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On the relative growth of Dirichlet series absolutely

Myroslav M. Sheremeta[1,*]

[1] Ivan Franko National University of Lviv, Lviv, Ukraine

* Correspondence: m.m.sheremeta@gmail.com

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Abstract: Let $\Lambda = (\lambda_n)$ be an increasing sequence of non-negative numbers tending to $+\infty$, with $\lambda_0 = 0$. We denote by $S(\Lambda, 0)$ a class of Dirichlet series $F(s) = \sum_{n=0}^{\infty} f_n \exp\{s\lambda_n\}, \quad s = \sigma + it$, which have an abscissa of absolute convergence $\sigma_a = 0$. For $\sigma < 0$, we define $M_F(\sigma) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$. The growth of the function $F \in S(\Lambda, 0)$ is analyzed in relation to the function $G(s) = \sum_{n=0}^{\infty} g_n \exp\{s\lambda_n\} \in S(\Lambda, 0)$, via the growth of the function $1/|M_G^{-1}(M_F(\sigma))|$ as $\sigma \uparrow 0$. We investigate the connection between this growth and the behavior of the coefficients f_n and g_n in terms of generalized orders.

Keywords: Dirichlet series, relative growth, generalized order.

MSC: 30B50

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1. Introduction

Let f and g be entire transcendental functions, and let $M_f(r) = \max\{|f(z)| : |z| = r\}$. To study the relative growth of the functions f and g, Ch. Roy [1] introduced the order

$$\varrho_g[f] = \overline{\lim_{r \to +\infty}} \frac{\ln M_g^{-1}(M_f(r))}{\ln r}$$

and the lower order

$$\lambda_g[f] = \underline{\lim_{r \to +\infty}} \frac{\ln M_g^{-1}(M_f(r))}{\ln r},$$

which identify the growth of f relative to g with the growth of the function $M_g^{-1}(M_f(r))$ as $r \to +\infty$.

Research on the relative growth of entire functions has been extended by S.K. Datta, T. Biswas, and other mathematicians (see, for example, [2–5]) using maximal terms, Nevanlinna's characteristic function, and k-logarithmic orders. In [6], the relative growth of entire functions of two complex variables is considered, and in [7], the relative growth of entire Dirichlet series is studied in terms of R-orders.

Suppose that $\Lambda = (\lambda_n)$ is an increasing sequence tending to $+\infty$ of non-negative numbers, with $\lambda_0 = 0$. By $S(\Lambda, A)$, we denote a class of Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} f_n \exp\{s\lambda_n\}, \quad s = \sigma + it, \tag{1}$$

with the abscissa of absolute convergence $\sigma_a = A \in (-\infty, +\infty]$. For $\sigma < A$, we define

$$M_F(\sigma) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\},$$

and note that the function $M_F(\sigma)$ is continuous and increases to $+\infty$ on $(-\infty, A)$. Therefore, there exists a function $M_F^{-1}(x)$ inverse to $M_F(\sigma)$, which increases to A on $(|a_0|, +\infty)$.

We denote by L a class of continuous non-negative functions α on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. We say that $\alpha \in L^0$ if $\alpha \in L$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$ if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$, i.e., α is a slowly increasing function. Clearly, $L_{si} \subset L^0$.

If $\alpha \in L$, $\beta \in L$, and $F \in S(\Lambda, +\infty)$, then the quantities

$$\varrho_{\alpha,\beta}[F] := \varlimsup_{\sigma\to+\infty} \frac{\alpha(\ln M_F(\sigma))}{\beta(\sigma)}, \quad \lambda_{\alpha,\beta}[F] := \varliminf_{\sigma\to+\infty} \frac{\alpha(\ln M_F(\sigma))}{\beta(\sigma)}$$

are called the generalized (α, β)-order and the generalized lower (α, β)-order of F, respectively [8,9]. We say that F has generalized regular (α, β)-growth if

$$0 < \lambda_{\alpha,\beta}[F] = \varrho_{\alpha,\beta}[F] < +\infty.$$

If $G \in S(\Lambda, +\infty)$ and

$$G(s) = \sum_{n=1}^{\infty} g_n \exp\{s\lambda_n\}, \tag{2}$$

then the growth of the function F with respect to the function G is identified with the growth of the function $M_G^{-1}(M_F(\sigma))$ as $\sigma \to +\infty$. The generalized (α, β)-order $\varrho_{\alpha,\beta}[F]_G$ and the generalized lower (α, β)-order $\lambda_{\alpha,\beta}[F]_G$ of the function $F \in S(\Lambda, +\infty)$ with respect to a function $G \in S(\Lambda, +\infty)$ are defined as follows:

$$\varrho_{\alpha,\beta}[F]_G := \varlimsup_{\sigma\to+\infty} \frac{\alpha(M_G^{-1}(M_F(\sigma)))}{\beta(\sigma)}, \quad \lambda_{\alpha,\beta}[F]_G := \varliminf \sigma \to +\infty \frac{\alpha(M_G^{-1}(M_F(\sigma)))}{\beta(\sigma)}.$$

In the articles [10] and [11], the relationship between $\varrho_{\alpha,\beta}[F]_G$, $\lambda_{\alpha,\beta}[F]_G$, and $\varrho_{\alpha,\beta}[F]$, $\lambda_{\alpha,\beta}[F]$, $\varrho_{\alpha,\beta}[G]$, and $\lambda_{\alpha,\beta}[G]$ is studied, and formulas are found for calculating $\varrho_{\alpha,\beta}[F]_G$ and $\lambda_{\alpha,\beta}[F]_G$ in terms of the coefficients f_n and g_n. In particular, the following theorem is proved in [10].

Theorem 1. *Let* $\alpha \in L_{si}$, $\beta \in L^0$, *and* $\frac{d\beta^{-1}(c\alpha(x))}{d\ln x} = O(1)$ *as* $x \to +\infty$. *Suppose that* $\alpha(\lambda_{n+1}) = (1 + o(1))\alpha(\lambda_n)$ *and* $\ln n = o(\lambda_n \beta^{-1}(c\alpha(\lambda_n)))$ *as* $n \to \infty$ *for each* $c \in (0, +\infty)$, *and that* $\varrho_{\alpha,\beta}[F] < +\infty$. *If the function* G *has generalized regular* (α, β)*-growth and*

$$\varkappa_n[G] := \frac{\ln|g_n| - \ln|g_{n+1}|}{\lambda_{n+1} - \lambda_n} \nearrow +\infty \quad as \quad n_0 \le n \to \infty,$$

then

$$\varrho_{\beta,\beta}[F]_G = \varlimsup_{n\to\infty} \frac{\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)}{\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)}$$

except for cases when either $\varrho_{\alpha,\beta}[F] = \varrho_{\alpha,\beta}[G] = 0$ *or* $\varrho_{\alpha,\beta}[F] = \varrho_{\alpha,\beta}[G] = +\infty$. *If, moreover,* $\varkappa_n[F] \nearrow +\infty$ *as* $n_0 \le n \to \infty$, *then*

$$\lambda_{\beta,\beta}[F]_G = \varliminf_{n\to\infty} \frac{\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|g_n|}\right)}{\beta\left(\frac{1}{\lambda_n}\ln\frac{1}{|f_n|}\right)}$$

except for cases when either $\lambda_{\alpha,\beta}[F] = \lambda_{\alpha,\beta}[G] = 0$ *or* $\lambda_{\alpha,\beta}[F] = \lambda_{\alpha,\beta}[G] = +\infty$.

In the proposed article we will study the growth of the function $F \in S(\Lambda, 0)$ with respect to the function $G \in S(\Lambda, 0)$.

2. Definitions and supporting results

For $F \in S(\Lambda, 0)$, $\alpha \in L$ and $\beta \in L$ the quantities

$$\varrho^0_{\alpha,\beta}[F] := \varlimsup_{\sigma\uparrow 0} \frac{\alpha(\ln M_F(\sigma))}{\beta(1/|\sigma|)}, \quad \lambda^0_{\alpha,\beta}[F] := \varliminf_{\sigma\uparrow 0} \frac{\alpha(\ln M_F(\sigma))}{\beta(1/|\sigma|)} \tag{3}$$

are called [12] the generalized (α, β)-order and the generalized lower (α, β)-order of F accordingly. If $G \in S(\Lambda, 0)$ then the function $M_G(\sigma)$ can be bounded on $(-\infty, 0)$, but if $\varlimsup\limits_{n\to\infty} |g_n| = +\infty$ then $M_G(\sigma)$ is continuous and increasing to $+\infty$ on $(-\infty, 0)$ and, thus, there exists the function $M_G^{-1}(x) < 0$ inverse to $M_G(\sigma)$, which increase to 0 on $(|g_0|, +\infty)$. In what follows we will assume that $\varlimsup\limits_{n\to\infty} |g_n| = \varlimsup\limits_{n\to\infty} |f_n| = +\infty$.

Since $M_G^{-1}(x) \uparrow 0$ as $|g_0| \le x \uparrow +\infty$, we have $|M_G^{-1}(x)| \downarrow 0$ as $|g_0| \le x \uparrow +\infty$, $|M_G^{-1}(M_F(\sigma))| \downarrow 0$ and, thus, $1/|M_G^{-1}(M_F(\sigma))| \uparrow +\infty$ as $\sigma_0 \le \sigma \uparrow 0$ for some $\sigma_0 < 0$. Therefore, we can identify the growth of the function $F \in S(\Lambda, 0)$ in respect to the function $G \in S(\Lambda, 0)$ with the growth of the function $1/|M_G^{-1}(M_F(\sigma))|$ as $\sigma_0 \le \sigma \uparrow 0$, i. e., determine (α, β)-order and lower (α, β)-order as

$$\varrho^{00}_{\alpha,\beta}[F]_G = \varlimsup_{\sigma\uparrow 0} \frac{\alpha(1/|M_G^{-1}(M_F(\sigma))|)}{\beta(1/|\sigma|)}, \quad \lambda^{00}_{\alpha,\beta}[F]_G = \varliminf_{\sigma\uparrow 0} \frac{\alpha(1/|M_G^{-1}(M_F(\sigma))|)}{\beta(1/|\sigma|)}. \tag{4}$$

Lemma 2. *Let $\alpha \in L$ and $\beta \in L$. Except for cases, when either $\varrho^0_{\alpha,\beta}[F] = \varrho^0_{\alpha,\beta}[G] = 0$ or $\varrho^0_{\alpha,\beta}[F] = \varrho^0_{\alpha,\beta}[G] = +\infty$, the inequality $\varrho^{00}_{\beta,\beta}[F]_G \ge \varrho^0_{\alpha,\beta}[F]/\varrho^0_{\alpha,\beta}[G]$ is true, and under the condition of generalized regularity of (α, β)-growth of G, this inequality turns into equality.*

Except for cases, when either $\lambda^0_{\alpha,\beta}[F] = \lambda^0_{\alpha,\beta}[G] = 0$ or $\lambda^0_{\alpha,\beta}[F] = \lambda^0_{\alpha,\beta}[G] = +\infty$, the inequality $\lambda^{00}_{\beta,\beta}[F]_G \le \lambda^0_{\alpha,\beta}[F]/\lambda^0_{\alpha,\beta}[G]$ is true, and under the condition of generalized regularity of (α, β)-growth of G, this inequality turns into equality.

Proof. Indeed,

$$\varrho^{00}_{\beta,\beta}[F]_G = \varlimsup_{x\to+\infty} \frac{\beta(1/|M_G^{-1}(x))|)}{\beta(1/|M_F^{-1}(x))|)} = \varlimsup_{x\to+\infty} \frac{\alpha(\ln x)}{\beta(1/|M_F^{-1}(x))|)} \frac{\beta(1/|M_G^{-1}(x))|)}{\alpha(\ln x)} \ge$$

$$\ge \varlimsup_{x\to+\infty} \frac{\alpha(\ln x)}{\beta(1/|M_F^{-1}(x))|)} \varliminf_{x\to+\infty} \frac{\beta(1/|M_G^{-1}(x))|)}{\alpha(\ln x)} =$$

$$= \varlimsup_{\sigma\uparrow 0} \frac{\alpha(\ln M_F(\sigma))}{\beta(1/|\sigma|)} \varliminf_{\sigma\uparrow 0} \frac{\beta(1/|\sigma|)}{\alpha(\ln M_G(\sigma))} = \frac{\varrho^0_{\alpha,\beta}[F]}{\varrho^0_{\alpha,\beta}[G]}$$

and, similarly,

$$\varrho^{00}_{\beta,\beta}[F]_G \le \varlimsup_{\sigma\uparrow 0} \frac{\alpha(\ln M_F(\sigma))}{\beta(1/|\sigma|)} \varlimsup_{\sigma\uparrow 0} \frac{\beta(1/|\sigma|)}{\alpha(\ln M_G(\sigma))} = \frac{\varrho^0_{\alpha,\beta}[F]}{\lambda^0_{\alpha,\beta}[G]}.$$

This implies the first part of Lemma 1. The proof of the second part is similar. You just need to use the inequalities $\varliminf a(x) \varliminf b(x) \le \varliminf a(x)b(x) \le \varliminf a(x) \varlimsup b(x)$. □

Remark 1. If the functions F and G have the generalized regular (α, β)-growth for some $\alpha \in L$ then $\lambda^{00}_{\beta,\beta}[F]_G = \varrho^{00}_{\beta,\beta}[F]_G$. To obtain estimates $\lambda^{00}_{\alpha,\beta}[F]_G$ and $\varrho^{00}_{\alpha,\beta}[F]_G$ with $\alpha \ne \beta$, you need to use an additional function $\gamma \in L$ as in [11].

Lemma 3. *If $\alpha \in L$ and $\beta \in L$, then for each function $\gamma \in L$, the following inequalities are true:*

$$\frac{\varrho^0_{\gamma,\beta}[F]}{\varrho^0_{\gamma,\alpha}[G]} \le \varrho^{00}_{\alpha,\beta}[F]_G \le \frac{\varrho^0_{\gamma,\beta}[F]}{\lambda^0_{\gamma,\alpha}[G]} \tag{5}$$

except for cases when $\varrho^0_{\gamma,\beta}[F] = \varrho^0_{\gamma,\alpha}[G] = 0$, $\varrho^0_{\gamma,\beta}[F] = \lambda^0_{\gamma,\alpha}[G] = 0$, $\varrho^0_{\gamma,\beta}[F] = \varrho^0_{\gamma,\alpha}[G] = +\infty$, or $\varrho^0_{\gamma,\beta}[F] = \lambda^0_{\gamma,\alpha}[G] = +\infty$.

Additionally,

$$\frac{\lambda^0_{\gamma,\beta}[F]}{\varrho^0_{\gamma,\alpha}[G]} \le \lambda^{00}_{\alpha,\beta}[F]_G \le \frac{\lambda^0_{\gamma,\beta}[F]}{\lambda^0_{\gamma,\alpha}[G]} \tag{6}$$

except for cases when $\lambda^0_{\gamma,\beta}[F]=\lambda^0_{\gamma,\alpha}[G]=0$, $\lambda^0_{\gamma,\beta}[F]=\varrho^0_{\gamma,\alpha}[G]=0$, $\lambda^0_{\gamma,\beta}[F]=\lambda^0_{\gamma,\alpha}[G]=+\infty$, *or* $\lambda^0_{\gamma,\beta}[F]=\varrho^0_{\gamma,\alpha}[G]=+\infty$.

Proof. As in the proof of Lemma 1, now we have

$$\varrho^{00}_{\alpha,\beta}[F]_G=\varlimsup_{x\to+\infty}\frac{\alpha(1/|M^{-1}_G(x))|)}{\beta(1/|M^{-1}_F(x))|)}=\varlimsup_{x\to+\infty}\frac{\gamma(\ln x)}{\beta(1/|M^{-1}_F(x))|)}\frac{\alpha(1/|M^{-1}_G(x))|)}{\gamma(\ln x)}\ge$$

$$\ge\varlimsup_{\sigma\uparrow 0}\frac{\gamma(\ln M_F(\sigma))}{\beta(1/|\sigma|)}\varliminf_{\sigma\uparrow 0}\frac{\beta(1/|\sigma|)}{\gamma(\ln M_G(\sigma))}=\frac{\varrho^0_{\gamma,\beta}[F]}{\varrho^0_{\gamma,\alpha}[G]}$$

and, similarly,

$$\varrho^{00}_{\alpha,\beta}[F]_G\le\varlimsup_{\sigma\uparrow 0}\frac{\gamma(\ln M_F(\sigma))}{\beta(1/|\sigma|)}\varlimsup_{\sigma\uparrow 0}\frac{\beta(1/|\sigma|)}{\gamma(\ln M_G(\sigma))}=\frac{\varrho^0_{\gamma,\beta}[F]}{\lambda^0_{\gamma,\alpha}[G]},$$

i. e., (5) is proved. The proof of (6) is similar. □

Remark 2. In the statements of Lemma 2 the conditions for the function γ hold if $0<\lambda^0_{\gamma,\alpha}[G]\le\varrho^0_{\gamma,\alpha}[G]<+\infty$. From Lemma 2 it follows that if G has the generalized regular (γ,α)-growth then $\varrho^{00}_{\alpha,\beta}[F]_G=\varrho^0_{\gamma,\beta}[F]/\varrho^0_{\gamma,\alpha}[G]$ and $\lambda^{00}_{\alpha,\beta}[F]_G=\lambda^0_{\gamma,\beta}[F]/\lambda^0_{\gamma,\alpha}[G]$.

3. Main results

We need the following lemmas [12,13].

Lemma 4. *Let* $F\in S(\Lambda,0)$, $\alpha\in L_{si}$ *and* $\beta\in L_{si}$, $x/\beta^{-1}(c\alpha(x))\uparrow+\infty$ *and* $\alpha(x/\beta^{-1}(c\alpha(x)))=(1+o(1))\alpha(x)$ *as* $x_0\le x\to+\infty$ *for each* $c\in(0,+\infty)$. *If* $\alpha(\lambda_n)=o(\beta(\lambda_n/\ln n))$ *as* $n\to+\infty$ *then*

$$\varrho^0_{\alpha,\beta}[F]=\varlimsup_{n\to\infty}\frac{\alpha(\lambda_n)}{\beta(\lambda_n/\ln|f_n|)}.\tag{7}$$

If, moreover, $\alpha(\lambda_{n+1})\sim\alpha(\lambda_n)$ *and* $\kappa_n[F]\nearrow 0$ *as* $n_0\le n\to\infty$ *then*

$$\lambda^0_{\alpha,\beta}[F]=\varliminf_{n\to\infty}\frac{\alpha(\lambda_n)}{\beta(\lambda_n/\ln|f_n|)}.$$

Lemma 5. *Let* $F\in S(\Lambda,0)$, $\alpha\in L_{si}$ *and* $\beta\in L_{si}$, $x/\alpha^{-1}(c\beta(x))\uparrow+\infty$ *and* $\beta(x/\alpha^{-1}(c\beta(x)))=(1+o(1))\beta(x)$ *as* $x_0\le x\to+\infty$ *for each* $c\in(0,+\infty)$. *If* $\alpha(\ln n)=o(\beta(\lambda_n))$ *as* $n\to+\infty$ *then*

$$\varrho^0_{\alpha,\beta}[F]=\varlimsup_{n\to\infty}\frac{\alpha(\ln|f_n|)}{\beta(\lambda_n)}.\tag{8}$$

If, moreover, $\alpha(\lambda_{n+1})\sim\alpha(\lambda_n)$ *and* $\kappa_n[F]\nearrow 0$ *as* $n_0\le n\to\infty$ *then*

$$\lambda^0_{\alpha,\beta}[F]=\varliminf_{n\to\infty}\frac{\alpha(\ln|f_n|)}{\beta(\lambda_n)}.$$

Using Lemmas 1 and 3 we prove at first the following analogue of Theorem A.

Theorem 6. *Let the functions* $\alpha\in L_{si}$, $\beta\in L_{si}$ *and the sequence* Λ *satisfy the conditions of Lemma 3. If the function* G *has generalized regular* (α,β)*-growth and* $\kappa_n[G]\nearrow 0$ *as* $n_0\le n\to\infty$ *then*

$$\varrho^{00}_{\beta,\beta}[F]_G=\varlimsup_{n\to\infty}\beta\left(\frac{\lambda_n}{\ln|g_n|}\right)/\beta\left(\frac{\lambda_n}{\ln|f_n|}\right).\tag{9}$$

If, moreover, $\alpha(\lambda_{n+1}) = (1+o(1))\alpha(\lambda_n)$ *and* $\kappa_n[F] \nearrow 0$ *as* $n_0 \le n \to \infty$ *then*

$$\lambda^{00}_{\beta,\beta}[F]_G = \lim_{n\to\infty} \beta\left(\frac{\lambda_n}{\ln |g_n|}\right) / \beta\left(\frac{\lambda_n}{\ln |f_n|}\right). \tag{10}$$

Proof. Since the function G has generalized regular (α,β)-growth, i. e. $0 < \lambda^0_{\alpha,\beta}[G] = \varrho^0_{\alpha,\beta}[G] < +\infty$, and $\kappa_n[G] \nearrow 0$ as $n_0 \le n \to \infty$ by Lemma 3 we get

$$\lambda^0_{\alpha,\beta}[G] = \varrho^0_{\alpha,\beta}[G] = \lim_{n\to\infty} \frac{\alpha(\lambda_n)}{\beta(\lambda_n / \ln |g_n|)},$$

and by Lemma 1 $\varrho^{00}_{\beta,\beta}[F]_G = \varrho^0_{\alpha,\beta}[F]/\varrho^0_{\alpha,\beta}[G]$ and $\lambda^{00}_{\beta,\beta}[F]_G = \lambda^0_{\alpha,\beta}[F]/\lambda^0_{\alpha,\beta}[G]$. Therefore,

$$\varrho^{00}_{\beta,\beta}[F]_G = \overline{\lim_{n\to\infty}} \frac{\alpha(\lambda_n)}{\beta(\lambda_n / \ln |f_n|)} \lim_{n\to\infty} \frac{\beta(\lambda_n / \ln |g_n|)}{\alpha(\lambda_n)} =$$

$$= \overline{\lim_{n\to\infty}} \frac{\alpha(\lambda_n)}{\beta(\lambda_n / \ln |f_n|)} \lim_{n\to\infty} \frac{\beta(\lambda_n / \ln |g_n|)}{\alpha(\lambda_n)} = \overline{\lim_{n\to\infty}} \frac{\beta(\lambda_n / \ln |g_n|)}{\beta(\lambda_n / \ln |f_n|)},$$

i. e. (9) is proved. The proof of (10) is similar.

Using Lemma 2 we arrive at the following statement. □

Theorem 7. *Let* $\alpha \in L_{si}$, $\beta \in L_{si}$, $\gamma \in L_{si}$, $x/\alpha^{-1}(c\gamma(x)) \uparrow +\infty$, $x/\beta^{-1}(c\gamma(x)) \uparrow +\infty$, $\gamma(x/\alpha^{-1}(c\gamma(x))) = (1+o(1))\gamma(x)$ *and* $\gamma(x/\beta^{-1}(c\gamma(x))) = (1+o(1))\gamma(x)$ *as* $x \to +\infty$ *for each* $c \in (0,+\infty)$. *Suppose that* $\gamma(\lambda_n) = o(\alpha(\lambda_n / \ln n))$ *and* $\gamma(\lambda_n) = o(\beta(\lambda_n / \ln n))$ *as* $n \to \infty$. *If* $0 < \lambda^0_{\gamma,\alpha}[G] \le \varrho^0_{\gamma,\alpha}[G] < +\infty$, $\gamma(\lambda_{n+1}) \sim \gamma(\lambda_n)$ *and* $\kappa_n[G] \nearrow 0$ *as* $n_0 \le n \to \infty$ *then*

$$\frac{\lambda^0_{\gamma,\alpha}[G]}{\varrho^0_{\gamma,\alpha}[G]} P_{\alpha,\beta} \le \varrho^{00}_{\alpha,\beta}[F]_G \le \frac{\varrho^0_{\gamma,\alpha}[G]}{\lambda^0_{\gamma,\alpha}[G]} P_{\alpha,\beta}, \quad P_{\alpha,\beta} := \overline{\lim_{n\to\infty}} \frac{\alpha(\lambda_n / \ln |g_n|)}{\beta(\lambda_n / \ln |f_n|)}. \tag{11}$$

If, moreover, $\kappa_n[F] \nearrow 0$ *as* $n_0 \le n \to \infty$ *then*

$$\frac{\lambda^0_{\gamma,\alpha}[G]}{\varrho^0_{\gamma,\alpha}[G]} p_{\alpha,\beta} \le \lambda^{00}_{\alpha,\beta}[F]_G \le \frac{\varrho^0_{\gamma,\alpha}[G]}{\lambda^0_{\gamma,\alpha}[G]} p_{\alpha,\beta}, \quad p_{\alpha,\beta} := \underline{\lim_{n\to\infty}} \frac{\alpha(\lambda_n / \ln |g_n|)}{\beta(\lambda_n / \ln |f_n|)}. \tag{12}$$

Proof. Since $0 < \lambda^0_{\gamma,\alpha}[G] \le \varrho^0_{\gamma,\alpha}[G] < +\infty$, Lemma 2 implies

$$\frac{\varrho^0_{\gamma,\beta}[F]}{\varrho^0_{\gamma,\alpha}[G]} \le \varrho^{00}_{\alpha,\beta}[F]_G \le \frac{\varrho^0_{\gamma,\beta}[F]}{\lambda^0_{\gamma,\alpha}[G]} = \frac{\varrho^0_{\gamma,\beta}[F]}{\varrho^0_{\gamma,\alpha}[G]} \frac{\varrho^0_{\gamma,\alpha}[G]}{\lambda^0_{\gamma,\alpha}[G]}. \tag{13}$$

We need to estimate the value $\varrho^0_{\gamma,\beta}[F]/\varrho^0_{\gamma,\alpha}[G]$. On the one hand, by Lemma 3

$$\frac{\varrho^0_{\gamma,\beta}[F]}{\varrho^0_{\gamma,\alpha}[G]} = \overline{\lim_{n\to\infty}} \frac{\gamma(\lambda_n)}{\beta(\lambda_n / \ln |f_n|)} \underline{\lim_{n\to\infty}} \frac{\alpha(\lambda_n / \ln |g_n|)}{\gamma(\lambda_n)} \le \overline{\lim_{n\to\infty}} \frac{\alpha(\lambda_n / \ln |g_n|)}{\beta(\lambda_n / \ln |f_n|)} = P_{\alpha,\beta}. \tag{14}$$

On the other hand, if $P_{\alpha,\beta} > 0$ then for every $\varepsilon \in (0, P_{\alpha,\beta})$ there exists an increasing to $+\infty$ sequence (n_k) such that $\alpha(\lambda_{n_k} / \ln |g_{n_k}|) \ge (P_{\alpha,\beta} - \varepsilon)\beta(\lambda_{n_k} / \ln |f_{n_k}|)$, whence

$$\frac{\gamma(\lambda_{n_k})}{\beta(\lambda_{n_k} / \ln |f_{n_k}|)} \ge (P_{\alpha,\beta} - \varepsilon) \frac{\gamma(\lambda_{n_k})}{\alpha(\lambda_{n_k} / \ln |g_{n_k}|)}$$

and, thus,

$$\varrho^0_{\gamma,\beta}[F] = \overline{\lim_{n\to\infty}} \frac{\gamma(\lambda_n)}{\beta(\lambda_n / \ln |f_n|)} \ge (P_{\alpha,\beta} - \varepsilon) \underline{\lim_{n\to\infty}} \frac{\gamma(\lambda_n)}{\alpha(\lambda_n / \ln |g_n|)} = (P_{\alpha,\beta} - \varepsilon)\lambda^0_{\gamma,\alpha}[G].$$

In view of the arbitrariness of ε we get

$$\frac{\varrho^0_{\gamma,\beta}[F]}{\varrho^0_{\gamma,\alpha}[G]} = \frac{\varrho^0_{\gamma,\beta}[F]}{\lambda^0_{\gamma,\alpha}[G]}\frac{\lambda^0_{\gamma,\alpha}[G]}{\varrho^0_{\gamma,\beta}[F]} \geq P_{\alpha,\beta}\frac{\lambda^0_{\gamma,\alpha}[G]}{\varrho^0_{\gamma,\beta}[F]}. \tag{15}$$

If $P_{\alpha,\beta} = 0$ then this inequality is obvious. From (13), (14) and (15) we obtain (11).

For the proof of (12) we remark that now by Lemmas 2 and 3

$$\lambda^{00}_{\alpha,\beta}[F]_G \geq \frac{\lambda^0_{\gamma,\beta}[F]}{\varrho^0_{\gamma,\alpha}[G]} = \frac{\lambda^0_{\gamma,\beta}[F]}{\lambda^0_{\gamma,\alpha}[G]}\frac{\lambda^0_{\gamma,\alpha}[G]}{\varrho^0_{\gamma,\alpha}[G]} =$$

$$= \frac{\lambda^0_{\gamma,\alpha}[G]}{\varrho^0_{\gamma,\alpha}[G]} \varliminf_{n\to\infty} \frac{\gamma(\lambda_n)}{\beta\left(\lambda_n/\ln|f_n|\right)} \varlimsup_{n\to\infty} \frac{\alpha(\lambda_n/\ln|g_n|)}{\gamma(\lambda_n)} \geq$$

$$\geq \frac{\lambda^0_{\gamma,\alpha}[G]}{\varrho^0_{\gamma,\alpha}[G]} \varliminf_{n\to\infty} \frac{\alpha(\lambda_n/\ln|g_n|)}{\beta\left(\lambda_n/\ln|f_n|\right)} = \frac{\lambda^0_{\gamma,\alpha}[G]}{\varrho^0_{\gamma,\alpha}[G]} p_{\alpha,\beta}.$$

On the other hand, if $p_{\alpha,\beta} < +\infty$ then for every $\varepsilon > 0$) there exists an increasing to $+\infty$ sequence (n_k) such that $\alpha(\lambda_{n_k}/\ln|g_{n_k}|) \leq (p_{\alpha,\beta}+\varepsilon)\beta(\lambda_{n_k}/\ln|f_{n_k}|)$, whence as above

$$\varliminf_{n\to\infty} \frac{\gamma(\lambda_n)}{\beta\left(\lambda_n/\ln|f_n|\right)} \leq (p_{\alpha,\beta}+\varepsilon) \varlimsup_{n\to\infty} \frac{\gamma(\lambda_n)}{\alpha\left(\lambda_n/\ln|g_n|\right)},$$

i. e., in view of the arbitrariness of ε we get $\lambda^0_{\gamma,\beta}[F] \leq p_{\alpha,\beta}\varrho^0_{\gamma,\alpha}[G]$ and by Theorem 2

$$\lambda^{00}_{\alpha,\beta}[F]_G \leq \frac{\lambda^0_{\gamma,\beta}[F]}{\lambda^0_{\gamma,\alpha}[G]} = \frac{\lambda^0_{\gamma,\beta}[F]}{\varrho^0_{\gamma,\alpha}[G]}\frac{\varrho^0_{\gamma,\alpha}[G]}{\lambda^0_{\gamma,\alpha}[G]} \leq \frac{\varrho^0_{\gamma,\alpha}[G]}{\lambda^0_{\gamma,\alpha}[G]} p_{\alpha,\beta}.$$

The last inequality holds if $p_{\alpha,\beta} = +\infty$. Therefore, inequalities (12) and Theorem 2 are proved. □

Remark 3. If the conditions of Theorem 2 completed and G has generalized regular (γ,α)-growth (i. e. $0 < \lambda^0_{\gamma,\alpha}[G] = \varrho^0_{\gamma,\alpha}[G] < +\infty$) then $\varrho^{00}_{\alpha,\beta}[F]_G = P_{\alpha,\beta}$ and $\lambda^{00}_{\alpha,\beta}[F] = p_{\alpha,\beta}$.

If we use Lemma 4 then we obtain the following two theorems.

Theorem 8. *Let the functions $\alpha \in L_{si}$, $\beta \in L_{si}$ and the sequence Λ satisfy the conditions of Lemma 4. If the function G has generalized regular (α,β)-growth and $\kappa_n[G] \nearrow 0$ as $n_0 \leq n \to \infty$ then*

$$\varrho^{00}_{\beta,\beta}[F]_G = \varlimsup_{n\to\infty} \frac{\alpha(\ln|f_n|)}{\alpha(\ln|g_n|)}. \tag{16}$$

If, moreover, $\alpha(\lambda_{n+1}) = (1+o(1))\alpha(\lambda_n)$ and $\kappa_n[F] \nearrow 0$ as $n_0 \leq n \to \infty$ then

$$\lambda^{00}_{\beta,\beta}[F]_G = \varliminf_{n\to\infty} \frac{\alpha(\ln|f_n|)}{\alpha(\ln|g_n|)}. \tag{17}$$

Proof. Since the function G has generalized regular (α,β)-growth and $\kappa_n[G] \nearrow 0$ as $n_0 \leq n \to \infty$ by Lemma 4 we get $\lambda^0_{\alpha,\beta}[G] = \varrho^0_{\alpha,\beta}[G] = \lim\limits_{n\to\infty} \dfrac{\alpha(\ln|g_n|)}{\beta(\lambda_n)}$ and, therefore, by Lemma 1

$$\varrho^{00}_{\beta,\beta}[F]_G = \frac{\varrho^0_{\alpha,\beta}[F]}{\varrho^0_{\alpha,\beta}[G]} = \varlimsup_{n\to\infty} \frac{\alpha(\ln|f_n|)}{\beta(\lambda_n)} \lim_{n\to\infty} \frac{\beta(\lambda_n)}{\alpha(\ln|g_n|)} = \varlimsup_{n\to\infty} \frac{\alpha(\ln|f_n|)}{\alpha(\ln|g_n|)}.$$

i. e. (16) is proved. The proof of (17) is similar. □

Theorem 9. *Let $\alpha \in L_{si}$, $\beta \in L_{si}$, $\gamma \in L_{si}$, $x/\gamma^{-1}(c\alpha(x)) \uparrow +\infty$, $x/\gamma^{-1}(c\beta(x)) \uparrow +\infty$, $\alpha(x/\gamma^{-1}(c\alpha(x))) = (1+o(1))\alpha(x)$ and $\beta(x/\gamma^{-1}(c\beta(x))) = (1+o(1))\beta(x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$. Suppose that $\gamma(\ln n) = o(\alpha(\lambda_n))$ and $\gamma(\ln n) = o(\beta(\lambda_n))$ as $n \to +\infty$. If $0 < \lambda^0_{\gamma,\alpha}[G] \le \varrho^0_{\gamma,\alpha}[G] < +\infty$, $\gamma(\lambda_{n+1}) \sim \gamma(\lambda_n)$ and $\varkappa_n[G] \nearrow 0$ as $n_0 \le n \to \infty$ then*

$$\frac{\lambda^0_{\gamma,\alpha}[G]}{\varrho^0_{\gamma,\alpha}[G]} Q_{\gamma,\alpha,\beta} \le \varrho^{00}_{\alpha,\beta}[F]_G \le \frac{\varrho^0_{\gamma,\alpha}[G]}{\lambda^0_{\gamma,\alpha}[G]} Q_{\gamma,\alpha,\beta}, \quad Q_{\gamma,\alpha,\beta} := \varlimsup_{n\to\infty} \frac{\alpha(\lambda_n)\gamma(\ln|f_n|)}{\beta(\lambda_n)\gamma(\ln|g_n|)}. \tag{18}$$

If, moreover, $\varkappa_n[F] \nearrow 0$ as $n_0 \le n \to \infty$ then

$$\frac{\lambda^0_{\gamma,\alpha}[G]}{\varrho^0_{\gamma,\alpha}[G]} q_{\gamma,\alpha,\beta} \le \lambda^{00}_{\alpha,\beta}[F]_G \le \frac{\varrho^0_{\gamma,\alpha}[G]}{\lambda^0_{\gamma,\alpha}[G]} q_{\gamma,\alpha,\beta}, \quad q_{\gamma,\alpha,\beta} := \varliminf_{n\to\infty} \frac{\alpha(\lambda_n)\gamma(\ln|f_n|)}{\beta(\lambda_n)\gamma(\ln|g_n|)}. \tag{19}$$

Proof. Using Lemmas 2 and 4 as in proof of Theorem 2 we obtain

$$\varrho^{00}_{\alpha,\beta}[F]_G \le \frac{\varrho^0_{\gamma,\beta}[F]}{\lambda^0_{\gamma,\alpha}[G]} = \frac{\varrho^0_{\gamma,\beta}[F]}{\varrho^0_{\gamma,\alpha}[G]}\frac{\varrho^0_{\gamma,\alpha}[G]}{\lambda^0_{\gamma,\alpha}[G]} = \frac{\varrho^0_{\gamma,\alpha}[G]}{\lambda^0_{\gamma,\alpha}[G]} \varlimsup_{n\to\infty} \frac{\gamma(\ln|f_n|)}{\beta(\lambda_n)} \varliminf_{n\to\infty} \frac{\alpha(\lambda_n)}{\gamma(\ln|g_n|)} \le$$

$$\le \frac{\varrho^0_{\gamma,\alpha}[G]}{\lambda^0_{\gamma,\alpha}[G]} \varlimsup_{n\to\infty} \frac{\alpha(\lambda_n)\gamma(\ln|f_n|)}{\beta(\lambda_n)\gamma(\ln|g_n|)} = \frac{\varrho^0_{\gamma,\alpha}[G]}{\lambda^0_{\gamma,\alpha}[G]} Q_{\gamma,\alpha,\beta}.$$

On the other hand, if $Q_{\gamma,\alpha,\beta} > 0$ then for every $\varepsilon \in (0, Q_{\gamma,\alpha,\beta})$ there exists an increasing to $+\infty$ sequence (n_k) such that $\alpha(\lambda_{n_k})\gamma(\ln|f_{n_k}|) \ge (Q_{\gamma,\alpha,\beta} - \varepsilon)\beta(\lambda_{n_k})\gamma(\ln|g_{n_k}|)$, i. e.

$$\frac{\gamma(\ln|f_{n_k}|)}{\beta(\lambda_{n_k})} \ge (Q_{\gamma,\alpha,\beta} - \varepsilon)\frac{\gamma(\ln|g_{n_k}|)}{\alpha(\lambda_{n_k})}$$

and, thus,

$$\varrho^0_{\gamma,\beta}[F] = \varlimsup_{n\to\infty} \frac{\gamma(\ln|f_n|)}{\beta(\lambda_n)} \ge (Q_{\gamma,\alpha,\beta} - \varepsilon) \varliminf_{n\to\infty} \frac{\gamma(\ln|g_n|)}{\alpha(\lambda_n)} = (Q_{\gamma,\alpha,\beta} - \varepsilon)\lambda^0_{\gamma,\alpha}[G].$$

Therefore, in view of the arbitrariness of ε we get by Lemma 2

$$\varrho^{00}_{\alpha,\beta}[F]_G \ge \frac{\varrho^0_{\gamma,\beta}[F]}{\varrho^0_{\gamma,\alpha}[G]} = \frac{\varrho^0_{\gamma,\beta}[F]}{\lambda^0_{\gamma,\alpha}[G]}\frac{\lambda^0_{\gamma,\alpha}[G]}{\varrho^0_{\gamma,\alpha}[G]} \ge \frac{\lambda^0_{\gamma,\alpha}[G]}{\varrho^0_{\gamma,\alpha}[G]} Q_{\gamma,\alpha,\beta}.$$

If $Q_{\gamma,\alpha,\beta} = 0$ then this inequality is obvious. Inequalities (18) are proved. □

Combining the proofs of the inequalities (12) and (18) we arrive at the validity of the inequalities (19). The proof of Theorem 4 is complete.

Remark 4. If the conditions of Theorem 2 completed and G has generalized regular (γ,α)-growth then $\varrho^{00}_{\alpha,\beta}[F]_G = Q_{\gamma,\alpha,\beta}$ and $\lambda^{00}_{\alpha,\beta}[F] = q_{\gamma,\alpha,\beta}$.

4. Dirichlet series of finite R-order

If we choose $\alpha(x) = \ln x$ and $\beta(x) = x$ for $x \ge 3$ then from the definitions of $\varrho^0_{\alpha,\beta}[F]$ and $\lambda^0_{\alpha,\beta}[F]$ we obtain the definitions of the R-order $\varrho^0_R[F]$ and the lower R-order $\lambda^0_R[F]$ of the function $F \in S(\Lambda, 0)$, introduced by A.M. Gaisin [14]. If we choose $\alpha(x) = \beta(x) = \ln x$ for $x \ge 3$ then we obtain the definitions of the logarithmic order $\varrho^0_l[F]$ and the logarithmic lower order $\lambda^0_l[F]$ of $F \in S(\Lambda, 0)$.

For the characteristic of the relative growth of the function $F \in S(\Lambda, 0)$ with respect to a function $G \in S(\Lambda, 0)$ in Gaisin's scale we use $\varrho^{00}_R[F]_G = \varrho^{00}_{\beta,\beta}[F]_G$ and $\lambda^{00}_R[F]_G = \lambda^{00}_{\beta,\beta}[F]_G$ with $\beta(x) = x$. In the logarithmic scale we use $\varrho^{00}_l[F]_G = \varrho^{00}_{\beta,\beta}[F]_G$ and $\lambda^{00}_l[F]_G = \lambda^{00}_{\beta,\beta}[F]_G$ with $\beta(x) = \ln x$. Then Lemma 1 implies the following statement.

Corollary 10. *If* $0 < \lambda_R^0[G] = \varrho_R^0[G] < +\infty$ *then* $\varrho_R^{00}[F]_G = \varrho_R^0[F]/\varrho_R^0[G]$ *and* $\lambda_R^{00}[F]_G = \lambda_R^0[F]/\lambda_R^0[G]$. *If* $0 < \lambda_l^0[G] = \varrho_l^0[G] < +\infty$ *then* $\varrho_l^{00}[F]_G = \varrho_l^0[F]/\varrho_l^0[G]$ *and* $\lambda_l^{00}[F]_G = \lambda_l^0[F]/\lambda_l^0[G]$.

If we choose $\gamma(x) = \ln x$ *and* $\alpha(x) = \beta(x) = x$ *for* $x \geq 3$ *then from Lemma 2 we obtain the following statement.*

Corollary 11. *If* $0 < \lambda_R^0[G] \leq \varrho_R^0[G] < +\infty$ *then* $\varrho_R^0[F]/\varrho_R^0[G] \leq \varrho_R^{00}[F]_G \leq \varrho_R^0[F]/\lambda_R^0[G]$ *and* $\lambda_R^0[F]/\varrho_R^0[G] \leq \lambda_R^{00}[F]_G \leq \lambda_R^0[F]/\lambda_R^0[G]$.

For $\gamma(x) = \alpha(x) = \beta(x) = \ln x$ *for* $x \geq 3$ *Lemma 2 implies the following statement.*

Corollary 12. *If* $0 < \lambda_l^0[G] \leq \varrho_l^0[G] < +\infty$ *then* $\varrho_l^0[F]/\varrho_l^0[G] \leq \varrho_l^{00}[F]_G \leq \varrho_l^0[F]/\lambda_l^0[G]$ *and* $\lambda_l^0[F]/\varrho_l^0[G] \leq \lambda_l^{00}[F]_G \leq \lambda_l^0[F]/\lambda_l^0[G]$.

Lemma 2 makes it possible to study the relative growth of the function $F \in S(\Lambda, 0)$ with respect to a function $G \in S(\Lambda, 0)$ in mixed scales. For this we use

$$\varrho_{R,l}^{00}[F]_G = \overline{\lim_{\sigma\uparrow 0}}\, |\sigma| \ln (1/|M_G^{-1}(M_F(\sigma))|), \quad \lambda_{R,l}^{00}[F]_G = \underline{\lim}_{\sigma\uparrow 0}\, |\sigma| \ln (1/|M_G^{-1}(M_F(\sigma))|)$$

if $\alpha(x) = \ln x$, $\beta(x) = x$, and

$$\varrho_{l,R}^{00}[F]_G = \overline{\lim_{\sigma\uparrow 0}} \frac{1}{|M_G^{-1}(M_F(\sigma))| \ln (1/|\sigma|)}, \quad \lambda_{l,R}^{00}[F]_G = \underline{\lim}_{\sigma\uparrow 0} \frac{1}{|M_G^{-1}(M_F(\sigma))| \ln (1/|\sigma|)}$$

if $\alpha(x) = x$, $\beta(x) = \ln x$. We choose $\gamma(x) = \ln x$ for $x \geq 3$. Then $\varrho_{\gamma,\beta}^0[F] = \varrho_R^0[F]$, $\varrho_{\gamma,\alpha}^0[G] = \varrho_l^0[F]$, $\lambda_{\gamma,\beta}^0[F] = \lambda_R^0[F]$ and $\lambda_{\gamma,\alpha}^0[G] = \lambda_l^0[F]$ for $\alpha(x) = \ln x$ and $\beta(x) = x$. If $\alpha(x) = x$ and $\beta(x) = \ln x$ then $\varrho_{\gamma,\beta}^0[F] = \varrho_l^0[F]$, $\varrho_{\gamma,\alpha}^0[G] = \varrho_R^0[F]$, $\lambda_{\gamma,\beta}^0[F] = \lambda_l^0[F]$ and $\lambda_{\gamma,\alpha}^0[G] = \lambda_R^0[F]$. Therefore, Lemma 2 implies the following corollary.

Corollary 13. *If* $0 < \lambda_l^0[G] \leq \varrho_l^0[G] < +\infty$ *then* $\varrho_R^0[F]/\varrho_l^0[G] \leq \varrho_{R,l}^{00}[F]_G \leq \varrho_R^0[F]/\lambda_l^0[G]$ *and* $\lambda_R^0[F]/\varrho_l^0[G] \leq \lambda_{R,l}^{00}[F]_G \leq \lambda_R^0[F]/\lambda_l^0[G]$. *If* $0 < \lambda_R^0[G] \leq \varrho_R^0[G] < +\infty$ *then* $\varrho_l^0[F]/\varrho_R^0[G] \leq \varrho_{l,R}^{00}[F]_G \leq \varrho_l^0[F]/\lambda_R^0[G]$ *and* $\lambda_l^0[F]/\varrho_R^0[G] \leq \lambda_{l,R}^{00}[F]_G \leq \lambda_l^0[F]/\lambda_R^0[G]$.

Since the function $\beta(x) = x$ for $x \geq 3$ does not belong to L_{si}, Theorems 3 and 4 do not lead to the corresponding result in Gaisin's scale. However, in this case the following lemma is true [14].

Lemma 14. *If* $G \in S(\Lambda, 0)$ *and* $\ln n = o(\lambda_n / \ln \lambda_n)$ *as* $n \to \infty$ *then* $\varrho_R^0[F] = \overline{\lim_{n\to\infty}} \dfrac{\ln \lambda_n}{\lambda_n} \ln |g_n|$. *If, moreover,* $\ln \lambda_{n+1} = (1 + o(1)) \ln \lambda_n$ *and* $\kappa_n[G] \nearrow 0$ *as* $n_0 \leq n \to \infty$ *then* $\lambda_R^0[F] = \underline{\lim}_{n\to\infty} \dfrac{\ln \lambda_n}{\lambda_n} \ln |g_n|$.

Using Corollary 1 and Lemma 5, the following statement is proved by the usual method.

Proposition 15. *If* $0 < \lambda_R^0[G] = \varrho_R^0[G] < +\infty$, $\ln n = o(\lambda_n / \ln \lambda_n)$ *and* $\kappa_n[G] \nearrow 0$ *as* $n_0 \leq n \to \infty$ *then* $\varrho_R^{00}[F]_G = \overline{\lim_{n\to\infty}} \dfrac{\ln |f_n|}{\ln |g_n|}$. *If, moreover,* $\ln \lambda_{n+1} = (1 + o(1)) \ln \lambda_n$ *and* $\kappa_n[F] \nearrow 0$ *as* $n_0 \leq n \to \infty$ *then* $\lambda_R^{00}[F]_G = \underline{\lim}_{n\to\infty} \dfrac{\ln |f_n|}{\ln |g_n|}$.

For logarithmic orders the following lemma is true [15].

Lemma 16. *If* $G \in S(\Lambda, 0)$ *and* $\overline{\lim_{n\to\infty}} \dfrac{\ln \ln n}{\ln \lambda_n} = 0$ *then* $\dfrac{\varrho_l^0[G]}{\varrho_l[G] + 1} = \overline{\lim_{n\to\infty}} \dfrac{\ln \ln |g_n|}{\ln \lambda_n}$. *If, moreover,* $\ln \lambda_{n+1} = (1 + o(1)) \ln \lambda_n$ *and* $\kappa_n[G] \nearrow 0$ *as* $n_0 \leq n \to \infty$ *then* $\dfrac{\lambda_l^0[G]}{\lambda_l[G] + 1} = \underline{\lim}_{n\to\infty} \dfrac{\ln \ln |g_n|}{\ln \lambda_n}$.

Using this lemma and Corollary 1 it is easy to prove the following statement.

Proposition 17. *If* $0 < \lambda_l^0[G] = \varrho_l^0[G] < +\infty$, $\ln\ln n = o(\ln \lambda_n)$ *and* $\kappa_n[G] \nearrow 0$ *as* $n_0 \le n \to \infty$ *then*

$$\varrho_l^{00}[F]_G = \overline{\lim_{n\to\infty}} \frac{(\ln \lambda_n - \ln\ln |g_n|)\ln\ln |f_n|}{(\ln \lambda_n - \ln\ln |f_n|)\ln\ln |g_n|}.$$

If, moreover, $\ln \lambda_{n+1} = (1+o(1))\ln \lambda_n$ *and* $\kappa_n[F] \nearrow 0$ *as* $n_0 \le n \to \infty$ *then*

$$\lambda_l^{00}[F]_G = \underline{\lim}_{n\to\infty} \frac{(\ln \lambda_n - \ln\ln |g_n|)\ln\ln |f_n|}{(\ln \lambda_n - \ln\ln |f_n|)\ln\ln |g_n|}.$$

In conclusion, consider the mixed scales. First of all, let us note the correctness of the following statement.

Proposition 18. *For every functions* $F \in S(\Lambda, 0)$, $G \in S(\Lambda, 0)$, $\alpha \in L$ *and* $\beta \in L$ *the general formula* $\varrho_{\alpha,\beta}^{00}[G]_F = 1/\lambda_{\beta,\alpha}^{00}[F]_G$ *is correct.*

Indeed,

$$\varrho_{\alpha,\beta}^{00}[G]_F = \overline{\lim_{\sigma\uparrow 0}} \frac{\alpha(1/|M_F^{-1}(M_G(\sigma))|)}{\beta(1/|\sigma|)} = \overline{\lim_{x\to+\infty}} \frac{\alpha(1/|M_F^{-1}(x)|}{\beta(1/|M_G^{-1}(x)|)} =$$

$$= \frac{1}{\underline{\lim}_{x\to+\infty} \frac{\beta(1/|M_G^{-1}(x)|)}{\alpha(1/|M_F^{-1}(x)|}} = \frac{1}{\underline{\lim}_{\sigma\uparrow 0} \frac{\beta(1/|M_G^{-1}(M_F(\sigma))|)}{\alpha(1/|\sigma|}} = \frac{1}{\lambda_{\beta,\alpha}^{00}[F]_G}.$$

Using Lemmas 5, 6 and Corollary 4 we obtain the following proposition in mixed scales.

Proposition 19. *Let* $F \in S(\Lambda, 0)$, $G \in S(\Lambda, 0)$ *and* $\ln\ln n = o(\ln \lambda_n)$ *as* $n \to \infty$. *If* $0 < \lambda_l^0[G] = \varrho_l^0[G] < +\infty$ *and* $\kappa_n[G] \nearrow 0$ *as* $n_0 \le n \to \infty$ *then*

$$\varrho_{R,l}^{00}[F]_G = \overline{\lim_{n\to\infty}} \frac{\ln \lambda_n(\ln \lambda_n - \ln\ln |g_n|)\ln\ln |f_n|}{\lambda_n \ln\ln |g_n|},$$

and if, moreover, $\ln \lambda_{n+1} = (1+o(1))\ln \lambda_n$ *and* $\kappa_n[F] \nearrow 0$ *as* $n_0 \le n \to \infty$ *then*

$$\lambda_{R,l}^{0,0}[F]_G = \underline{\lim}_{n\to\infty} \frac{\ln \lambda_n(\ln \lambda_n - \ln\ln |g_n|)\ln\ln |f_n|}{\lambda_n \ln\ln |g_n|}.$$

On the other hand, if $0 < \lambda_R^0[F] = \varrho_R^0[F] < +\infty$ *and* $\kappa_n[F] \nearrow 0$ *as* $n_0 \le n \to \infty$ *then*

$$\varrho_{l,R}^{00}[G]_F = \overline{\lim_{n\to\infty}} \frac{\lambda_n \ln\ln |g_n|}{\ln \lambda_n(\ln \lambda_n - \ln\ln |g_n|)\ln\ln |f_n|},$$

and if, moreover, $\ln \lambda_{n+1} = (1+o(1))\ln \lambda_n$ *and* $\kappa_n[G] \nearrow 0$ *as* $n_0 \le n \to \infty$ *then*

$$\lambda_{l,R}^{00}[G]_F = \underline{\lim}_{n\to\infty} \frac{\lambda_n \ln\ln |g_n|}{\ln \lambda_n(\ln \lambda_n - \ln\ln |g_n|)\ln\ln |f_n|}.$$

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