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Article

Multidual Gamma function

Farid Messelmi[1,*]

[1] Department of Mathematics and LDMM Laboratory, University of Djelfa, Algeria
* Correspondence: foudimath@yahoo.fr

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Abstract: The purpose of this paper is to contribute to the development of the multidual Gamma function. For this aim, we start by defining the multidual Gamma and we propose a multidual analysis technics of in order to show a result regarding real Gamma function.

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1. Introduction

Multidual numbers were first introduced by F. Messelmi in [1] as a generalization of dual numbers to higher dimensions. This concept involves a unit number satisfying $\varepsilon^{n+1} = 0$, thereby creating an $(n+1)$-dimensional associative, commutative, and unitary generalized Clifford algebra generated by ε, known as multidual algebra. The author explored functions of multidual variables, generalizing the Cauchy-Riemann formulas and presenting results on the continuation of multidual functions.

In [2], the concept was extended to complex numbers, resulting in multidual complex numbers, and the study encompassed multidual complex functions and their inverses. The algebraic properties of multidual numbers were thoroughly discussed in [1,3,4], and differential calculus of multidual functions was the subject of [5]. This paper introduced anti-hyperholomorphic and co-hyperholomorphic functions, generalized Dirac operators, and established several significant results. Furthermore, multidual analysis has been applied in various technological fields, including Mechanics, Robotics, Aeronautics, and Electronics, as detailed in [6–14].

The primary aim of this paper is to define the multidual Gamma function as a multidual continuation of the real Gamma function and to investigate its properties.

The paper is organized as follows: The second section reviews the basic properties of multidual analysis, including hyperholomorphic functions and the continuation of real functions to the algebra of multidual numbers. The third section extends the real Gamma function to multidual numbers and examines its properties. An intriguing result concerning the real Gamma function, utilizing multidual analysis and involving harmonic numbers, will also be established.

2. Prliminaries

A multidual number z is defined according to the work in [1] as an ordered $(n+1)$-tuple of real numbers $(x_0, x_1, \ldots, x_n)$ associated with the real unit 1 and the powers of the multidual unit ε, where ε is an $(n+1)$-nilpotent number, i.e., $\varepsilon^{n+1} = 0$ and $\varepsilon^i \neq 0$ for $i = 1, \ldots, n$. Specifically, a multidual number is typically denoted in the form

$$z = \sum_{i=0}^{n} x_i \varepsilon^i. \tag{1}$$

Here, we assume that $\varepsilon^0 = 1$.

The set of multidual numbers is denoted by $\mathbb{D}_n$ and is defined as

$$\mathbb{D}_n = \left\{ z = \sum_{i=0}^{n} x_i \varepsilon^i \mid x_i \in \mathbb{R}, \text{ where } \varepsilon^{n+1} = 0 \text{ and } \varepsilon^i \neq 0 \text{ for } i = 1, \ldots, n \right\}. \tag{2}$$

There are various ways to choose the multidual unit ε. A basic example is given by the matrix

$$\varepsilon = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 1 & 0 \end{bmatrix}.$$

If $z = \sum_{i=0}^{n} x_i \varepsilon^i$ is a multidual number, we denote by real(z) the real part of z, given by

$$\text{real}(z) = x_0. \tag{3}$$

The multidual numbers form a commutative ring with characteristic 0. Moreover, the inherited multiplication gives the multidual numbers the structure of an $(n+1)$-dimensional generalized Clifford Algebra. For $n = 1$, $\mathbb{D}_1$ represents the Clifford algebra of dual numbers. For more details regarding dual numbers, see references [2,16,17]. In abstract algebra terms, the multidual ring can be obtained as the quotient of the polynomial ring $\mathbb{R}[X]$ by the ideal generated by the polynomial X^{n+1}, i.e.,

$$\mathbb{D}_n \simeq \frac{\mathbb{R}[X]}{\langle X^{n+1} \rangle}. \tag{4}$$

It is also important to point out that every multidual number possesses a matrix representation that can be formulated as follows:

Let us denote by $\mathcal{G}_{n+1}(\mathbb{R})$ the subset of $\mathcal{M}_{n+1}(\mathbb{R})$ given by

$$\mathcal{G}_{n+1}(\mathbb{R}) = \left\{ A = (x_{ij}) \in \mathcal{M}_{n+1}(\mathbb{R}) \mid \begin{array}{l} x_{ij} = 0 \text{ if } i < j, \\ x_{i+1,j+1} = x_{ij} \text{ if } j \leq i \leq n \end{array} \right\}. \tag{5}$$

An element A of $\mathcal{G}_{n+1}(\mathbb{R})$ can be written as

$$A = \begin{bmatrix} a_0 & 0 & \dots & 0 \\ a_1 & a_0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ a_n & \dots & a_1 & a_0 \end{bmatrix}. \tag{6}$$

It is clear that $\mathcal{G}_{n+1}(\mathbb{R})$ is a subring of $\mathcal{M}_{n+1}(\mathbb{R})$ having the structure of an $(n+1)$-dimensional associative, commutative, and unitary algebra. If $a_0 \neq 0$, $\mathcal{G}_{n+1}$ becomes a field. In particular, the set $\mathcal{G}_{n+1}(\mathbb{R})$ can also be seen as a subgroup of $GL(n+1)$.

Introducing now the following mapping

$$\begin{cases} \mathcal{R} : \mathbb{D}_n \longrightarrow \mathcal{G}_{n+1}(\mathbb{R}), \\ \mathcal{R}\left(\sum_{i=0}^{n} x_i \varepsilon^i \right) = A = \begin{bmatrix} x_0 & 0 & \dots & 0 \\ x_1 & x_0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ x_n & \dots & x_1 & x_0 \end{bmatrix} \end{cases} \tag{7}$$

The result below shows the relationship between the sets $\mathbb{D}_n$ and $\mathcal{G}_{n+1}(\mathbb{R})$.

Theorem 1. *$\mathcal{R}$ is an isomorphism of algebras.*

If z is a multidual number, the conjugate of z denoted by $\bar{z}$ is the multidual number given by

$$z\bar{z} = \det \mathcal{R}(z) = \text{real}(z)^{n+1}. \tag{8}$$

Hence, $z = \sum_{i=0}^{n} x_i \varepsilon^i$ has a unique conjugate if and only if real $(z) = x_0 \neq 0$. If $x_0 = 0$ the number $\sum_{i=1}^{n} x_i \varepsilon^i$ is a divisor of zero in the ring $\mathbb{D}_n$. Denote by D the set of zero divisors of the ring $\mathbb{D}_n$, i.e.

$$D = \left\{ \sum_{i=1}^{n} x_i \varepsilon^i \mid x_i \in \mathbb{R} \right\}. \tag{9}$$

For the sequel we admit that $\mathbb{D}_n$ is endowed with the usual topology of $\mathbb{R}^{n+1}$. We recall now, according to the work [1], some results regarding multidual functions.

Let Ω be an open subset of $\mathbb{D}_n$, $z = \sum_{i=0}^{n} x_i \varepsilon^i \in \Omega$ and $f : \Omega \longrightarrow \mathbb{D}_n$ a multidual function. The Cauchy-Riemann conditions can be generalized for multidual function as follows.

Theorem 2. *Let f be a multidual function in $\Omega \subset \mathbb{D}_n$, which can be written in terms of its real and multidual parts as*

$$f(z) = \sum_{i=0}^{n} f_i(x_0, x_1, ..., x_n)\, \varepsilon^i. \tag{10}$$

and suppose that the partial derivatives of f exist. Then,

***1.** f is hyperholomorphic in Ω if and only if the following formulas hold*

$$\begin{cases} \dfrac{\partial f_i}{\partial x_j} = \dfrac{\partial f_{i-j}}{\partial x_0} \text{ if } j \leq i, \\ \dfrac{\partial f_i}{\partial x_j} = 0 \text{ if } j > i. \end{cases} \tag{11}$$

***2.** f is hyperholomorphic in Ω if and only if its partial derivatives satisfy*

$$\frac{\partial f}{\partial x_j} = \varepsilon^j \frac{\partial f}{\partial x_0}, \quad j = 0, ..., n. \tag{12}$$

This allows us to deduce in particluar that if the function f is hyperholomorphic then

$$\frac{df}{dz} = \frac{\partial f}{\partial x_0}. \tag{13}$$

A multidual function defined in $\Omega \subset \mathbb{D}_n$ is said to be homogeneous if

$$f(\text{real}(z)) \in \mathbb{R}. \tag{14}$$

The following Theorem asserts us that we can extend any homogeneous hyperholomorphic function defined in a subset $\Omega \subset \mathbb{D}_n$ to the whole multidual subset $\mathcal{P}_1(\Omega) \times \mathbb{R}^n \subset \mathbb{D}_n$, where $\mathcal{P}_1(\Omega)$ represents the first projection of Ω on $\mathbb{R}$.

Theorem 3 (*Continuation of* hyperholomorphic functions)**.** *Let f be an homogeneous multidual function in $\Omega \subset \mathbb{D}_n$, which can be written in terms of its real and multidual parts as in the expression (10) and suppose that the partial derivatives of f exist. If f is hyperholomorphic in Ω, then the functions f_i verify*

***1.** $f_0 \in \mathcal{C}^{n+1}(\mathcal{P}_1(\Omega))$.*

***2.** $f_i \in \mathcal{C}^{n-i+1}(\mathcal{P}_1(\Omega) \times \mathbb{R}^i)$, $i = 1, ..., n$.*

***3.** f can be holomorphically extended to the multidual subset $\mathcal{P}_1(\Omega) \times \mathbb{R}^n \subset \mathbb{D}_n$.*

The following proposition ensures that every regular real function can be extended to the algebra of multidual numbers.

Proposition 4 (Continuation of real functions)**.** *Let $f : O \longrightarrow \mathbb{R}$ be a real function, where O is an open connected domain of $\mathbb{R}$.*

1. *Suppose that* $f \in \mathcal{C}^{n+1}(O)$. *Then, there exists a unique homogeneous hyperholomorphic multidual function* $\widetilde{f} : O \times \mathbb{R}^n \subset \mathbb{D}_n \longrightarrow \mathbb{D}_n$ *satisfying*

$$\widetilde{f}(x_0) = f(x_0) \quad \forall x_0 \in O. \tag{15}$$

2. *For* $i = 1, ..., n$ *and* $j = 1, ..., i$, *there exists polynomials* $P_{ij} \in \mathbb{R}[x_1, ..., x_i]$ *where* $\deg(P_{ij}) \le i$, *such that*

$$\widetilde{f}(z) = f(x_0) + \sum_{i=1}^{n} \sum_{j=1}^{i} P_{ij}(x_1, ..., x_i) f^{(i-j+1)}(x_0) \varepsilon^i. \tag{16}$$

If in addition $f \in \mathcal{C}^q(O)$, $q \ge n+1$, *then* $\widetilde{f} \in \mathcal{C}^{q-n}(O \times \mathbb{R}^n)$. *In Particular, if* $f \in \mathcal{C}^\infty(O)$, *then* $\widetilde{f} \in \mathcal{C}^\infty(O \times \mathbb{R}^n)$, *we say in such case that* f *is an analytic function in* $O \times \mathbb{R}^n$.

In the following proposition, we give some properties regarding the generator polynomials P_{ij} appearing in formula (16).

Proposition 5. *The generator polynomials verify the following statements:*

$$\begin{cases} P_{ij} = 0 \ \ \forall i = 1, ..., n \text{ and } j = i+1, ..., n, \\ \frac{\partial P_{ij}}{\partial x_k} = 0 \ \ \forall i = 1, ..., n, \ k = 1, ..., i \text{ and } j = 1, ..., k-1, \\ \frac{\partial P_{ij}}{\partial x_k} = P_{i-k, j-k+1} \ \ \forall i = 2, ..., n, \ k = 1, ..., i-1 \text{ and } j = k, ..., i-1, \\ P_{ii}(x_1, ..., x_i) = x_i \ \ \forall i = 1, ..., n. \end{cases} \tag{17}$$

3. Multidual Gamma Function

We focus in this sectionr on the generalization of the real Gamma function to multidual numbers and we will intereste to show some results regading real Gamma function making use the properties of multidual Gamma function.

Let Γ be the real Gamma function given by

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt, \tag{18}$$

It is will known that $\Gamma \in \mathcal{C}^\infty(]0, +\infty[)$, then by Proposition 4 there exists a unique multidual continuation function still denoted by Γ, called multidual Gamma function, defined in the subset $(\mathbb{D}_n)^*_+ = \left\{ z = \sum_{i=0}^{n} x_i \varepsilon^i \in \mathbb{D}_n \mid x_i > 0 \right\}$ by

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt \ \ \forall z \in (\mathbb{D}_n)^*_+. \tag{19}$$

We will sketch in the following some of the main properties of the multidual Gamma function.

Proposition 6. **1.** $\forall z \in (\mathbb{D}_n)^*_+$ *the multidual Gamma function satisfies the functional equation*

$$\Gamma(z+1) = z\Gamma(z). \tag{20}$$

2. $\forall k \in \mathbb{N}^*$ *we have*

$$\Gamma^{(k)}(z) = \int_0^{+\infty} (\log t)^k t^{z-1} e^{-t} dt. \tag{21}$$

The proof is an immediate consequence of the formula (19).

We will need the following notations

$B_0 = (\mathbb{D}_n)^*_+ = \left\{ z = \sum_{i=1}^{n} x_i \varepsilon^i \in \mathbb{D}_n \mid x_0 > 0 \right\},$

$$B_1 = \left\{ z = \sum_{i=1}^{n} x_i \varepsilon^i \in \mathbb{D}_n \mid x_0 > -1 \text{ and } x_0 \neq 0 \right\},$$
$$B_2 = \left\{ z = \sum_{i=1}^{n} x_i \varepsilon^i \in \mathbb{D}_n \mid x_0 > -2 \text{ and } x_0 \neq -1,\ 0 \right\},$$
$$\vdots$$
$$B_m = \left\{ z = \sum_{i=1}^{n} x_i \varepsilon^i \in \mathbb{D}_n \mid x_0 > -m \text{ and } x_0 \neq 0, -1, ..., -m+1 \right\}.$$

So far we know that $\Gamma(z)$ is defined and hyperholomorphic on B_0. Our strategy is to hyperholomorphically extende Γ from B_0 to B_m. To this aim, proposition allows us to write

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}.$$

The right side of this equation is hyperholomorphic on B_1. Since it agrees with Γ on B_0 it represents a hyperholomorphic continuation from B_0 to B_1.

Similarly, equation can be expressed as $\Gamma(z+1) = \frac{\Gamma(z+2)}{(z+1)}$. So,

$$\Gamma(z) = \frac{\Gamma(z+2)}{z(z+1)}.$$

The right side of this equation is hyperholomorphic on B_2. Since it agrees with Γ on B_0 it is a hyperholomorphic continuation from B_0 to B_2.

We can so iterate this procedure as far as to get

$$\Gamma(z) = \frac{\Gamma(z+m)}{z(z+1)...(z+m-1)}. \tag{22}$$

The right side of this equation is analytic on B_m. Since it agrees with Γ on B_0 it is an hyperholomorphic continuation from B_0 to B_m. We conclude that the function Γ can be extended to the set $B_m = \left\{ z = \sum_{i=1}^{n} x_i \varepsilon^i \in \mathbb{D}_n \mid x_0 \neq -m,\ m \in \mathbb{N} \right\}$.

Theorem 7. *For every $k \in \mathbb{N}^*$ and $x_0 \neq -m$, $m \in \mathbb{N}$. The following formulas hold*

$$\Gamma'(x_0+k) = \prod_{r=0}^{k-1} (x_0+r) \left(\Gamma'(x_0) + \Gamma(x_0) \sum_{r=0}^{k-1} \frac{1}{x_0+r} \right), \tag{23}$$

and

$$\begin{aligned} &\sum_{j=1}^{i} P_{ij}(x_1, ..., x_i)\, \Gamma^{(i-j+1)}(x_0+k) \\ &= \prod_{r=0}^{k-1} (x_0+r) \sum_{j=1}^{i} \left(P_{ij}(x_1, ..., x_i)\, \Gamma^{(i-j+1)}(x_0) + \Gamma(x_0)\, P_{ij}(y_1, ..., y_i) \right) + \\ &\sum_{j=1}^{i-1} \left(\sum_{m=1}^{j} P_{jm}(x_1, ..., x_i)\, \Gamma^{(j-m-1)}(x_0) \right) \left(\sum_{m=1}^{i-j} P_{i-j,m}(y_1, ..., y_{i-j}) \right), \end{aligned} \tag{24}$$

where

$$y_i = \sum_{r=0}^{k-1} \sum_{j=1}^{i} P_{ij} \left(\frac{x_1}{x_0+r}, ..., \frac{x_i}{x_0+r} \right) (-1)^{i-j} (i-j)!. \tag{25}$$

Proof. We know from formula (20) that for all $k \in \mathbb{N}^*$ we have

$$\Gamma(z+k) = \Gamma(z) \prod_{r=0}^{k-1} (z+r). \tag{26}$$

So, if $z = \sum\limits_{i=0}^{n} x_i \varepsilon^i$, we can write

$$\begin{aligned}\Gamma(z+k) &= \Gamma(z) \prod_{r=0}^{k-1} \left(x_0 + r + \sum_{i=1}^{n} x_i \varepsilon^i \right) \\ &= \Gamma(z) \prod_{r=0}^{k-1} (x_0 + r) \prod_{r=0}^{k-1} \left(1 + \sum_{i=1}^{n} \frac{x_i}{x_0 + r} \varepsilon^i \right) \\ &= \Gamma(z) \prod_{r=0}^{k-1} (x_0 + r) \prod_{r=0}^{k-1} e^{\sum\limits_{i=1}^{n} y_{ri} \varepsilon^i}, \end{aligned} \tag{27}$$

where

$$e^{\sum\limits_{i=1}^{n} y_{ri} \varepsilon^i} = 1 + \sum_{i=1}^{n} \frac{x_i}{x_0 + r} \varepsilon^i. \tag{28}$$

This leads to

$$\Gamma(z+k) = \Gamma(z) \prod_{r=0}^{k-1} (x_0 + r) \, e^{\sum\limits_{r=o}^{k-1} \sum\limits_{i=1}^{n} y_{ri} \varepsilon^i}. \tag{29}$$

Denoting now by y_i the sum $y_i = \sum\limits_{r=o}^{k-1} y_{ri}$, we obtain keeping in mind (16)

$$\Gamma(z+k) = \Gamma(z) \prod_{r=0}^{k-1} (x_0 + r) \left(1 + \sum_{i=1}^{n} \sum_{j=1}^{i} P_{ij}(y_1, ..., y_i) \varepsilon^i \right). \tag{30}$$

Using again (16), we find

$$\begin{aligned} &\Gamma(x_0 + k) + \sum_{i=1}^{n} \sum_{j=1}^{i} P_{ij}(x_1, ..., x_i) \Gamma^{(i-j+1)}(x_0 + k) \varepsilon^i \\ = &\left(\Gamma(x_0) + \sum_{i=1}^{n} \sum_{j=1}^{i} P_{ij}(x_1, ..., x_i) \Gamma^{(i-j+1)}(x_0) \varepsilon^i \right) \times \\ &\prod_{r=0}^{k-1} (x_0 + r) \left(1 + \sum_{i=1}^{n} \sum_{j=1}^{i} P_{ij}(y_1, ..., y_i) \varepsilon^i \right). \end{aligned} \tag{31}$$

Further, (11) gives

$$\sum_{i=1}^{n} y_i \varepsilon^i = \log \left(1 + \sum_{i=1}^{n} \frac{x_i}{x_0 + r} \varepsilon^i \right).$$

Then, we can infer thinks to (16)

$$\sum_{i=1}^{n} y_i \varepsilon^i = \sum_{i=1}^{n} \sum_{j=1}^{i} P_{ij} \left(\frac{x_1}{x_0 + r}, ..., \frac{x_i}{x_0 + r} \right) \log^{(i-j+1)}(1) \varepsilon^i. \tag{32}$$

This yields

$$y_i = \sum_{r=0}^{k-1} \sum_{j=1}^{i} P_{ij} \left(\frac{x_1}{x_0 + r}, ..., \frac{x_i}{x_0 + r} \right) (-1)^{i-j} (i-j)!.$$

Which implis that

$$\Gamma\left(x_0+k\right)+\sum_{i=1}^{n}\sum_{j=1}^{i}P_{ij}\left(x_1,...,x_i\right)\Gamma^{(i-j+1)}\left(x_0+k\right)\varepsilon^i$$
$$=\Gamma\left(x_0\right)\prod_{r=0}^{k-1}\left(x_0+r\right)+\prod_{r=0}^{k-1}\left(x_0+r\right)\sum_{i=1}^{n}\sum_{j=1}^{i}P_{ij}\left(x_1,...,x_i\right)\Gamma^{(i-j+1)}\left(x_0\right)\varepsilon^i+$$
$$\Gamma\left(x_0\right)\prod_{r=0}^{k-1}\left(x_0+r\right)\sum_{i=1}^{n}\sum_{j=1}^{i}P_{ij}\left(y_1,...,y_i\right)\varepsilon^i+$$
$$\prod_{r=0}^{k-1}\left(x_0+r\right)\sum_{i=2}^{n}\sum_{j=2}^{i-1}\left(\sum_{m=1}^{j}P_{jm}\left(x_1,...,x_i\right)\Gamma^{(j-m-1)}\left(x_0\right)\right)\times$$
$$\left(\sum_{m=1}^{i-j}P_{i-j,m}\left(y_1,...,y_{i-j}\right)\right)\varepsilon^i.$$

One can easily, using the propertis of the generator polynomials conclude, that

$$\begin{cases}\Gamma'\left(x_0+k\right)=\prod_{r=0}^{k-1}\left(x_0+r\right)\left(\Gamma'\left(x_0\right)+\Gamma\left(x_0\right)\sum_{r=0}^{k-1}\frac{1}{x_0+r}\right),\\ \text{and}\\ \sum_{j=1}^{i}P_{ij}\left(x_1,...,x_i\right)\Gamma^{(i-j+1)}\left(x_0+k\right)\\ =\prod_{r=0}^{k-1}\left(x_0+r\right)\sum_{j=1}^{i}\left(P_{ij}\left(x_1,...,x_i\right)\Gamma^{(i-j+1)}\left(x_0\right)+\Gamma\left(x_0\right)P_{ij}\left(y_1,...,y_i\right)\right)+\\ \sum_{j=1}^{i-1}\left(\sum_{m=1}^{j}P_{jm}\left(x_1,...,x_i\right)\Gamma^{(j-m-1)}\left(x_0\right)\right)\left(\sum_{m=1}^{i-j}P_{i-j,m}\left(y_1,...,y_{i-j}\right)\right)\end{cases}$$

This allows us to achieve the proof. □

Example 1. We investigate the particular case $n=2$. We have for $k\in\mathbb{N}^*$

$$\Gamma\left(z+k\right)=\Gamma\left(z\right)\prod_{r=0}^{k-1}\left(z+r\right).$$

So, for $z=x_0+x_1\varepsilon+x_2\varepsilon^2$, we can write

$$\Gamma\left(z+k\right)=\Gamma\left(z\right)\prod_{r=0}^{k-1}\left(x_0+r+x_1\varepsilon+x_2\varepsilon^2\right)$$
$$=\Gamma\left(z\right)\prod_{r=0}^{k-1}\left(x_0+r\right)\prod_{r=0}^{k-1}\left(1+\frac{x_1}{x_0+r}\varepsilon+\frac{x_2}{x_0+r}\varepsilon^2\right)$$
$$=\Gamma\left(z\right)\prod_{r=0}^{k-1}\left(x_0+r\right)\prod_{r=0}^{k-1}e^{y_{r1}+y_{r2}\varepsilon^2},\tag{33}$$

where

$$e^{y_{r1}\varepsilon+y_{r2}\varepsilon^2}=1+\frac{x_1}{x_0+r}\varepsilon+\frac{x_2}{x_0+r}\varepsilon^2.\tag{34}$$

Equation gives

$$\begin{cases}y_{r1}=\frac{x_1}{x_0+r},\\ y_{r2}=\frac{x_2}{x_0+r}-\frac{1}{2}\left(\frac{x_1}{x_0+r}\right)^2.\end{cases}\tag{35}$$

Thus, one finds

$$\Gamma(z+k)=\prod_{r=0}^{k-1}(x_0+r)\Gamma(z)$$
$$\exp\left(x_1\left(\sum_{r=0}^{k-1}\frac{1}{x_0+r}\right)\varepsilon+\left(x_2\left(\sum_{r=0}^{k-1}\frac{1}{x_0+r}\right)-\frac{x_1^2}{2}\sum_{r=0}^{k-1}\frac{1}{(x_0+r)^2}\right)\varepsilon^2\right). \tag{36}$$

For $x\in\mathbb{R}^*$, let us denote by $H_{p,q}(x)$ the $p-$th harmonic number given by

$$H_{p,q}(x)=\sum_{r=0}^{p}\frac{1}{(x+r)^q}. \tag{37}$$

Hence, equation can be witten making use

$$\begin{aligned}\Gamma(z+k)&=\prod_{r=0}^{k-1}(x_0+r)\Gamma(z)\times\\ &\exp\left(x_1H_{k-1,1}(x_0)\varepsilon+\left(x_2H_{k-1,1}(x_0)-\frac{x_1^2}{2}H_{k-1,2}(x_0)\right)\varepsilon^2\right)\\ &=\prod_{r=0}^{k-1}(x_0+r)\Gamma(z)\left[1+x_1H_{k-1,1}(x_0)\varepsilon+\right.\\ &\left.\left(x_1^2H_{k-1,1}(x_0)^2+x_2H_{k-1,1}(x_0)-\frac{x_1^2}{2}H_{k-1,2}(x_0)\right)\varepsilon^2\right].\end{aligned} \tag{38}$$

So, we can infer

$$\begin{aligned}&\Gamma(x_0+k)+\sum_{i=1}^{n}\sum_{j=1}^{i}P_{ij}(x_1,...,x_i)\Gamma^{(i-j+1)}(x_0+k)\varepsilon^i\\ =&\left[\Gamma(x_0)+\sum_{i=1}^{n}\sum_{j=1}^{i}P_{ij}(x_1,...,x_i)\Gamma^{(i-j+1)}(x_0)\varepsilon^i\right]\\ =&\prod_{r=0}^{k-1}(x_0+r)\left[1+x_1H_{k-1,1}(x_0)\varepsilon+\right.\\ &\left.\left(x_1^2H_{k-1,1}(x_0)^2+x_2H_{k-1,1}(x_0)-\frac{x_1^2}{2}H_{k-1,2}(x_0)\right)\varepsilon^2\right].\end{aligned}$$

Consequently, we deduce making use some algebraic manipulations

$$\Gamma(x_0+k)=\prod_{r=0}^{k-1}(x_0+r)\Gamma(x_0), \tag{39}$$

$$\Gamma'(x_0+k)=\prod_{r=0}^{k-1}(x_0+r)\left(\Gamma'(x_0)+\Gamma(x_0)H_{k-1,1}(x_0)\right), \tag{40}$$

$$\begin{aligned}\Gamma''(x_0+k)=&\prod_{r=0}^{k-1}(x_0+r)\left(\Gamma''(x_0)+\Gamma'(x_0)H_{k-1,1}(x_0)+\right.\\ &\left.\Gamma(x_0)\left(H_{k-1,1}(x_0)^2-H_{k-1,2}(x_0)\right)\right).\end{aligned} \tag{41}$$

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