\right]\right\}^2 \\
&= \left[\lim_{n \to +\infty} \frac{1}{n^{\beta(1-\alpha)}}\right] \left[\lim_{n \to +\infty} \frac{1}{n^{1-\alpha}} \sum_{k=1}^{n} \left\{g\left[\frac{1}{n^{\alpha}} h\left(\frac{k}{n}\right)\right]\right\}^2\right] = \iota \times 0 = 0,
\end{aligned}$$

where we have set $\iota = 1$ if $\beta = 0$ and $\iota = 0$ otherwise.

By combining these limit results into Equation (5), we obtain

$$\begin{aligned}
g'(0) \int_0^1 h(t) dt &= \lim_{n \to +\infty} C_n - \frac{1}{2} \lim_{n \to +\infty} D_n = \lim_{n \to +\infty} \left[C_n - \frac{1}{2(1+a_n)^2} D_n\right] \\
&\leq \lim_{n \to +\infty} z_n(g, h; \alpha, \beta) \leq \lim_{n \to +\infty} C_n = g'(0) \int_0^1 h(t) dt.
\end{aligned}$$

Therefore, by the continuity of the exponential function, we have

$$\lim_{n\to+\infty} z_n(g,h;\alpha,\beta) = g'(0)\int_0^1 h(t)dt$$

and

$$\begin{aligned}\lim_{n\to+\infty} u_n(g,h;\alpha,\beta) &= \lim_{n\to+\infty} \exp\left[z_n(g,h;\alpha,\beta)\right] = \exp\left[\lim_{n\to+\infty} z_n(g,h;\alpha,\beta)\right] \\ &= \exp\left[g'(0)\int_0^1 h(t)dt\right].\end{aligned}$$

The desired formula is demonstrated.

3.2.3. For Case 2

In the case $\alpha = 0$ and $\beta > 0$, based on Equation (4), we have

$$z_n(g,h;\alpha,\beta) = \frac{1}{n^{1-\beta}}\sum_{k=1}^{n}\log\left\{1+\frac{1}{n^\beta}g\left[h\left(\frac{k}{n}\right)\right]\right\}.$$

By following line lines the proof in Subsection 3.2.2, we directly arrive at Equation (5) with

$$C_n = \frac{1}{n}\sum_{k=1}^{n} g\left[h\left(\frac{k}{n}\right)\right]$$

and

$$D_n = \frac{1}{n^{1+\beta}}\sum_{k=1}^{n}\left\{g\left[h\left(\frac{k}{n}\right)\right]\right\}^2.$$

Under the considered assumptions, it follows from the second item in Proposition 4, i.e., with $\gamma = 0$, that

$$\lim_{n\to+\infty} C_n = \lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n} g\left[h\left(\frac{k}{n}\right)\right] = \int_0^1 h[g(t)]dt.$$

Furthermore, it follows from the second item in Proposition 5, i.e., with $\gamma = 0$, and $T(x) = x^2$, and $\beta > 0$, that

$$\begin{aligned}\lim_{n\to+\infty} D_n &= \lim_{n\to+\infty}\frac{1}{n^{1+\beta}}\sum_{k=1}^{n}\left\{g\left[h\left(\frac{k}{n}\right)\right]\right\}^2 \\ &= \left[\lim_{n\to+\infty}\frac{1}{n^\beta}\right]\left[\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\left\{g\left[h\left(\frac{k}{n}\right)\right]\right\}^2\right] = 0\times\int_0^1\{g[h(t)]\}^2 dt = 0.\end{aligned}$$

By combining these limit results into Equation (5), we obtain

$$\begin{aligned}\int_0^1 h[g(t)]dt &= \lim_{n\to+\infty} C_n - \frac{1}{2}\lim_{n\to+\infty} D_n = \lim_{n\to+\infty}\left[C_n - \frac{1}{2(1+a_n)^2}D_n\right] \\ &\leq \lim_{n\to+\infty} z_n(g,h;\alpha,\beta) \leq \lim_{n\to+\infty} C_n = \int_0^1 h[g(t)]dt.\end{aligned}$$

Therefore, we have

$$\lim_{n\to+\infty} z_n(g,h;\alpha,\beta) = \int_0^1 h[g(t)]dt$$

and

$$\lim_{n\to+\infty} u_n(g,h;\alpha,\beta) = \lim_{n\to+\infty} \exp\left[z_n(g,h;\alpha,\beta)\right] = \exp\left[\lim_{n\to+\infty} z_n(g,h;\alpha,\beta)\right]$$
$$= \exp\left[\int_0^1 h[g(t)]dt\right].$$

The desired formula is proved.

3.2.4. For Case 3

For the case $\alpha = 0$ and $\beta = 0$, based on Equation (4), we can write

$$z_n(g,h;\alpha,\beta) = \frac{1}{n}\sum_{k=1}^{n} \log\left\{1+g\left[h\left(\frac{k}{n}\right)\right]\right\}.$$

With a direct application of the Riemann sum formula, we get

$$\lim_{n\to+\infty} z_n(g,h;\alpha,\beta) = \int_0^1 \log\left\{1+g[h(t)]\right\} dt.$$

Hence, we have

$$\lim_{n\to+\infty} u_n(g,h;\alpha,\beta) = \lim_{n\to+\infty} \exp\left[z_n(g,h;\alpha,\beta)\right] = \exp\left[\lim_{n\to+\infty} z_n(g,h;\alpha,\beta)\right]$$
$$= \exp\left[\int_0^1 \log\left\{1+g[h(t)]\right\} dt\right].$$

The desired formula is proved. The proof of Theorem 2 ends. □

4. Proof of Theorem 3

By adopting the setting of Theorem 2, we can write

$$\begin{aligned}\Xi(g,h;\alpha,\beta) &= \sum_{n=1}^{+\infty}\prod_{k=1}^{n}\left\{1+\frac{1}{n^{\beta(1-\alpha)}}g\left[\frac{1}{n^{\alpha}}h\left(\frac{k}{n}\right)\right]\right\}^{1/n^{\alpha\beta-\alpha-\beta}}\\
&= \sum_{n=1}^{+\infty}\prod_{k=1}^{n}\left\{1+\frac{1}{n^{\beta(1-\alpha)}}g\left[\frac{1}{n^{\alpha}}h\left(\frac{k}{n}\right)\right]\right\}^{1/n^{(1-\alpha)(1-\beta)-1}}\\
&= \sum_{n=1}^{+\infty}\left[\prod_{k=1}^{n}\left\{1+\frac{1}{n^{\beta(1-\alpha)}}g\left[\frac{1}{n^{\alpha}}h\left(\frac{k}{n}\right)\right]\right\}^{1/n^{(1-\alpha)(1-\beta)}}\right]^{n}\\
&= \sum_{n=1}^{+\infty} v_n(g,h;\alpha,\beta),\end{aligned}$$

where

$$v_n = \left[u_n(g,h;\alpha,\beta)\right]^n.$$

Hence, $\Xi(g,h;\alpha,\beta)$ is well defined if $u_n(g,h;\alpha,\beta)$ is well-defined for any $n \in \mathbb{N}\setminus\{0\}$, which explains the consideration of **Cond 1** or **Cond 3**, and adds the conditions $K_g < 1$ to **Cond 2** to ensure that u_n is defined for all $n \in \mathbb{N}\setminus\{0\}$; we have $n \geq 1 > K_g^{1/[\beta(1-\alpha)]}$, by taking into account that $\beta(1-\alpha) \geq 0$.

The rest of the proof is a consequence of Theorem 2 and the Cauchy root convergence test. The details are given below.

For Case I: If $\alpha > 0$, $\beta(1-\alpha) \geq 0$, $g(0) = 0$, g is twice differentiable with $\sup_{t\in[\min(0,I_h),\max(0,J_h)]} |g''(t)| < +\infty$, and either $g'(0) < 0$ and $\int_0^1 h(t)dt > 0$, or $g'(0) > 0$ and $\int_0^1 h(t)dt < 0$, then **Case 1** of Theorem 2 gives

$$\lim_{n\to+\infty} [\nu_n(g,h;\alpha,\beta)]^{1/n} = \lim_{n\to+\infty} u_n(g,h;\alpha,\beta) = \exp\left[g'(0)\int_0^1 h(t)dt\right] < 1.$$

It follows from the Cauchy root convergence test that $\Xi(g,h;\alpha,\beta)$ converges.

For Case II: If $\alpha = 0$ and $\beta > 0$, and $\int_0^1 g[h(t)]dt < 0$, then **Case 2** of Theorem 2 implies that

$$\lim_{n\to+\infty} [\nu_n(g,h;\alpha,\beta)]^{1/n} = \lim_{n\to+\infty} u_n(g,h;\alpha,\beta) = \exp\left\{\int_0^1 g[h(t)]dt\right\} < 1.$$

The Cauchy root convergence test ensures that $\Xi(g,h;\alpha,\beta)$ converges.

For Case III: If $\alpha = 0$, $\beta = 0$, and $\int_0^1 \log\{1+g[h(t)]\}\,dt < 0$, then **Case 3** of Theorem 2 gives

$$\lim_{n\to+\infty} [\nu_n(g,h;\alpha,\beta)]^{1/n} = \lim_{n\to+\infty} u_n(g,h;\alpha,\beta) = \exp\left[\int_0^1 \log\{1+g[h(t)]\}\,dt\right] < 1.$$

It follows from the Cauchy root convergence test that $\Xi(g,h;\alpha,\beta)$ converges.

This concludes the proof of Theorem 3. □

5. Some examples

This section is devoted to some special examples illustrating Theorems 2 and 3.

Example 1: By selecting $h(x) = \log(1+x)$ for $x \in [0,1]$ and $g(x) = \sin(\pi x)$ for $x \in [0,\log(2)]$, both satisfying the required fist conditions in Theorem 2, and $\alpha = 1/2$ and $\beta = 1/2$, we have

$$\begin{aligned} u_n(g,h;\alpha,\beta) &= \prod_{k=1}^{n}\left\{1+\frac{1}{n^{\beta(1-\alpha)}}g\left[\frac{1}{n^{\alpha}}h\left(\frac{k}{n}\right)\right]\right\}^{1/n^{(1-\alpha)(1-\beta)}} \\ &= \prod_{k=1}^{n}\left\{1+\frac{1}{n^{1/4}}\sin\left[\frac{\pi}{n^{1/2}}\log\left(1+\frac{k}{n}\right)\right]\right\}^{1/n^{1/4}}. \end{aligned}$$

Since g is non-negative, **Cond 1** holds. Furthermore, since $\alpha = 1/2 > 0$, $\beta(1-\alpha) = 1/4 \geq 0$, $g(0) = 0$, and g is twice differentiable with $\sup_{t\in[0,\log(2)]} |g''(t)| = \pi^2 < +\infty$, **Case 1** of Theorem 2 gives

$$\begin{aligned} &\lim_{n\to+\infty} u_n(g,h;\alpha,\beta) = \exp\left[g'(0)\int_0^1 h(t)dt\right] = \exp\left[\pi\cos(\pi x)|_{x=0}\int_0^1 \log(1+t)dt\right] \\ &= \exp\{\pi[\log(4)-1]\} = 4^{\pi}\exp(-\pi). \end{aligned}$$

Figure 1 illustrates graphically this convergence by considering the curve of the following function of n, defined as the difference between $u_n(g,h;\alpha,\beta)$ and its limit:

$$\phi_n = u_n(g,h;\alpha,\beta) - 4^{\pi}\exp(-\pi), \tag{6}$$

for numerous values of n. The aim is to show that ϕ_n approaches 0 when n increases.

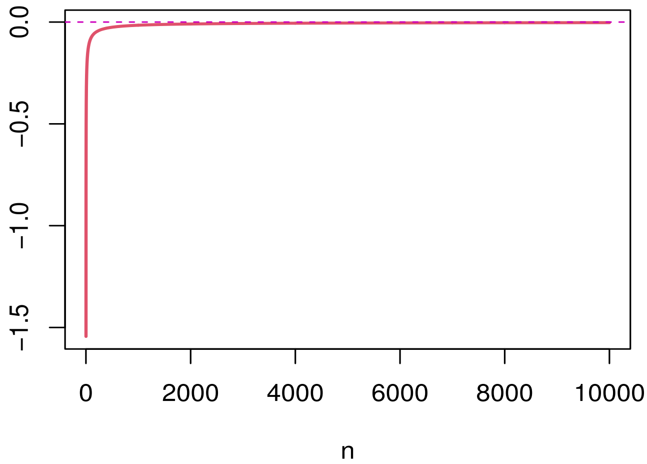

Figure 1. Curve of ϕ_n as given in Equation (6) with respect to n for $n = 1, 2, \ldots, 10, 000$.

From this figure, the fact that $\lim_{n\to+\infty} \phi_n = 0$ is observable.

Example 2: By selecting $h(x) = -(1/2)x^2$ for $x \in [0, 1]$ and $g(x) = (2/\pi)\arctan(x)$ for $x \in [-1/2, 0]$ both satisfying the required first conditions in Theorem 2, and $\alpha = 2$ and $\beta = -3/2$, we have

$$\begin{aligned} u_n(g, h; \alpha, \beta) &= \prod_{k=1}^{n} \left\{1 + \frac{1}{n^{\beta(1-\alpha)}} g\left[\frac{1}{n^{\alpha}} h\left(\frac{k}{n}\right)\right]\right\}^{1/n^{(1-\alpha)(1-\beta)}} \\ &= \prod_{k=1}^{n} \left[1 - \frac{1}{n^{3/2}} \left\{\frac{2}{\pi} \arctan\left[\frac{1}{2n^2}\left(\frac{k}{n}\right)^2\right]\right\}\right]^{n^{5/2}}. \end{aligned}$$

Let us notice that g is negative, so **Cond 1** is not satisfied (contrary to the previous example). However, we have $\beta(1-\alpha) = 3/2 \geq 0$ and

$$K_g = \sup_{t\in[-1/2,0]} |g(t)| = \frac{2}{\pi}\arctan\left(\frac{1}{2}\right) < 1,$$

so that $n \geq 1 > K_g^{1/[\beta(1-\alpha)]}$. Hence, **Cond 2** is satisfied. Since $\alpha = 2 > 0$, $\beta(1-\alpha) = 3/2 \geq 0$, $g(0) = 0$, and g is twice differentiable with $\sup_{t\in[0,\log(2)]} |g''(t)| \leq 2/\pi < +\infty$, **Case 1** of Theorem 2 gives

$$\begin{aligned} &\lim_{n\to+\infty} u_n(g, h; \alpha, \beta) = \exp\left[g'(0)\int_0^1 h(t)dt\right] = \exp\left[\frac{2}{\pi}\left.\frac{1}{1+x^2}\right|_{x=0} \int_0^1 \frac{1}{2}(-t^2)dt\right] \\ &= \exp\left(-\frac{1}{3\pi}\right). \end{aligned}$$

Figure 2 supports graphically this convergence by considering the following function with respect to n:

$$\psi_n = u_n(g, h; \alpha, \beta) - \exp\left(-\frac{1}{3\pi}\right), \tag{7}$$

and show how it tends to 0 when n increases.

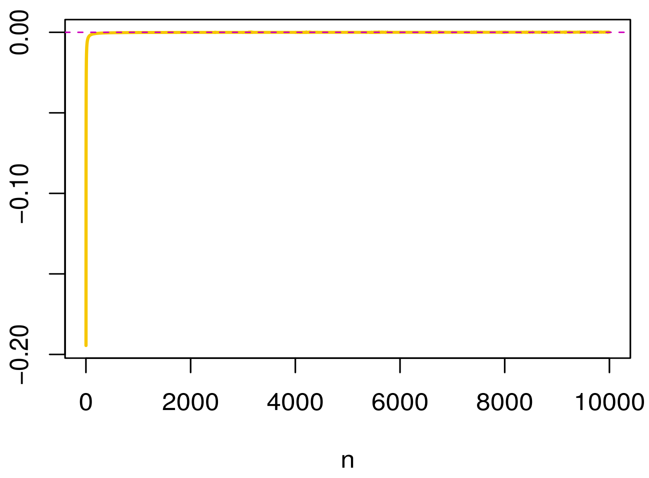

Figure 2. Curve of ψ_n as given in Equation (7) for $n = 1, 2, \ldots, 10,000$

This figure clearly shows the desired convergence.

On the other hand, since **Cond 2** is satisfied with $K_g < 1$, **Cond 3** is also satisfied. Furthermore, since $\alpha = 2 > 0$, $\beta(1-\alpha) = 3/2 \geq 0$, $g'(0) = 2/\pi > 0$ and $\int_0^1 h(t)dt = -1/6 < 0$, **Case I** of Theorem 3 ensures that the following series is convergent:

$$\Xi(g, h; \alpha, \beta) = \sum_{n=1}^{+\infty} \prod_{k=1}^{n} \left\{1 + \frac{1}{n^{\beta(1-\alpha)}} g\left[\frac{1}{n^{\alpha}} h\left(\frac{k}{n}\right)\right]\right\}^{1/n^{\alpha\beta-\alpha-\beta}}$$
$$= \sum_{n=1}^{+\infty} \prod_{k=1}^{n} \left[1 - \frac{1}{n^{3/2}} \left\{\frac{2}{\pi} \arctan\left[\frac{1}{2n^2}\left(\frac{k}{n}\right)^2\right]\right\}\right]^{n^{7/2}}.$$

Figure 3 illustrates graphically this convergence by considering the following truncated series with respect to m:

$$\varphi_m = \sum_{n=1}^{m} \prod_{k=1}^{n} \left[1 - \frac{1}{n^{3/2}} \left\{\frac{2}{\pi} \arctan\left[\frac{1}{2n^2}\left(\frac{k}{n}\right)^2\right]\right\}\right]^{n^{7/2}}. \quad (8)$$

We want to show how φ_m stabilizes up to a certain limit as m increases.

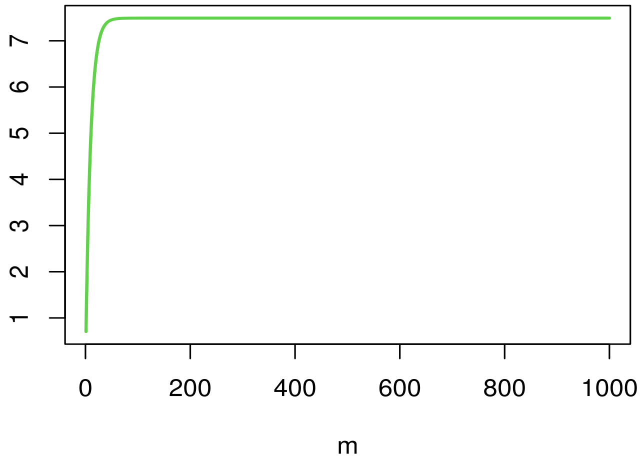

Figure 3. Plots of φ_m as given in Equation (8) for $m = 1, 2, \ldots, 1,000$

From this figure, the desired convergence is evident. More precisely, numerically, we observe that

$$\lim_{m\to+\infty} \varphi_m = 7.4910683.$$

Example 3: By selecting $h(x) = 1/[2(1+x)]$ for $x \in [0,1]$ and $g(x) = -x^2$ for $x \in [0,1/2]$ both satisfying the required first conditions in Theorem 2, and $\alpha = 0$ and $\beta = 2$, we have

$$\begin{aligned} u_n(g,h;\alpha,\beta) &= \prod_{k=1}^{n} \left\{1 + \frac{1}{n^{\beta(1-\alpha)}} g\left[\frac{1}{n^\alpha} h\left(\frac{k}{n}\right)\right]\right\}^{1/n^{(1-\alpha)(1-\beta)}} \\ &= \prod_{k=1}^{n} \left[1 - \frac{1}{4(n+k)^2}\right]^n. \end{aligned}$$

With mathematical efforts, we can express $u_n(g,h;\alpha,\beta)$ in terms of the gamma function as

$$u_n(g,h;\alpha,\beta) = \pi^n \left[\frac{2^{-4n}\Gamma(2n+1/2)\Gamma(2n+3/2)}{\Gamma(n+1/2)^3\Gamma(n+3/2)}\right]^n,$$

where, classically, $\Gamma(x) = \int_0^{+\infty} t^{x-1}\exp(-t)dt$.

Let us notice that g is negative, so **Cond 1** is not satisfied. However, we have $\beta(1-\alpha) = \beta = 2 \geq 0$ and

$$K_g = \sup_{t\in[0,1/2]} |g(t)| = \frac{1}{4} < 1,$$

so that $n \geq 1 > K_g^{1/[\beta(1-\alpha)]}$. Hence, **Cond 2** is satisfied. Since $\alpha = 0$ and $\beta = 2 > 0$, **Case 2** of Theorem 2 gives

$$\begin{aligned} &\lim_{n\to+\infty} u_n(g,h;\alpha,\beta) = \exp\left\{\int_0^1 g[h(t)]dt\right\} = \exp\left[-\frac{1}{4}\int_0^1 \frac{1}{(1+t)^2}dt\right] \\ &= \exp\left[-\frac{1}{4}\left(-\frac{1}{1+t}\right)\Bigg|_{t=0}^{t=1}\right] = \exp\left(-\frac{1}{8}\right). \end{aligned}$$

Figure 4 supports graphically this convergence by considering the following function with respect to n:

$$v_n = u_n(g, h; \alpha, \beta) - \exp\left(-\frac{1}{8}\right), \tag{9}$$

We want to show how it tends to 0 when n increases.

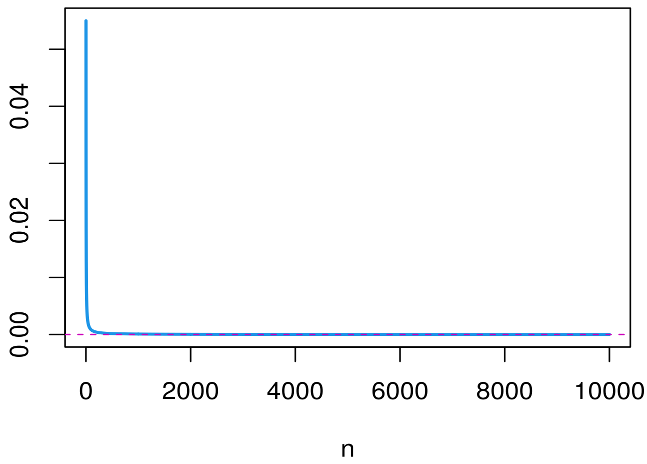

Figure 4. Plots of v_n as given in Equation (9) for $n = 1, 2, \ldots, 10,000$

The desired convergence is displayed.

On the other hand, since **Cond 2** is satisfied with $K_g < 1$, **Cond 3** is also satisfied. Furthermore, since $\alpha = 0$, $\beta = 2 \geq 0$, and $\int_0^1 g[h(t)]dt = -1/8 < 0$, **Case II** of Theorem 3 ensures that the following series is convergent:

$$\begin{aligned}\Xi(g, h; \alpha, \beta) &= \sum_{n=1}^{+\infty} \prod_{k=1}^{n} \left\{1 + \frac{1}{n^{\beta(1-\alpha)}} g\left[\frac{1}{n^{\alpha}} h\left(\frac{k}{n}\right)\right]\right\}^{1/n^{\alpha\beta-\alpha-\beta}} \\ &= \sum_{n=1}^{+\infty} \prod_{k=1}^{n} \left[1 - \frac{1}{4(n+k)^2}\right]^{n^2}.\end{aligned}$$

Figure 5 illustrates graphically this convergence by considering the following truncated series with respect to m:

$$\omega_m = \sum_{n=1}^{m} \prod_{k=1}^{n} \left[1 - \frac{1}{4(n+k)^2}\right]^{n^2} \tag{10}$$

and show how it stabilizes when m increases.

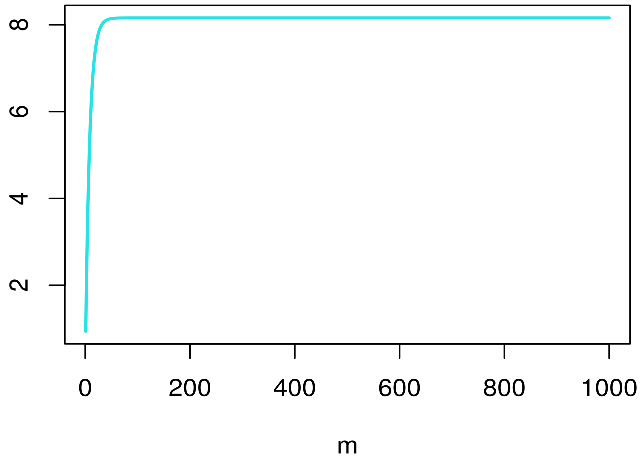

Figure 5. Plots of ω_m as given in Equation (10) for $m = 1, 2, \ldots, 1,000$

From this figure, it is clear that $\lim_{m\to+\infty} \omega_m$ is finite. More precisely, numerically, we observe that

$$\lim_{m\to+\infty} \omega_m = 8.159798.$$

6. Conclusion

Finally, this article presents a detailed analysis of the convergence properties of certain adaptable product sequences and series governed by two tuning parameters and two functions. Several results are of independent interest, including an extended Riemann sum formula. By studying their convergence behavior in different configurations, we contribute to a broader understanding of mathematical sequences and series. The potential applications are in various scientific fields. A natural perspective of this article is the consideration of a sequence with terms of the following form:

$$"u_n = \prod_{k=1}^{n} \left\{1 + a_{k,n} g\left[b_{k,n} h\left(c_{k,n}\right)\right]\right\}^{d_{k,n}}",$$

where g and h denote two general functions, and $a_{k,n}$, $b_{k,n}$, $c_{k,n}$, and $d_{k,n}$ are certain terms depending on k and n whose properties must be explored to guarantee the convergence of this sequence. It is possible that the combination of the techniques developed in this article and those elaborated in [6] and [7] may be useful. A first step in this direction is offered in the appendix below. However, the challenge is certain and requires a deeper analysis, which we leave for future work.

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Conflicts of Interest: "The author declares no conflict of interest."

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: "The authors declare no conflict of interest."

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Appendix

The result below is an extension of Proposition 4, that also extends [6, Theorem 1].

Proposition 7. *Let h be a continuous function defined on $[0,1]$. Let $\gamma > 0$, $\theta > 0$, $D > 0$ and $\tau = (\tau_k)_{k\in\mathbb{N}\setminus\{0\}}$ be a sequence of positive real numbers such that $\sup_{k\in\{1,\ldots,n\}} \tau_k \leq Dn^{\theta\gamma}$, there exists a constant $\ell > 0$ satisfying*

$$\lim_{n\to+\infty} \frac{1}{n^\theta} \sum_{k=1}^{n} \tau_k = \ell$$

and

$$\lim_{n\to+\infty} \frac{1}{n^{\theta(1+\gamma)}} \sum_{k=1}^{n} \tau_k^2 = 0. \tag{11}$$

Let g be a continuous function defined on $[D\min(0,I_h), D\max(0,J_h)]$, where $I_h = \inf_{t\in[0,1]} h(t)$ and $J_h = \sup_{t\in[0,1]} h(t)$. In this setting, we set

$$v_n(g,h;\gamma;\tau) = \frac{1}{n^{\theta(1-\gamma)}} \sum_{k=1}^{n} g\left[\frac{\tau_k}{n^{\theta\gamma}} h\left(\frac{k}{n}\right)\right].$$

If $g(0) = 0$, g is twice differentiable, and $\sup_{t\in[\min(0,I_h),\max(0,J_h)]} |g''(t)| < +\infty$, then we have

$$\lim_{n\to+\infty} v_n(g,h;\gamma;\tau) = g'(0)\ell\theta \int_0^1 t^{\theta-1} h(t)dt.$$

Proof. We follow the lines of the proof of Proposition 4. A suitable decomposition and the triangular inequality give

$$\begin{aligned}
&\left| v_n(g,h;\gamma;\tau) - g'(0)\ell\theta \int_0^1 t^{\theta-1} h(t)dt \right| \\
&= \left| \frac{1}{n^{\theta(1-\gamma)}} \sum_{k=1}^{n} g\left[\frac{\tau_k}{n^{\theta\gamma}} h\left(\frac{k}{n}\right)\right] - g'(0)\frac{1}{n^\theta} \sum_{k=1}^{n} \tau_k h\left(\frac{k}{n}\right) \right. \\
&\left. + g'(0)\frac{1}{n^\theta} \sum_{k=1}^{n} \tau_k h\left(\frac{k}{n}\right) - g'(0)\ell\theta \int_0^1 t^{\theta-1} h(t)dt \right| \\
&\leq E_n + |g'(0)|F_n,
\end{aligned} \tag{12}$$

where

$$E_n = \left| \frac{1}{n^{\theta(1-\gamma)}} \sum_{k=1}^{n} g\left[\frac{\tau_k}{n^{\theta\gamma}} h\left(\frac{k}{n}\right)\right] - g'(0)\frac{1}{n^\theta} \sum_{k=1}^{n} \tau_k h\left(\frac{k}{n}\right) \right|$$

and

$$F_n = \left| \frac{1}{n^\theta} \sum_{k=1}^{n} \tau_k h\left(\frac{k}{n}\right) - \ell\theta \int_0^1 t^{\theta-1} h(t)dt \right|.$$

Let us determine an upper bound for E_n. By using again the triangular inequality, we have

$$E_n = \frac{1}{n^{\theta(1-\gamma)}} \left| \sum_{k=1}^{n} \left\{ g\left[\frac{\tau_k}{n^{\theta\gamma}} h\left(\frac{k}{n}\right) \right] - g'(0) \frac{\tau_k}{n^{\theta\gamma}} h\left(\frac{k}{n}\right) \right\} \right|$$
$$\leq \frac{1}{n^{\theta(1-\gamma)}} \sum_{k=1}^{n} \left| g\left[\frac{\tau_k}{n^{\theta\gamma}} h\left(\frac{k}{n}\right) \right] - g'(0) \frac{\tau_k}{n^{\theta\gamma}} h\left(\frac{k}{n}\right) \right|. \tag{13}$$

Since $g(0) = 0$ and $\sup_{t\in[D\min(0,I_h),D\max(0,J_h)]} |g''(t)| < \infty$, by setting $N = \sup_{t\in[D\min(0,I_h),D\max(0,J_h)]} |g''(t)|$, the Taylor inequality at the point 0 implies that, for any $t \in [D\min(0, I_h), D\max(0, J_h)]$,

$$|g(t) - tg'(0)| = |g(t) - g(0) - tg'(0)| \leq \frac{N}{2} t^2.$$

Therefore, by applying this inequality with $t = (\tau_k/n^{\theta\gamma})h(k/n) \in [D\min(0, I_h), D\max(0, J_h)]$ into Equation (13) and introducing $L_h = \sup_{t\in[0,1]} |h(t)|$, we have

$$E_n \leq \frac{N}{2} \left\{ \frac{1}{n^{\theta(1-\gamma)}} \sum_{k=1}^{n} \left[\frac{\tau_k}{n^{\theta\gamma}} h\left(\frac{k}{n}\right) \right]^2 \right\} \leq \frac{N}{2} L_h^2 \left(\frac{1}{n^{\theta(1+\gamma)}} \sum_{k=1}^{n} \tau_k^2 \right).$$

Using Equation (11), we have

$$0 \leq \lim_{n\to+\infty} E_n \leq \frac{N}{2} L_h^2 \left(\lim_{n\to+\infty} \frac{1}{n^{\theta(1+\gamma)}} \sum_{k=1}^{n} \tau_k^2 \right) = 0,$$

so $\lim_{n\to+\infty} E_n = 0$.

Concerning F_n, it is a direct application of [6, Theorem 1], which states that

$$\lim_{n\to+\infty} \frac{1}{n^{\theta}} \sum_{k=1}^{n} \tau_k h\left(\frac{k}{n}\right) = \ell\theta \int_0^1 t^{\theta-1} h(t) dt.$$

As a result, we have $\lim_{n\to+\infty} F_n = 0$.

Therefore, based on Equation (12), we have

$$0 \leq \lim_{n\to+\infty} \left| v_n(g, h; \gamma; \tau) - g'(0)\ell\theta \int_0^1 t^{\theta-1} h(t) dt \right| \leq \lim_{n\to+\infty} E_n + |g'(0)| \lim_{n\to+\infty} F_n = 0,$$

implying that

$$\lim_{n\to+\infty} v_n(g, h; \gamma; \tau) = g'(0)\ell\theta \int_0^1 t^{\theta-1} h(t) dt.$$

This ends the proof. □

By taking $\tau_k = 1$ for any $k = 1, \ldots, n$, $\theta = 1$, and $D = 1$, we get the first item in Proposition 4. On the other hand, by choosing $g(x) = x$, we re-obtain [6, Theorem 1], noticing that in this case, the condition in Equation (11) is unnecessary. This ends the appendix.