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On Schur power convexity of generalized invariant contra harmonic means with respect to geometric means

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Abstract: In this article, we investigate the power convexity of two generalized forms of the invariant of the contra harmonic mean with respect to the geometric mean, and establish several inequalities involving bivariate power mean as applications. Some open problems related to the Schur power convexity and concavity are also given.

Keywords: Schur power convexity; generalized invariant contra harmonic mean; majorization; binary power mean.

MSC: 26E60, 26D15, 26A51,34k38.

1. Introduction

Over the past century, convex analysis has emerged as a pivotal research area across various scientific fields, including applied mathematics, physics, optimization, communications and networks, economics and finance, and automatic control systems. Numerous new concepts of generalized convexity and concavity have been extensively studied by researchers; for more details, we refer readers to the research monographs and papers [1–9] and the references therein. Majorization theory has significantly contributed to convex theory, particularly in the study of inequalities and various mathematical means. The properties of various binary means are a crucial topic in contemporary inequality research. In recent years, the study of the Schur convexity of means has garnered increasing attention from scholars (see, e.g., [10–42]).

Definition 1. Let a and b be two positive numbers.

(i) Two means $M(a,b)$ and $N(a,b)$ of a and b are said to be inverses with respect to the geometric mean $G(a,b)=\sqrt{ab}$ if

$$M(a,b)\cdot N(a,b)=\frac{[G(a,b)]^2}{M(a,b)}.$$

In other words, $\frac{[G(a,b)]^2}{M(a,b)}$ is a mean which is inverse of $M(a,b)$.

(ii) The invariant of the contra harmonic mean $C(a,b)=\frac{a^2+b^2}{a+b}$ with respect to the geometric mean $G(a,b)=\sqrt{ab}$ is defined by

$$V(a,b)=\frac{ab(a+b)}{a^2+b^2}. \tag{1}$$

Remark 1. It is worth noting that this mean $V(a,b)$ can be expressed as

$$V(a,b)=\frac{ab(a+b)}{a^2+b^2}=\frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a^2}+\frac{1}{b^2}}=\frac{H(a^2,b^2)}{H(a,b)},$$

where $H(a,b)=\frac{2}{\frac{1}{a}+\frac{1}{b}}$ is the harmonic mean of two positive numbers a and b.

In 2017, R. Sampath Kumar and K M. Nagaraja [43] studied the Schur power convexity of $V(a,b)$, and obtained the following result.

Theorem 2 (see [43]). *For $a<b$, $z=\frac{b}{a}>1$, then $V(a,b)$ is*

(a) *Schur m-power convex, if $-\frac{2}{3}<m<\frac{1}{2}$;*
(b) *Schur m-power concave if $m\in\left(-\infty,\frac{-3}{2}\right)\cup\left(\frac{1}{2},\infty\right)$.*

Remark 2. In fact, the condition "$a<b$, $z=\frac{b}{a}>1$" in Theorem 2 is equivalent to the condition "$0<a<b$".

Inspired by Theorem 2, we will investigate the Schur power convexity of the following generalized forms of the invariant contra harmonic mean $V_k(a,b)$ and $V_f(a,b)$ and establish new generalizations of Theorem 2.

Definition 3. Let $(a,b)\in\mathbb{R}^2_+$ and f be a positive function on $\mathbb{R}_+$ (i.e., $f(x)>0$ for all $x\in\mathbb{R}_+$). Define

$$V_k(a,b)=\frac{H(a^{k+1},b^{k+1})}{H(a^k,b^k)}$$

and

$$V_f(a,b)=\frac{f(a)+f(b)}{(f(a))^2+(f(b))^2}.$$

The following is a generalizations of Theorem 2 which is one of the main results of this paper.

Theorem 4. *Let $(a,b)\in\mathbb{R}^2_+$ and $k\in\mathbb{N}$. Then the following statements hold:*

(a) *If $0\le m\le 1$, then $V_k(a,b)$ is Schur m-power concave with $(a,b)\in\mathbb{R}^2_+$;*
(b) *$V_k(a,b)$ is Schur geometrically concave with $(a,b)\in\mathbb{R}^2_+$;*
(c) *$V_k(a,b)$ is Schur harmonically concave with $(a,b)\in\mathbb{R}^2_+$.*

Definition 5. Let $(a,b)\in\mathbb{R}^2_+$. Define

$$V_{p,q}(a,b)=\frac{H(a^q,b^q)}{H(a^p,b^p)}. \tag{2}$$

Remark 3. It is easy to see that

$$V_{p,q}(a,b)=\frac{H(a^q,b^q)}{H(a^{q-1},b^{q-1})}\frac{H(a^{q-1},b^{q-1})}{H(a^{q-2},b^{q-2})}\cdots\frac{H(a^{p+1},b^{p+1})}{H(a^p,b^p)}.$$

As a direct consequence of Theorem 4, we obtain the following corollary which is also a generalizations of Theorem 2.

Corollary 6. *Let $(a,b)\in\mathbb{R}^2_+$ and $p,q\in\mathbb{N}$ with $p<q$. Then the following statements hold:*

(a) *If $0\le m\le 1$, then $V_{p,q}(a,b))$ is Schur m-power concave with $(a,b)\in\mathbb{R}^2_+$.*
(b) *$V_{p,q}(a,b)$ is Schur geometrically concave with $(a,b)\in\mathbb{R}^2_+$.*
(c) *$V_{p,q}(a,b)$ is Schur harmonically concave with $(a,b)\in\mathbb{R}^2_+$.*

The sufficient conditions for $V_f(a,b)$ to have the Schur-m power concavity are stated in the following theorem.

Theorem 7. *Let $(a,b)\in\mathbb{R}^2_+$. Then the following statements hold:*

(a) *If the function $f(t)$ is a decreasing positive convex function and $m\ge 1$, then $V_f(a,b)$ is Schur-m power concave with $(a,b)\in\mathbb{R}^2_+$.*

(*b*) *If the function* $f(t)$ *is a increasing positive convex function and* $0 \le m \le 1$*, then* $V_f(a,b)$ *is Schur-m power concave with* $(a,b) \in \mathbb{R}_+^2$.

The detailed proofs of Theorems 4 and 7 will be given in Sections 3. Finally, some applications are presented in Sections 4.

2. Preliminaries

We first recall some notations, definitions and well-known results, which will be used in this paper. For $n \in \mathbb{N}$ (the set of positive integers), we write

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}} = \{\mathbf{a} = (a_1, a_2, \ldots, a_n) : a_i \in \mathbb{R}, i = 1, \ldots, n\}$$

and

$$\mathbb{R}_+^n = \underbrace{\mathbb{R}_+ \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+}_{n \text{ times}} = \{\mathbf{a} = (a_1, a_2, \ldots, a_n) : a_i > 0, i = 1, \ldots, n\}$$

where $\mathbb{R} := (-\infty, +\infty)$ and $\mathbb{R}_+ := (0, +\infty)$. In particular, we denote $\mathbb{R}^1$ and $\mathbb{R}_+^1$ simply as $\mathbb{R}$ and $\mathbb{R}_+$ respectively. Recall that a set $\Omega \subset \mathbb{R}^n$ is called *convex* if for any $\mathbf{a} = (a_1, a_2, \ldots, a_n), \mathbf{b} = (b_1, b_2, \ldots, b_n) \in \Omega$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, we have

$$\alpha \mathbf{a} + \beta \mathbf{b} = (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \ldots, \alpha a_n + \beta b_n) \in \Omega.$$

Definition 8 (see [44,45])**.** Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$. The vector $\boldsymbol{a}$ is said to be majorized by $\boldsymbol{b}$, denoted by $\boldsymbol{a} \prec \boldsymbol{b}$, if

$$\sum_{i=1}^{t} a_{[i]} \le \sum_{i=1}^{t} b_{[i]} \quad \text{for } 1 \le t \le n-1,$$

and

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i,$$

where $a_{[1]} \ge \cdots \ge a_{[n]}$ and $b_{[1]} \ge \cdots \ge b_{[n]}$ are rearrangements of $\boldsymbol{a}$ and $\boldsymbol{b}$ in a descending order.

We need the following known definitions and lemmas.

Definition 9 (see [44,45])**.** Let $D \subset \mathbb{R}^n$. A function $f : D \to \mathbb{R}$ is said to be

(*i*) *Schur-convex* on D if $\boldsymbol{x} \prec \boldsymbol{y}$ on D implies $f(\boldsymbol{x}) \le f(\boldsymbol{y})$;
(*ii*) *Schur-concave* on D if and only if $-f$ is Schur-convex.

Definition 10 (see [46,47])**.** Let $D \subset \mathbb{R}_+^n$ and $\boldsymbol{x} = (x_1, \ldots, x_n), \boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}_+^n$.

(*i*) D is called a *geometrically convex* set if $(x_1^\alpha y_1^\beta, \ldots, x_n^\alpha y_n^\beta) \in D$ for any $\boldsymbol{x}, \boldsymbol{y} \in D$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$.
(*ii*) $f : D \to \mathbb{R}_+$ is said to be a *Schur-geometrically convex* function on D if

$$(\log x_1, \ldots, \log x_n) \prec (\log y_1, \ldots, \log y_n) \quad \text{on } D$$

implies $f(\boldsymbol{x}) \le f(\boldsymbol{y})$.
(*iii*) $f : D \to \mathbb{R}_+$ is said to be a *Schur-geometrically concave* function on D if and only if $-f$ is Schur-geometrically convex function.

Definition 11 (see [48,49])**.** Let $D \subset \mathbb{R}_+^n$.

(i) D is said to be a *harmonically convex* set if

$$\frac{\boldsymbol{ab}}{t\boldsymbol{a}+(1-t)\boldsymbol{b}} \in D$$

for every $\boldsymbol{a} = (a_1, \ldots, a_n), \boldsymbol{b} = (b_1, \ldots, b_n) \in D$ and $t \in [0, 1]$, where $\boldsymbol{ab} = \sum_{i=1}^{n} a_i b_i$ and $\frac{1}{\boldsymbol{a}} = \left(\frac{1}{a_1}, \cdots, \frac{1}{a_n}\right)$.

(ii) A function $f : D \to \mathbb{R}_+$ is said to be *Schur harmonically convex* on D if $\frac{1}{\boldsymbol{a}} \prec \frac{1}{\boldsymbol{b}}$ implies $f(\boldsymbol{a}) \le f(\boldsymbol{b})$.

(iii) A function $f : D \to \mathbb{R}_+$ is said to be *Schur harmonically concave* on D if and only if $-f$ is a Schur harmonically convex function.

Definition 12 (see [11,12]). Let $D \subset \mathbb{R}_+^n$ and $h : \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$h(x) = \begin{cases} \dfrac{x^k - 1}{k}, & k \neq 0, \\ \log x, & k = 0. \end{cases} \tag{3}$$

Then a function $g : D \to \mathbb{R}$ is said to be *Schur m-power convex* on D if

$$(h(a_1), \ldots, h(a_n)) \prec (h(b_1), \ldots, h(b_n))$$

for all $\boldsymbol{a} = (a_1, \ldots, a_n) \in D$ and $\boldsymbol{b} = (b_1, \ldots, b_n) \in D$ implies $g(\boldsymbol{a}) \le g(\boldsymbol{b})$.

If $-g$ is Schur m-power convex, then we say that g is *Schur m-power concave*.

Remark 4. If we respectively take $h(x) = x$, $h(x) = \log x$ and $h(x) = \frac{1}{x}$ in Definition 12, then the definitions of Schur-convex, Schur-geometrically convex, and Schur-harmonically convex functions can be deduced respectively.

Lemma 13 (see [44,45]). *Let $D \subset \mathbb{R}^n$ be a convex set with a nonempty interior set D°. Let $f : D \to \mathbb{R}$ be continuous on D and differentiable in D°. Then the following statements hold:*

(i) *f is Schur-convex if and only if it is symmetric on D and if*

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \ge 0$$

holds for any $\boldsymbol{x} = (x_1, \cdots, x_n) \in D^\circ$.

(ii) *f is Schur-concave if and only if it is symmetric on D and if*

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \le 0$$

holds for any $\boldsymbol{x} = (x_1, \cdots, x_n) \in D^\circ$.

Lemma 14 (see [46,47]). *Let $D \subset \mathbb{R}_+^n$ be a symmetric geometrically convex set with a nonempty interior D°. Let $f : D \to \mathbb{R}_+$ be continuous on Ω and differentiable on D°. Then the following statements hold:*

(i) *f is a Schur geometrically convex function if and only if f is symmetric on D and*

$$(x_1 - x_2)\left(x_1\frac{\partial f}{\partial x_1} - x_2\frac{\partial f}{\partial x_2}\right) \ge 0$$

holds for any $\boldsymbol{x} = (x_1, \cdots, x_n) \in D^\circ$.

(ii) *f is a Schur geometrically concave function if and only if f is symmetric on D and*

$$(x_1 - x_2)\left(x_1\frac{\partial f}{\partial x_1} - x_2\frac{\partial f}{\partial x_2}\right) \le 0$$

holds for any $\boldsymbol{x} = (x_1, \cdots, x_n) \in D^\circ$.

Lemma 15 (see [48,49]). *Let $D \subset \mathbb{R}_+^n$ be a symmetric harmonically convex set with a nonempty interior D°. Let $f : D \to \mathbb{R}_+$ be continuous on D and differentiable on D°. Then the following statements hold:*

(*i*) *f is a Schur harmonically convex function if and only if f is symmetric on D and*

$$(x_1 - x_2)\left(x_1^2\frac{\partial f}{\partial x_1} - x_2^2\frac{\partial f}{\partial x_2}\right) \geq 0$$

holds for any $\boldsymbol{x} = (x_1, \cdots, x_n) \in D^\circ$.

(*ii*) *f is a Schur harmonically concave function if and only if f is symmetric on D and*

$$(x_1 - x_2)\left(x_1^2\frac{\partial f}{\partial x_1} - x_2^2\frac{\partial f}{\partial x_2}\right) \leq 0$$

holds for any $\boldsymbol{x} = (x_1, \cdots, x_n) \in D^\circ$.

Lemma 16 (see [11,12]). *Let $D \subset \mathbb{R}_+^n$ be a symmetric set with a nonempty interior D° and $f : D \to \mathbb{R}_+$ be continuous on D and differentiable in D°. Then f is Schur m-power convex on D if and only if f is symmetric on D and*

$$\frac{x_1^k - x_2^k}{k}\left[x_1^{1-k}\frac{\partial f(\boldsymbol{x})}{\partial x_1} - x_2^{1-k}\frac{\partial f(\boldsymbol{x})}{\partial x_2}\right] \geq 0, \quad \text{if } k \neq 0 \tag{4}$$

and

$$(\log x_1 - \log x_2)\left[x_1\frac{\partial f(\boldsymbol{x})}{\partial x_1} - x_2\frac{\partial f(\boldsymbol{x})}{\partial x_2}\right] \geq 0, \quad \text{if } k = 0 \tag{5}$$

for all $\boldsymbol{x} = (x_1, \cdots, x_n) \in D^\circ$.

Lemma 17 (see [12]). *Let $(a, b) \in \mathbb{R}_+^2$. Then we have*

$$\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \prec (a, b), \tag{6}$$

$$\left(\log\sqrt{ab}, \log\sqrt{ab}\right) \prec (\log a, \log b), \tag{7}$$

and

$$\left(\frac{(M_m(a,b))^m - 1}{m}, \frac{(M_m(a,b))^m - 1}{m}\right) \prec \left(\frac{a^m - 1}{m}, \frac{b^m - 1}{m}\right). \tag{8}$$

where $M_m(a, b) = \left(\frac{a^m+b^m}{2}\right)^{\frac{1}{m}}$.

3. Proofs of Theorems 4 and 7

First, we show Theorem 4 as follows:

3.1. Proof of Theorem 4.

It is not difficult to verify that

$$V_k(a, b) = \frac{H(a^{k+1}, b^{k+1})}{H(a^k, b^k)} = \frac{a^{k+1}b + ab^{k+1}}{a^{k+1} + b^{k+1}}.$$

Then

$$\frac{\partial V_k(a,b)}{\partial a} = \frac{A}{(a^{k+1} + b^{k+1})^2}$$

and

$$\frac{\partial V_k(a,b)}{\partial b} = \frac{B}{(a^{k+1} + b^{k+1})^2},$$

where

$$\begin{aligned}A =&[(k+1)a^k b + b^{k+1}](a^{k+1}+b^{k+1}) - (a^{k+1}b + ab^{k+1})[(k+1)a^k]\\ =&(k+1)a^{2k+1}b + (k+1)a^k b^{k+2} + b^{k+1}a^{k+1} + b^{2k+2}\\ &- (k+1)a^{2k+1}b - (k+1)a^{k+1}b^{k+1}\\ =&(k+1)a^k b^{k+2} + b^{2k+2} - ka^{k+1}b^{k+1}\end{aligned}$$

and

$$B = (k+1)b^k a^{k+2} + a^{2k+2} - kb^{k+1}a^{k+1} \quad \text{(by the symmetry of } V_k(a,b)).$$

So, we have

$$\begin{aligned}&a^{1-m}A - b^{1-m}B\\ =&a^{1-m}[(k+1)a^k b^{k+2} + b^{2k+2} - ka^{k+1}b^{k+1}]\\ &- b^{1-m}[(k+1)b^k a^{k+2} + a^{2k+2} - kb^{k+1}a^{k+1}]\\ =&a^{1-m}b^{1-m}[(k+1)a^k b^{k+m+1} + b^{2k+m+1} - ka^{k+1}b^{k+m}]\\ &- a^{1-m}b^{1-m}[(k+1)b^k a^{k+m+1} + a^{2k+m+1} - kb^{k+1}a^{k+m}]\\ =&a^{1-m}b^{1-m}[(k+1)a^k b^k(b^{m+1} - a^{m+1})\\ &+ (b^{2k+m+1} - a^{2k+m+1}) + ka^{k+1}b^{k+1}(a^{m-1} - b^{m-1})].\end{aligned}$$

(a). Let

$$\begin{aligned}\Delta :=& \frac{a^m - b^m}{m}\left(a^{1-m}\frac{\partial V_k(a,b)}{\partial a} - b^{1-m}\frac{\partial V_k(a,b)}{\partial b}\right)\\ =& \frac{a^m - b^m}{m}\cdot\frac{a^{1-m}A - b^{1-m}B}{(a^{k+1}+b^{k+1})^2}.\end{aligned}$$

For $0 \le m \le 1$, noting that $(a^m - b^m)(b^{m+1} - a^{m+1}) \le 0$, $(a^m - b^m)(b^{2k+m+1} - a^{2k+m+1}) \le 0$ and $(a^m - b^m)(a^{m-1} - b^{m-1}) \le 0$ are true. So $\Delta \le 0$. Applying Lemma 15, it follows that $V_k(a,b)$ is Schur-m power concave with $(a,b) \in \mathbb{R}^2_+$.

(b). We first calculate

$$\begin{aligned}aA - bB =&(k+1)a^{k+1}b^{k+2} + ab^{2k+2} - ka^{k+2}b^{k+1}\\ &- [(k+1)b^{k+1}a^{k+2} + ba^{2k+2} - kb^{k+2}a^{k+1}]\\ =&(2k+1)a^{k+1}b^{k+1}(b-a) + ab(b^{2k+1} - a^{2k+1}).\end{aligned}$$

So, we obtain

$$\begin{aligned}\Delta_0 :=& (\log a - \log b)\left(a\frac{\partial V_k(a,b)}{\partial a} - b\frac{\partial V_k(a,b)}{\partial b}\right)\\ =& (\log a - \log b)\cdot\frac{aA - bB}{(a^{k+1}+b^{k+1})^2}\\ =& (\log a - \log b)\cdot\frac{(2k+1)a^{k+1}b^{k+1}(b-a) + ab(b^{2k+1} - a^{2k+1})}{(a^{k+1}+b^{k+1})^2} \le 0.\end{aligned}$$

By Lemma 14, $V_k(a,b)$ is Schur geometrically concave with $(a,b) \in \mathbb{R}^2_+$.

(c). Since

$$\begin{aligned}a^2A-b^2B=&(k+1)a^{k+2}b^{k+2}+a^2b^{2k+2}-ka^{k+3}b^{k+1}\\&-[(k+1)b^{k+2}a^{k+2}+b^2a^{2k+2}-kb^{k+3}a^{k+1}]\\=&a^2b^2(b^{k+2}-a^{k+2})+ka^{k+1}b^{k+1}(b^2-a^2),\end{aligned}$$

we get

$$\begin{aligned}\Delta_1:=&(a-b)\left(a^2\frac{\partial V_k(a,b)}{\partial a}-b^2\frac{\partial V_k(a,b)}{\partial b}\right)\\=&(a-b)\cdot\frac{a^2A-b^2B}{(a^{k+1}+b^{k+1})^2}\\=&(a-b)\cdot\frac{a^2b^2(b^{2k+2}-a^{2k+2})+ka^{k+1}b^{k+1}(b^2-a^2)}{(a^{k+1}+b^{k+1})^2}\le 0.\end{aligned}$$

By Lemma 15, $V_k(a,b)$ is Schur harmonically concave with $(a,b)\in\mathbb{R}^2_+$.
The proof of Theorem 4 is completed. □

Next, we prove Theorem 7.

3.2. Proof of Theorem 7.

Since

$$\frac{\partial V_f(a,b)}{\partial a}=\frac{A}{[(f(a))^2+(f(b))^2]^2},\quad \frac{\partial V_f(a,b)}{\partial b}=\frac{B}{[(f(a))^2+(f(b))^2]^2},$$

where

$$\begin{aligned}A=&f'(a)[(f(a))^2+(f(b))^2]-2f(a)[(f(a)+(f(b)]f'(a)\\=&f'(a)[(f(a))^2+(f(b))^2-2f(a)(f(a)+f(b))]\\=&f'(a)[(f(b))^2-(f(a))^2-2f(a)f(b)]\end{aligned}$$

and

$$B=f'(b)[(f(a))^2-(f(b))^2-2f(a)f(b)],$$

we obtain

$$\begin{aligned}&a^{1-m}A-b^{1-m}B\\=&a^{1-m}f'(a)[(f(b))^2-(f(a))^2-2f(a)f(b)]\\=&-b^{1-m}f'(b)[(f(a))^2-(f(b))^2-2f(a)f(b)]\\=&[(f(b))^2-(f(a))^2][a^{1-m}f'(a)+b^{1-m}f'(b)]-2f(a)f(b)(a^{1-m}f'(a)-b^{1-m}f'(b))\end{aligned}$$

and

$$\begin{aligned}\Delta_f:=&\frac{a^m-b^m}{m}\left(a^{1-m}\frac{\partial V_f(a,b)}{\partial a}-b^{1-m}\frac{\partial V_f(a,b)}{\partial b}\right)\\=&\frac{a^m-b^m}{m}\cdot\frac{a^{1-m}A-b^{1-m}B}{(f(a)+f(b))^2}.\end{aligned}$$

From the symmetry of function $V_f(a,b)$ with respect to a and b, it can be assumed that $a\le b$. Let $u(t)=t^{1-m}f'(t)$. Then

$$u'(t)=(1-m)t^{-m}f'(t)+t^{1-m}f''(t).$$

(a). If $f(t)$ is a decreasing convex function and $m \geq 1$, then $f'(t) \leq 0$ and $f''(t) \geq 0$. So it follows that $u'(t) \geq 0$, $(f(b))^2 - (f(a))^2 \leq 0$ and

$$a^{1-m} f'(a) + b^{1-m} f'(b) \leq 0,$$

which imply

$$-2f(a)f(b)(a^{1-m} f'(a) - b^{1-m} f'(b)) \geq 0.$$

Hence we have $a^{1-m}A - b^{1-m}B \geq 0$. Since $a^m - b^m \leq 0$, we get $\Delta_f \leq 0$. Applying Lemma 16, we show that $V_f(a,b)$ is Schur-m power concave with $(a,b) \in \mathbb{R}^2_+$.

(b). If $f(t)$ is a increasing convex function and $0 \leq m \leq 1$, we have $u'(t) \geq 0$, and then $a^{1-m} f'(a) - b^{1-m} f'(b) \leq 0$. Since $f(t)$ is increasing, it follows that

$$[(f(b))^2 - (f(a))^2][a^{1-m} f'(a) + b^{1-m} f'(b)] \geq 0.$$

Therefore $a^{1-m}A - b^{1-m}B \geq 0$, which implies $\Delta_f \leq 0$. By Lemma 16, we prove that $V_f(a,b)$ is Schur-m power concave with $(a,b) \in \mathbb{R}^2_+$.

The proof of Theorem 7 is complete. □

4. Applications and open problems

In this section, we will give some interesting applications of Theorems 4 and 7.

Theorem 18. *Let $(a,b) \in \mathbb{R}^2_+$ and $p,q \in \mathbb{N}$ with $p < q$. If $0 \leq m \leq 1$, then*

$$(M_m(a,b))^{q-p} \leq \frac{a^{-q} + b^{-q}}{a^{-p} + b^{-p}}. \tag{9}$$

Proof. Since $0 \leq m \leq 1$, by Theorem 4 and (8), we have

$$V_{p,q}\left(M_m(a,b), M_m(a,b)\right) \geq V_{p,q}(a,b),$$

this is

$$\frac{\frac{2}{\frac{2}{(M_m(a,b))^p}}}{\frac{2}{\frac{2}{(M_m(a,b))^q}}} \geq \frac{\frac{2}{\frac{1}{a^p} + \frac{1}{b^p}}}{\frac{2}{\frac{1}{a^q} + \frac{1}{b^q}}}, \tag{10}$$

rearranging gives the inequality (9). □

Theorem 19. *Let $0 < a, b \leq \frac{1}{2}$. If $m \geq 1$, then*

$$\frac{\left(\log\left(\frac{1}{a} - 1\right)\right)^2 + \left(\log\left(\frac{1}{b} - 1\right)\right)^2}{\log\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)} \geq \log\left(\frac{1}{M_m(a,b)} - 1\right). \tag{11}$$

Proof. Let $g(t) = \log\left(\frac{1}{t} - 1\right)$ for $0 < t < \frac{1}{2}$. Since $g'(t) = \frac{-1}{(1-t)t} \leq 0$ and $g''(t) = \frac{1-2t}{(1-t)^2 x^2)} \geq 0$, $g(t)$ is a decreasing convex function on $\mathbb{R}$. For $m \geq 1$, from (8) and Theorem 7(a), we have

$$V_g\left(M_m(a,b), M_m(a,b)\right) \geq V_g(a,b),$$

this is

$$\frac{\log\left(\frac{1}{a}-1\right)+\log\left(\frac{1}{b}-1\right)}{\left(\log\left(\frac{1}{a}-1\right)\right)^2+\left(\log\left(\frac{1}{b}-1\right)\right)^2} \le \frac{\log\left(\frac{1}{M_m(a,b)}-1\right)+\log\left(\frac{1}{M_m(a,b)}-1\right)}{\left(\log\left(\frac{1}{M_m(a,b)}-1\right)\right)^2+\left(\log\left(\frac{1}{M_m(a,b)}-1\right)\right)^2} = \frac{1}{\log\left(\frac{1}{M_m(a,b)}-1\right)},$$

rearranging gives the inequality (11). □

Theorem 20. *Let $(a,b)\in\mathbb{R}^2_+$. If $0\le m\le 1$, then*

$$\frac{e^{2a}+e^{2b}}{e^a+e^b} \ge M_m(a,b). \tag{12}$$

Proof. It is known that $f(t)=e^t$ is a increasing convex function on $\mathbb{R}$. For $0\le m\le 1$, using (8) and applying Theorem 7(b), we get

$$V_f\left(M_m(a,b),M_m(a,b)\right) \ge V_f(a,b),$$

this is

$$\frac{e^a+e^b}{e^{2a}+e^{2b}} \le \frac{e^{M_m(a,b)}+e^{M_m(a,b)}}{(e^{M_m(a,b)})^2+(e^{M_m(a,b)})^2} = \frac{1}{e^{M_m(a,b)}}, \tag{13}$$

rearranging gives the inequality (14). □

Theorem 21. *Let $(a,b)\in\mathbb{R}^2_+$. If $m\ge 1$, then*

$$\frac{\left(1-\int_0^a e^{-\frac{x^2}{2}}dx\right)^2+\left(1-\int_0^b e^{-\frac{x^2}{2}}dx\right)^2}{\left(1-\int_0^a e^{-\frac{x^2}{2}}dx\right)+\left(1-\int_0^b e^{-\frac{x^2}{2}}dx\right)} \ge 1-\int_0^{M_m(a,b)} e^{-\frac{x^2}{2}}dx \tag{14}$$

Proof. It is known that $f(t)=1-\int_0^t e^{-\frac{x^2}{2}}dx$ for $t\ge 0$ is a decreasing convex function on $\mathbb{R}$. For $m\ge 1$, from (8) and using Theorem 7 (a), we obtain

$$V_f\left(M_m(a,b),M_m(a,b)\right) \ge V_f(a,b),$$

this is

$$\frac{\left(1-\int_0^a e^{-\frac{x^2}{2}}dx\right)+\left(1-\int_0^b e^{-\frac{x^2}{2}}dx\right)}{\left(1-\int_0^a e^{-\frac{x^2}{2}}dx\right)^2+\left(1-\int_0^b e^{-\frac{x^2}{2}}dx\right)^2}$$
$$\le \frac{\left(1-\int_0^{M_m(a,b)} e^{-\frac{x^2}{2}}dx\right)+\left(1-\int_0^{M_m(a,b)} e^{-\frac{x^2}{2}}dx\right)}{(1-\int_0^{M_m(a,b)} e^{-\frac{x^2}{2}}dx)^2+(1-\int_0^{M_m(a,b)} e^{-\frac{x^2}{2}}dx)^2} = \frac{1}{1-\int_0^{M_m(a,b)} e^{-\frac{x^2}{2}}dx},$$

rearranging gives the inequality (14). □

Here we present a selection of open problems that are related to the Schur power convexity and concavity.

Proposition 22. *What is the Schur power convexity of $V_k(a,b)$ when $m>1$ or $m<0(m\ne -1)$?*

Proposition 23. *If the function $f(t)$ is a decreasing concave function or increasing concave function and $0\le m\le 1$, then what is the Schur power convexity of $V_f(a,b)$?*

5. Conclusions

In this paper, we establish new generalizations of Sampath Kumar-Nagaraja Theorem (i.e. Theorem 2) as follows:

- (See Theorem 4):
  Let $(a,b) \in \mathbb{R}^2_+$ and $k \in \mathbb{N}$. Then the following statements hold:

  (a) If $0 \le m \le 1$, then $V_k(a,b)$ is Schur m-power concave with $(a,b) \in \mathbb{R}^2_+$;
  (b) $V_k(a,b)$ is Schur geometrically concave with $(a,b) \in \mathbb{R}^2_+$;
  (c) $V_k(a,b)$ is Schur harmonically concave with $(a,b) \in \mathbb{R}^2_+$.

- (See Corollary 6):
  Let $(a,b) \in \mathbb{R}^2_+$ and $p,q \in \mathbb{N}$ with $p < q$. Then the following statements hold:

  (a) If $0 \le m \le 1$, then $V_{p,q}(a,b))$ is Schur m-power concave with $(a,b) \in \mathbb{R}^2_+$.
  (b) $V_{p,q}(a,b)$ is Schur geometrically concave with $(a,b) \in \mathbb{R}^2_+$.
  (c) $V_{p,q}(a,b)$ is Schur harmonically concave with $(a,b) \in \mathbb{R}^2_+$.

- (See Theorem 7):
  Let $(a,b) \in \mathbb{R}^2_+$. Then the following statements hold:

  (a) If the function $f(t)$ is a decreasing positive convex function and $m \ge 1$, then $V_f(a,b)$ is Schur-m power concave with $(a,b) \in \mathbb{R}^2_+$.
  (b) If the function $f(t)$ is a increasing positive convex function and $0 \le m \le 1$, then $V_f(a,b)$ is Schur-m power concave with $(a,b) \in \mathbb{R}^2_+$.

Finally, some interesting applications are presented in Section 4.

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