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A collection of new integral inequalities involving sub-multiplicative functions

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Abstract: In this article, we establish new integral inequalities involving sub-multiplicative functions. We first derive several inequalities of primitive type, followed by new inequalities of the convolution product type. We also obtain integral bounds for functions evaluated on the product of two variables. Finally, we study double integral inequalities and their variations. Simple examples are used to illustrate the theory. The understanding of integral inequalities under submultiplicative assumptions is thus deepened, and some new ideas for further research in mathematical analysis are provided.

Keywords: sub-multiplicativity, integral inequalities, primitives, convolution product, double integral, polar change of variables

MSC: 26D15, 28A25.

1. Introduction

Integral inequalities are diverse in nature and scope. They play a central role in many areas of mathematical analysis, including functional analysis, probability theory, differential equations, numerical analysis and optimal control theory. In functional analysis, for example, they are used to establish bounds for operators on Banach and Hilbert spaces. In probability theory, they help to derive moment bounds and concentration inequalities involving random variables. In differential equations, integral inequalities are often used to derive stability conditions and to estimate solutions of integral equations. In numerical analysis, they provide error bounds for numerical integration and approximation methods. Finally, in optimal control theory, integral inequalities are crucial for proving existence and uniqueness results for optimal control problems, as well as for formulating necessary conditions for optimality. For a rigorous treatment, we refer to the following books: [1–4].

While many of these inequalities are well established, the search for new results remains an active area of research. In particular, refining existing bounds, extending results to broader classes of functions, and exploring less studied assumptions can lead to important advances. In this article, we focus on the third aspect by establishing new integral inequalities in a setting that has received limited attention, where certain functions satisfy the sub-multiplicative property. In order to develop our contributions further, a formal definition of this property is needed. Note that we restrict our attention to non-negative functions defined on $[0, +\infty)$ (or, without loss of generality, $(0, +\infty)$). A function $f : [0, +\infty) \mapsto [0, +\infty)$ is said to be sub-multiplicative (or to satisfy the sub-multiplicative property) if and only if, for any $s, t \in [0, +\infty)$, we have

$$f(st) \leq f(s)f(t).$$

Basic examples of such functions include $f(t) = t^{\alpha}$ with $\alpha \in \mathbb{R}$, $f(t) = \log(\beta + t)$ with $\beta \geq e$ (and $e = \exp(1)$), $f(t) = 1/\tanh(\gamma t)$ with $\gamma > 0$, i.e., $f(t) = (e^{\gamma t} + e^{-\gamma t})/(e^{\gamma t} - e^{-\gamma t})$, $f(t) = t^{\eta}(1 + |\log(t)|)$ with $\eta \geq 1$, and $f(t) = t^{\iota}(1 + |\sin(\log(t))|)$ with $\iota \geq 1$. Notably, the product of two sub-multiplicative functions is also sub-multiplicative, and the composition of a non-decreasing sub-multiplicative function with a sub-multiplicative function is also sub-multiplicative, leading to numerous additional examples. Further details on this property can be found in [5–7].

Some studies have explored integral inequalities under sub-multiplicative assumptions, including those in [8–14]. Specifically, Hardy-Hilbert-type integral inequalities for certain sub-multiplicative functions are established in [8,9], while Hardy-type integral inequalities are investigated in [10]. Generalizations and variations of some results in [10] are proposed in [11]. Hermite-Hadamard-type integral inequalities are examined in [12], and Hardy-type integral inequalities on time scales using Delta calculus are developed in [13]. Additionally, [14] explores various integral inequalities, leading to several divergence results.

Despite these advances, certain aspects of integral inequalities under sub-multiplicative assumptions remain underexplored. In this article, we aim to address these gaps by establishing new integral inequalities within several complementary frameworks. Specifically, we derive

- primitive-type integral inequalities, involving expressions of the form "$\int_0^x f(t)dt$" and their variations,
- convolution product-type integral inequalities, which include expressions of the following form:

$$"\int_0^x f(t)g(x-t)dt",$$

  and related formulations,
- integral bounds for a function evaluated at the product of two variables, say "$f(xy)$",
- double integral inequalities, which include expressions of the following form:

$$"\int_0^b \int_0^a f(s)h(t)g(s+t)dsdt",$$

  and their variations.

In particular, inequalities for such double integrals can be related in form to Hardy-Hilbert-type integral inequalities, which have attracted much attention under classical assumptions. See [3]. Several examples are given to illustrate the theory, with an emphasis on the use of the simple sub-multiplicative function $f(t) = \log(e+t)$ and the power function. Our results thus apply to a new collection of integral inequalities established under sub-multiplicative assumptions. They also open up directions for possible applications and generalizations in mathematical analysis and related fields.

The rest of this article is structured as follows: §2 is devoted to the primitive-type integral inequalities. §3 focuses on the convolution product-type integral inequalities. In §4, technical integral bounds for a sub-multiplicative function taken at a product of two variables are derived. Several double integral inequalities are studied in §5. Finally, §6 contains concluding remarks.

2. Primitive-type integral inequalities

The proposition below gives basic primitive-type integral inequalities. In particular, lower and upper bounds are given for $\int_0^x f(t)dt$, assuming that f is sub-multiplicative.

Proposition 1. *Let $f : [0,+\infty) \mapsto [0,+\infty)$ be a sub-multiplicative function. Then, for any $x \in (0,+\infty)$,*
1. we have

$$\int_0^x f(t)dt \leq xf(x) \int_0^1 f(t)dt,$$

2. we have

$$\int_1^x f(t)dt \leq (xf(x)-1) \int_0^1 f(t)dt,$$

3. we have

$$\int_0^x f(t)dt \geq f(1)\frac{x}{f(1/x)} \int_1^{+\infty} \frac{1}{t^2 f(t)}dt,$$

provided that the integrals involved converge.

Proof. 1. Using the change of variables $t = xu$ and the sub-multiplicative property of f (followed by an uniformization of the notation), we get

$$\int_0^x f(t)dt = x\int_0^1 f(xu)du \leq x\int_0^1 f(x)f(u)du = xf(x)\int_0^1 f(u)du = xf(x)\int_0^1 f(t)dt.$$

2. It follows from the Chasles integral relation and the result in the first item that

$$\int_1^x f(t)dt = \int_0^x f(t)dt - \int_0^1 f(t)dt \leq xf(x)\int_0^1 f(t)dt - \int_0^1 f(t)dt = (xf(x)-1)\int_0^1 f(t)dt.$$

3. The sub-multiplicative property of f implies that $f(1) = f(t/t) \leq f(t)f(1/t)$. Since f is non-negative, we get

$$f(t) \geq f(1)\frac{1}{f(1/t)}.$$

Integrating both sides and using the change of variables $t = x/u$, the sub-multiplicative property of f combined with the non-negative property of f (followed by an uniformization of the notation), we get

$$\begin{aligned}\int_0^x f(t)dt \geq f(1)\int_0^x \frac{1}{f(1/t)}dt &= f(1)\int_1^{+\infty} \frac{x}{u^2 f(u/x)}du\\ &\geq f(1)\int_1^{+\infty} \frac{x}{u^2 f(u)f(1/x)}du = f(1)\frac{x}{f(1/x)}\int_1^{+\infty}\frac{1}{u^2 f(u)}du\\ &= f(1)\frac{x}{f(1/x)}\int_1^{+\infty}\frac{1}{t^2 f(t)}dt.\end{aligned}$$

This concludes the proof of the proposition. □

For example, if we take the sub-multiplicative function $f(t) = \log(e+t)$, the result in the first item gives, for any $x \in (0,+\infty)$,

$$(e+x)(\log(e+x)-1) = \int_0^x f(t)dt \leq xf(x)\int_0^1 f(t)dt = x\log(e+x)(e+1)(\log(e+1)-1).$$

The result in the third item gives, for any $x \in (0,+\infty)$,

$$(e+x)(\log(e+x)-1) = \int_0^x f(t)dt \geq f(1)\frac{x}{f(1/x)}\int_1^{+\infty}\frac{1}{t^2 f(t)}dt \approx 0.79302 \times \frac{x}{\log(e+1/x)}.$$

These are some examples among so much possibilities.

By setting $m(x) = (1/x)\int_0^x f(t)dt$, which corresponds to the mean integral of f over $[0,x]$, the result in the first item implies that, for any $x \in (0,+\infty)$,

$$f(x) \geq \frac{1}{\int_0^1 f(t)dt}m(x).$$

The result in the third item gives, for any $x \in (0,+\infty)$,

$$f\left(\frac{1}{x}\right) \geq f(1)\left(\int_1^{+\infty}\frac{1}{t^2 f(t)}dt\right)\frac{1}{m(x)}.$$

To the best of our knowledge, despite their simplicity, the inequalities in Proposition 1 have not been mentioned as such in the literature. They are completed in the following proposition with the consideration of the product variable xy.

Proposition 2. *Let $f : [0,+\infty) \mapsto [0,+\infty)$ be a sub-multiplicative function. Then, for any $x \in (0,+\infty)$*
1. and $y \in (0,+\infty)$, we have

$$\int_0^{xy} f(t)dt \leq xyf(x)f(y) \int_0^1 f(t)dt,$$

2. and $y \in [1,+\infty)$, we have

$$\int_x^{xy} f(t)dt \leq xf(x)(yf(y)-1) \int_0^1 f(t)dt,$$

3. and $y \in (0,+\infty)$, we have

$$\int_0^{xy} f(t)dt \geq f(1)\frac{xy}{f(1/x)f(1/y)} \int_1^{+\infty} \frac{1}{t^2 f(t)}dt,$$

provided that the integrals involved converge.

Proof. 1. Applying the result in the first item in Proposition 1 and using the sub-multiplicative property of f combined with its non-negative property, we have

$$\int_0^{xy} f(t)dt \leq xyf(xy) \int_0^1 f(t)dt \leq xyf(x)f(y) \int_0^1 f(t)dt.$$

2. Using the change of variables $t = xu$, the sub-multiplicative property of f, the result in the second item in Proposition 1, the non-negative property of f and $y \in (1,+\infty)$, we obtain

$$\int_x^{xy} f(t)dt = x\int_1^{y} f(xu)du \leq x\int_1^{y} f(x)f(u)du = xf(x)\int_1^{y} f(u)du \leq xf(x)(yf(y)-1)\int_0^1 f(t)dt.$$

3. Applying the the result in the third item in Proposition 1 and using the sub-multiplicative property of f combined with its non-negative property, we have

$$\begin{aligned}\int_0^{xy} f(t)dt &\geq f(1)\frac{xy}{f(1/(xy))}\int_1^{+\infty}\frac{1}{t^2 f(t)}dt = f(1)\frac{xy}{f((1/x)(1/y))}\int_1^{+\infty}\frac{1}{t^2 f(t)}dt \\ &\geq f(1)\frac{xy}{f(1/x)f(1/y)}\int_1^{+\infty}\frac{1}{t^2 f(t)}dt.\end{aligned}$$

This concludes the proof of the proposition. □

For example, if we take the sub-multiplicative function $f(t) = \log(e+t)$, the result in the first item gives, for any $x, y \in (0,+\infty)$,

$$\begin{aligned}(e+xy)(\log(e+xy)-1) &= \int_0^{xy} f(t)dt \leq xyf(x)f(y)\int_0^1 f(t)dt \\ &= xy\log(e+x)\log(e+y)(e+1)(\log(e+1)-1).\end{aligned}$$

The result in the third item gives, for any $x, y \in (0,+\infty)$,

$$\begin{aligned}(e+xy)(\log(e+xy)-1) &= \int_0^{xy} f(t)dt \geq f(1)\frac{xy}{f(1/x)f(1/y)}\int_1^{+\infty}\frac{1}{t^2 f(t)}dt \\ &\approx 0.79302 \times \frac{xy}{\log(e+1/x)\log(e+1/y)}.\end{aligned}$$

The proposition below presents an original integral primitive-type inequality involving the power of sub-multiplicative function.

Proposition 3. *Let $m \in \mathbb{N}\backslash\{0\}$, $f : [0,+\infty) \mapsto [0,+\infty)$ be a sub-multiplicative function. Then, for any $x \in (0,+\infty)$ and $y \in (x,+\infty)$, we have*

$$\int_x^y f^m(t)dt \geq \frac{1}{m}\int_{x^m}^{y^m} t^{1/m-1}f(t)dt,$$

provided that the integrals involved converge.

Proof. The fact that $m \in \mathbb{N}\backslash\{0\}$ and the sub-multiplicative property of f combined with its non-negative property imply that

$$f^m(t) = f^{m-2}(t)f(t)f(t) \geq f^{m-2}(t)f(t^2) \geq \ldots \geq f(t^m).$$

Using this and the change of variables $u = t^m$ (followed by an uniformization of the notation), we obtain

$$\int_x^y f^m(t)dt \geq \int_x^y f(t^m)dt = \int_{x^m}^{y^m} f(u)\frac{1}{m}u^{1/m-1}du = \frac{1}{m}\int_{x^m}^{y^m} t^{1/m-1}f(t)dt.$$

This concludes the proof of the proposition. □

For example, if we take the sub-multiplicative function $f(t) = \log(e+t)$, $m = 2$, $x = 0$ and $y = 1$, we have

$$\begin{aligned} 1.3649 &\approx 2 + e + (1+e)(\log(1+e) - 2)\log(1+e) = \int_x^y f^m(t)dt \\ &\geq \frac{1}{m}\int_{x^m}^{y^m} t^{1/m-1}f(t)dt = \log(1+e) + 2\sqrt{e}\operatorname{arccot}(\sqrt{e}) - 2 \approx 1.111052. \end{aligned}$$

It is thus verified numerically for these specific values. Of course, taking x and y as variables makes the result much more interesting.

In the framework of Proposition 3, note that, since f is sub-multiplicative, $f^{1/m}$ is also sub-multiplicative. We therefore have, for any $x \in (0,+\infty)$ and $y \in (x,+\infty)$,

$$\int_x^y f(t)dt = \int_x^y (f^{1/m})^m(t)dt \geq \frac{1}{m}\int_{x^m}^{y^m} t^{1/m-1}f^{1/m}(t)dt.$$

The primitive-type inequalities presented have potential applications in mathematical analysis, particularly in the study of integral bounds, function approximations, and stability estimates in various functional spaces.

3. Convolution product-type integral inequalities

The proposition below is our first convolution product-type integral inequality involving a sub-multiplicative function f.

Proposition 4. *Let $p \in (1,+\infty)$, $q = p/(p-1)$, $f : [0,+\infty) \mapsto [0,+\infty)$ be a sub-multiplicative function, $g : [0,+\infty) \mapsto [0,+\infty)$ be the function defined by $g(x) = f(1/x)$ for any $x \in (0,+\infty)$ and $(f \star g) : [0,+\infty) \mapsto [0,+\infty)$ be the convolution product of f and g defined by*

$$(f \star g)(x) = \int_0^x f(t)g(x-t)dt.$$

Then, for any $x \in (0,+\infty)$, we have

$$(f \star g)(x) \geq \kappa x,$$

where

$$\kappa = \int_0^{+\infty} \frac{f(t)}{(1+t)^2}dt,$$

provided that the integrals involved converge.

Proof. Using the definition of g, the sub-multiplicative property of f and the change of variables $u = t/(x-t)$ satisfying $t = xu/(1+u)$ and $dt = (x/(1+u)^2)du$ (followed by an uniformization of the notation), we get

$$\begin{aligned}(f \star g)(x) &= \int_0^x f(t)g(x-t)dt = \int_0^x f(t)f\left(\frac{1}{x-t}\right)dt \geq \int_0^x f\left(\frac{t}{x-t}\right)dt \\ &= \int_0^{+\infty} f(u)\frac{x}{(1+u)^2}du = x\int_0^{+\infty}\frac{f(u)}{(1+u)^2}du = x\int_0^{+\infty}\frac{f(t)}{(1+t)^2}dt = \kappa x.\end{aligned}$$

This concludes the proof of the proposition. □

For example, if we take the sub-multiplicative function $f(t) = \log(e+t)$, we have, for any $x \in (0,+\infty)$,

$$(f \star g)(x) \geq \kappa x,$$

where

$$\kappa = \int_0^{+\infty}\frac{f(t)}{(1+t)^2}dt = \frac{e}{e-1}.$$

The proposition below is a generalization of Proposition 4 with the introduction of another sub-multiplicative function.

Proposition 5. *Let $p \in (1,+\infty)$, $q = p/(p-1)$, $f, g : [0,+\infty) \mapsto [0,+\infty)$ be two sub-multiplicative functions, $h : [0,+\infty) \mapsto [0,+\infty)$ be the function defined by $h(x) = f(1/x)$ for any $x \in (0,+\infty)$ and $((f/g) \star h) : [0,+\infty) \mapsto [0,+\infty)$ be the convolution product of f/g and h defined by*

$$\left(\frac{f}{g}\star h\right)(x) = \int_0^x \frac{f(t)}{g(t)}h(x-t)dt.$$

Then, for any $x \in (0,+\infty)$, we have

$$\left(\frac{f}{g}\star h\right)(x) \geq \tau\frac{x}{g(x)},$$

where

$$\tau = \int_0^{+\infty}\frac{f(t)}{(1+t)^2 g(t/(1+t))}dt,$$

provided that the integrals involved converge.

Proof. Using the definition of h, the sub-multiplicative property of f combined with the non-negative property of g, the change of variables $u = t/(x-t)$ satisfying $t = xu/(1+u)$ and $dt = (x/(1+u)^2)du$ and the sub-multiplicative property of g combined with the non-negative property of f (followed by an uniformization of the notation), we get

$$\begin{aligned}\left(\frac{f}{g}\star h\right)(x) &= \int_0^x \frac{f(t)}{g(t)}h(x-t)dt = \int_0^x \frac{f(t)}{g(t)}f\left(\frac{1}{x-t}\right)dt \geq \int_0^x \frac{1}{g(t)}f\left(\frac{t}{x-t}\right)dt \\ &= \int_0^{+\infty}\frac{1}{g(xu/(1+u))}f(u)\frac{x}{(1+u)^2}du \geq \int_0^{+\infty}\frac{1}{g(x)g(u/(1+u))}f(u)\frac{x}{(1+u)^2}du \\ &= \frac{x}{g(x)}\int_0^{+\infty}\frac{f(u)}{(1+u)^2 g(u/(1+u))}du = \frac{x}{g(x)}\int_0^{+\infty}\frac{f(t)}{(1+t)^2 g(t/(1+t))}dt = \tau\frac{x}{g(x)}.\end{aligned}$$

This concludes the proof of the proposition. □

For example, if we take the sub-multiplicative function $f(t) = \log(e+t)$ and $g(t) = \sqrt{t}$, we have, for any $x \in (0, +\infty)$,

$$\left(\frac{f}{g} \star h\right)(x) \geq \tau\sqrt{x},$$

where

$$\tau = \int_0^{+\infty} \frac{f(t)}{(1+t)^2 g(t/(1+t))} dt \approx 2.68631.$$

The convolution product-type inequalities presented can find potential applications in fields such as signal processing, where convolution products are used in both theory and practice.

4. Integral upper bounds for a sub-multiplicative function at a product of two variables

If a function $f : [0, +\infty) \mapsto [0, +\infty)$ is sub-multiplicative, then, for any $x, y \in (0, +\infty)$, we have $f(xy) \leq f(x)f(y)$. Some of our investigations lead to more technical or refined integral bounds for $f(xy)$. Some of these are presented in this section, starting with the proposition below.

Proposition 6. *Let $p \in (1, +\infty)$, $q = p/(p-1)$ and $f : [0, +\infty) \mapsto [0, +\infty)$ be a sub-multiplicative function. Then, for any $x, y \in (0, +\infty)$, we have*

$$f(xy) \leq \frac{1}{|\log(x) - \log(y)|} \left| \int_x^y \frac{f^p(t)}{t} dt \right|^{1/p} \left| \int_x^y \frac{f^q(t)}{t} dt \right|^{1/q},$$

provided that the integrals involved converge.

Proof. Using a suitable decomposition of xy, the sub-multiplicative property of f, the Hölder integral inequality with the parameter p (taking into account that f is non-negative), the changes of variables $u = x^\lambda y^{1-\lambda}$, i.e., $\lambda = (\log(u) - \log(y))/(\log(x) - \log(y))$, and $v = x^{1-\lambda} y^\lambda$, i.e., $\lambda = (\log(v) - \log(x))/(\log(y) - \log(x))$ (followed by an uniformization of the notation) and $1/p + 1/q = 1$, we obtain

$$\begin{aligned}
f(xy) &= f(xy) \int_0^1 d\lambda = \int_0^1 f(xy) d\lambda = \int_0^1 f(x^\lambda y^{1-\lambda} x^{1-\lambda} y^\lambda) d\lambda \\
&\leq \int_0^1 f(x^\lambda y^{1-\lambda}) f(x^{1-\lambda} y^\lambda) d\lambda \leq \left(\int_0^1 f^p(x^\lambda y^{1-\lambda}) d\lambda \right)^{1/p} \left(\int_0^1 f^q(x^{1-\lambda} y^\lambda) d\lambda \right)^{1/q} \\
&= \left(\frac{1}{\log(x) - \log(y)} \int_y^x \frac{f^p(u)}{u} du \right)^{1/p} \left(\frac{1}{\log(y) - \log(x)} \int_x^y \frac{f^q(v)}{v} dv \right)^{1/q} \\
&= \left(\frac{1}{|\log(x) - \log(y)|} \left| \int_y^x \frac{f^p(t)}{t} dt \right| \right)^{1/p} \left(\frac{1}{|\log(y) - \log(x)|} \left| \int_x^y \frac{f^q(t)}{t} dt \right| \right)^{1/q} \\
&= \frac{1}{|\log(x) - \log(y)|} \left| \int_x^y \frac{f^p(t)}{t} dt \right|^{1/p} \left| \int_x^y \frac{f^q(t)}{t} dt \right|^{1/q}.
\end{aligned}$$

This ends the proof of the proposition. □

For example, if we take the sub-multiplicative function $f(t) = \log(e+t)$, we have, for any $x, y \in (0, +\infty)$,

$$\log(e+xy) \leq \frac{1}{|\log(x) - \log(y)|} \left| \int_x^y \frac{\log^p(e+t)}{t} dt \right|^{1/p} \left| \int_x^y \frac{\log^q(e+t)}{t} dt \right|^{1/q}.$$

Equivalent formulations of the inequality in Proposition 6 are

$$|\log(x) - \log(y)| f(xy) \leq \left| \int_x^y \frac{f^p(t)}{t} dt \right|^{1/p} \left| \int_x^y \frac{f^q(t)}{t} dt \right|^{1/q},$$

or, with multiple logarithmic terms,

$$\log\left(f(xy)\right) \leq \frac{1}{p}\log\left|\int_x^y \frac{f^p(t)}{t}dt\right| + \frac{1}{q}\log\left|\int_x^y \frac{f^q(t)}{t}dt\right| - \log\left|\log(x)-\log(y)\right|.$$

As a special case of interest of Proposition 6, if we take $y=1$, for any $x\in(0,+\infty)$, we have

$$f(x) \leq \frac{1}{|\log(x)|}\left|\int_1^x \frac{f^p(t)}{t}dt\right|^{1/p}\left|\int_1^x \frac{f^q(t)}{t}dt\right|^{1/q}.$$

This can also be reformulated as

$$|\log(x)|f(x) \leq \left|\int_1^x \frac{f^p(t)}{t}dt\right|^{1/p}\left|\int_1^x \frac{f^q(t)}{t}dt\right|^{1/q},$$

or

$$\log\left(f(x)\right) \leq \frac{1}{p}\log\left|\int_1^x \frac{f^p(t)}{t}dt\right| + \frac{1}{q}\log\left|\int_1^x \frac{f^q(t)}{t}dt\right| - \log\left|\log(x)\right|.$$

The proposition below is a generalization of Proposition 6 with the introduction of an intermediate function.

Proposition 7. *Let $p\in(1,+\infty)$, $q=p/(p-1)$, $f:[0,+\infty)\mapsto[0,+\infty)$ be a sub-multiplicative function and $g:[0,1]\to[0,+\infty)$ be a function. Then, for any $x,y\in(0,+\infty)$ and $\theta\in[0,1]$, we have*

$$\begin{aligned} f(xy)\int_0^1 g(t)dt \leq & \frac{1}{|\log(x)-\log(y)|}\left|\int_y^x \frac{f^p(t)}{t}g^{\theta p}\left(\frac{\log(y)-\log(t)}{\log(y)-\log(x)}\right)dt\right|^{1/p} \\ & \times\left|\int_x^y \frac{f^q(t)}{t}g^{(1-\theta)q}\left(\frac{\log(t)-\log(x)}{\log(y)-\log(x)}\right)dt\right|^{1/q}, \end{aligned}$$

provided that the integrals involved converge.

Proof. The proof uses the main tools in that of Proposition 6, with the manage of the new function g. Using a suitable decomposition of xy, the sub-multiplicative property of f combined with the non-negative property of g, a suitable decomposition of $g(\lambda)$, the Hölder integral inequality with the parameter p (taking into account that f and g are non-negative), the changes of variables $u=x^{\lambda}y^{1-\lambda}$, i.e., $\lambda=(\log(u)-\log(y))/(\log(x)-\log(y))$, and $v=x^{1-\lambda}y^{\lambda}$, i.e., $\lambda=(\log(v)-\log(x))/(\log(y)-\log(x))$ (followed by an uniformization of the notation) and $1/p+1/q=1$, we obtain

$$\begin{aligned} f(xy)\int_0^1 g(t)dt =& f(xy)\int_0^1 g(\lambda)d\lambda = \int_0^1 f(xy)g(\lambda)d\lambda \\ &= \int_0^1 f(x^{\lambda}y^{1-\lambda}x^{1-\lambda}y^{\lambda})g(\lambda)d\lambda \leq \int_0^1 f(x^{\lambda}y^{1-\lambda})f(x^{1-\lambda}y^{\lambda})g(\lambda)d\lambda \\ &= \int_0^1 f(x^{\lambda}y^{1-\lambda})f(x^{1-\lambda}y^{\lambda})g^{\theta}(\lambda)g^{1-\theta}(\lambda)d\lambda \\ &\leq \left(\int_0^1 f^p(x^{\lambda}y^{1-\lambda})g^{\theta p}(\lambda)d\lambda\right)^{1/p}\left(\int_0^1 f^q(x^{1-\lambda}y^{\lambda})g^{(1-\theta)q}(\lambda)d\lambda\right)^{1/q} \\ &= \left(\frac{1}{\log(x)-\log(y)}\int_y^x \frac{f^p(u)}{u}g^{\theta p}\left(\frac{\log(u)-\log(y)}{\log(x)-\log(y)}\right)du\right)^{1/p} \\ &\times\left(\frac{1}{\log(y)-\log(x)}\int_x^y \frac{f^q(v)}{v}g^{(1-\theta)q}\left(\frac{\log(v)-\log(x)}{\log(y)-\log(x)}\right)dv\right)^{1/q} \\ &= \left(\frac{1}{|\log(x)-\log(y)|}\left|\int_y^x \frac{f^p(t)}{t}g^{\theta p}\left(\frac{\log(y)-\log(t)}{\log(y)-\log(x)}\right)dt\right|\right)^{1/p} \end{aligned}$$

$$\times \left(\frac{1}{|\log(y) - \log(x)|} \left| \int_x^y \frac{f^q(t)}{t} g^{(1-\theta)q} \left(\frac{\log(t) - \log(x)}{\log(y) - \log(x)} \right) dt \right| \right)^{1/q}$$
$$= \frac{1}{|\log(x) - \log(y)|} \left| \int_y^x \frac{f^p(t)}{t} g^{\theta p} \left(\frac{\log(y) - \log(t)}{\log(y) - \log(x)} \right) dt \right|^{1/p}$$
$$\times \left| \int_x^y \frac{f^q(t)}{t} g^{(1-\theta)q} \left(\frac{\log(t) - \log(x)}{\log(y) - \log(x)} \right) dt \right|^{1/q}.$$

This completes the proof of the proposition. □

An alternative integral bound for $f(xy)$ is presented in the proposition below.

Proposition 8. *Let $p \in (1, +\infty)$, $q = p/(p-1)$ and $f : [0, +\infty) \mapsto [0, +\infty)$ be a sub-multiplicative function. Then, for any $x, y \in (0, +\infty)$, we have*

$$f(xy) \leq x^{-1/p} y^{1/q} \left(\int_0^x f^p(t) dt \right)^{1/p} \left(\int_y^{+\infty} \frac{f^q(t)}{t^2} dt \right)^{1/q},$$

provided that the integrals involved converge.

Proof. Using a suitable decomposition of xy, the sub-multiplicative property of f, the Hölder integral inequality with the parameter p (taking into account that f is non-negative), and the changes of variables $u = \lambda x$ and $v = y/\lambda$ (followed by an uniformization of the notation), we get

$$f(xy) = f(xy) \int_0^1 d\lambda = \int_0^1 f(xy) d\lambda = \int_0^1 f \left(\lambda x \frac{y}{\lambda} \right) d\lambda$$
$$\leq \int_0^1 f(\lambda x) f \left(\frac{y}{\lambda} \right) d\lambda \leq \left(\int_0^1 f^p(\lambda x) d\lambda \right)^{1/p} \left(\int_0^1 f^q \left(\frac{y}{\lambda} \right) d\lambda \right)^{1/q}$$
$$= \left(\frac{1}{x} \int_0^x f^p(u) du \right)^{1/p} \left(y \int_y^{+\infty} \frac{f^q(v)}{v^2} dv \right)^{1/q}$$
$$= x^{-1/p} y^{1/q} \left(\int_0^x f^p(t) dt \right)^{1/p} \left(\int_y^{+\infty} \frac{f^q(t)}{t^2} dt \right)^{1/q}.$$

This concludes the proof of the proposition. □

For example, if we take the sub-multiplicative function $f(t) = \log(e + t)$, we have, for any $x, y \in (0, +\infty)$,

$$\log(e + xy) \leq x^{-1/p} y^{1/q} \left(\int_0^x \log^p(e + t) dt \right)^{1/p} \left(\int_y^{+\infty} \frac{\log^q(e + t)}{t^2} dt \right)^{1/q}.$$

The proposition below is a generalization of Proposition 8 with the introduction of an intermediate function.

Proposition 9. *Let $p \in (1, +\infty)$, $q = p/(p-1)$, $f : [0, +\infty) \mapsto [0, +\infty)$ be a sub-multiplicative function and $g : [0, 1] \to [0, +\infty)$ be a function. Then, for any $x, y \in (0, +\infty)$ and $\theta \in [0, 1]$, we have*

$$f(xy) \int_0^1 g(t) dt \leq x^{-1/p} y^{1/q} \left(\int_0^x f^p(t) g^{\theta p} \left(\frac{t}{x} \right) dt \right)^{1/p} \left(\int_y^{+\infty} \frac{f^q(t)}{t^2} g^{(1-\theta)q}(ty) dt \right)^{1/q},$$

provided that the integrals involved converge.

Proof. The proof uses the main tools in that of Proposition 8, with the manage of the new function g. Using a suitable decomposition of xy, the sub-multiplicative property of f combined with the non-negative property of g, a suitable decomposition of g, the Hölder integral inequality with the parameter p (taking into account that

f and g are non-negative), and the changes of variables $u = \lambda x$ and $v = y/\lambda$ (followed by an uniformization of the notation), we obtain

$$\begin{aligned} f(xy)\int_0^1 g(t)dt &= f(xy)\int_0^1 g(\lambda)d\lambda = \int_0^1 f(xy)g(\lambda)d\lambda \\ &= \int_0^1 f\left(\lambda x\frac{y}{\lambda}\right)g(\lambda)d\lambda \leq \int_0^1 f(\lambda x)f\left(\frac{y}{\lambda}\right)g(\lambda)d\lambda \\ &= \int_0^1 f(\lambda x)f\left(\frac{y}{\lambda}\right)g^{\theta}(\lambda)g^{1-\theta}(\lambda)d\lambda \\ &\leq \left(\int_0^1 f^p(\lambda x)g^{\theta p}(\lambda)d\lambda\right)^{1/p}\left(\int_0^1 f^q\left(\frac{y}{\lambda}\right)g^{(1-\theta)q}(\lambda)d\lambda\right)^{1/q} \\ &= \left(\frac{1}{x}\int_0^x f^p(u)g^{\theta p}\left(\frac{u}{x}\right)du\right)^{1/p}\left(y\int_y^{+\infty}\frac{f^q(v)}{v^2}g^{(1-\theta)q}(vy)dv\right)^{1/q} \\ &= x^{-1/p}y^{1/q}\left(\int_0^x f^p(t)g^{\theta p}\left(\frac{t}{x}\right)dt\right)^{1/p}\left(\int_y^{+\infty}\frac{f^q(t)}{t^2}g^{(1-\theta)q}(ty)dt\right)^{1/q}. \end{aligned}$$

This concludes the proof of the proposition. □

The integral bounds presented for $f(xy)$ are mainly of theoretical interest. They can be used to find new two-dimensional inequalities with different mathematical scopes.

5. Double integral inequalities

Some double integral inequalities using sub-multiplicative functions are established in [8,9]. They belong to the large topic of Hardy-Hilbert-type integral inequalities. In this section, we present new and general ones involving products and sums of variables, starting with the proposition below.

Proposition 10. *Let $a, b \in (0,+\infty) \cup \{+\infty\}$, $f : [0,+\infty) \mapsto [0,+\infty)$ be a sub-multiplicative function and $g, h : [0,+\infty) \to [0,+\infty)$ be two functions. Then*

1. we have

$$\int_0^b\int_0^a g(s)h(t)f(st)dsdt \leq \left(\int_0^a f(t)g(t)dt\right)\left(\int_0^b f(t)h(t)dt\right),$$

2. we have

$$\int_0^b\int_0^a g(s)h(t)f(st)dsdt \geq f(1)\left(\int_0^a \frac{1}{f(1/t)}g(t)dt\right)\left(\int_0^b \frac{1}{f(1/t)}h(t)dt\right),$$

provided that the integrals involved converge.

Proof. 1. Using the sub-multiplicative property of f combined with the non-negative property of g and h (followed by an uniformization of the notation), we have

$$\begin{aligned} &\int_0^b\int_0^a g(s)h(t)f(st)dsdt \leq \int_0^b\int_0^a g(s)h(t)f(s)f(t)dsdt \\ &= \left(\int_0^a f(s)g(s)ds\right)\left(\int_0^b f(t)h(t)dt\right) = \left(\int_0^a f(t)g(t)dt\right)\left(\int_0^b f(t)h(t)dt\right). \end{aligned}$$

2. The sub-multiplicative property of f combined with its non-negative property gives

$$f(1) = f\left(st\frac{1}{st}\right) \leq f(st)f\left(\frac{1}{st}\right) \leq f(st)f\left(\frac{1}{s}\right)f\left(\frac{1}{t}\right),$$

so that

$$f(st) \geq \frac{f(1)}{f(1/s)f(1/t)}.$$

Using this and the non-negative property of g and h (followed by an uniformization of the notation), we have

$$\begin{aligned}\int_0^b \int_0^a g(s)h(t)f(st)dsdt &\geq \int_0^b \int_0^a g(s)h(t)\frac{f(1)}{f(1/s)f(1/t)}dsdt \\ &= f(1)\left(\int_0^a \frac{1}{f(1/s)}g(s)ds\right)\left(\int_0^b \frac{1}{f(1/t)}h(t)dt\right) \\ &= f(1)\left(\int_0^a \frac{1}{f(1/t)}g(t)dt\right)\left(\int_0^b \frac{1}{f(1/t)}h(t)dt\right).\end{aligned}$$

This ends the proof of the proposition. □

For example, if we take the sub-multiplicative function $f(t) = \log(e+t)$, the result in the first item gives

$$\int_0^b \int_0^a g(s)h(t)\log(e+st)dsdt \leq \left(\int_0^a \log(e+t)g(t)dt\right)\left(\int_0^b \log(e+t)h(t)dt\right).$$

The result in the second item also gives

$$\int_0^b \int_0^a g(s)h(t)\log(e+st)dsdt \geq \log(e+1)\left(\int_0^a \frac{1}{\log(e+1/t)}g(t)dt\right)\left(\int_0^b \frac{1}{\log(e+1/t)}h(t)dt\right).$$

Another similar double integral inequality is presented below.

Proposition 11. *Let $a,b \in (0,+\infty) \cup \{+\infty\}$, $f : [0,+\infty) \mapsto [0,+\infty)$ be a sub-multiplicative function and $g,h : [0,+\infty) \to [0,+\infty)$ be two functions. Then we have*

$$\int_0^b \int_0^a sg(s)h(st)f(s)f(t)dsdt \geq \int_0^a g(s)\phi(s)ds,$$

where

$$\phi(s) = \int_0^{bs} f(t)h(t)dt,$$

provided that the integrals involved converge. In particular, when $b \to +\infty$, we have

$$\int_0^{+\infty} \int_0^a sg(s)h(st)f(s)f(t)dsdt \geq \left(\int_0^a g(t)dt\right)\left(\int_0^{+\infty} f(t)h(t)dt\right),$$

provided that the integrals involved converge.

Proof. Using the sub-multiplicative property of f combined with the non-negative property of g and h, and the change of variables $u = st$ (followed by an uniformization of the notation), we have

$$\begin{aligned}\int_0^b \int_0^a sg(s)h(st)f(s)f(t)dsdt &\geq \int_0^b \int_0^a sg(s)h(st)f(st)dsdt \\ &= \int_0^a g(s)\left(\int_0^b f(st)h(st)sdt\right)ds = \int_0^a g(s)\left(\int_0^{bs} f(u)h(u)du\right)ds \\ &= \int_0^a g(s)\phi(s)ds.\end{aligned}$$

Applying $b \to +\infty$, we have

$$\phi(s) = \int_0^{+\infty} f(t)h(t)dt,$$

for any $s \in (0,+\infty)$, so that, with an uniformization of the notation,

$$\int_0^{+\infty} \int_0^a sg(s)h(st)f(s)f(t)dsdt \geq \int_0^a g(s)\left(\int_0^{+\infty} f(t)h(t)dt\right)ds$$

$$= \left(\int_0^a g(s)ds\right)\left(\int_0^{+\infty} f(t)h(t)dt\right) = \left(\int_0^a g(t)dt\right)\left(\int_0^{+\infty} f(t)h(t)dt\right).$$

This ends the proof of the proposition. □

For example, if we take the sub-multiplicative function $f(t) = \log(e+t)$, we have

$$\int_0^{+\infty}\int_0^a sg(s)h(st)\log(e+s)\log(e+t)dsdt \geq \left(\int_0^a g(t)dt\right)\left(\int_0^{+\infty} \log(e+t)h(t)dt\right).$$

The proposition below establishes another double integral inequality. It has the originality of considering a general function taken at the sum of the variables, multiplied by two sub-multiplicative functions.

Proposition 12. *Let $a, b \in (0,+\infty)\cup\{+\infty\}$, $f, g : [0,+\infty) \mapsto [0,+\infty)$ be two sub-multiplicative functions and $h : [0,+\infty) \to [0,+\infty)$ be a function. Then we have*

$$\int_0^b\int_0^a f(s)h(t)g(s+t)dsdt \leq \int_0^b tf(t)g(t)h(t)\psi(t)dt,$$

with

$$\psi(t) = \int_0^{a/t} f(s)g(1+s)ds,$$

provided that the integrals involved converge. In particular, when $a \to +\infty$, we have

$$\int_0^b\int_0^{+\infty} f(s)h(t)g(s+t)dsdt \leq \left(\int_0^{+\infty} f(t)g(1+t)dt\right)\left(\int_0^b tf(t)g(t)h(t)dt\right),$$

provided that the integrals involved converge.

Proof. Using the change of variables $s = ut$ and the sub-multiplicative property of f and g combined with the non-negative property of h, we have

$$\begin{aligned}
\int_0^b\int_0^a f(s)h(t)g(s+t)dsdt &= \int_0^b\left(\int_0^a f(s)h(t)g(s+t)ds\right)dt \\
&= \int_0^b\left(\int_0^{a/t} f(ut)h(t)g(ut+t)tdu\right)dt = \int_0^b\left(\int_0^{a/t} f(ut)h(t)g(t(1+u))tdu\right)dt \\
&\leq \int_0^b\left(\int_0^{a/t} f(u)f(t)h(t)g(t)g(1+u)tdu\right)dt \\
&= \int_0^b tf(t)g(t)h(t)\left(\int_0^{a/t} f(u)g(1+u)du\right)dt = \int_0^b tf(t)g(t)h(t)\psi(t)dt.
\end{aligned}$$

Applying $a \to +\infty$, we have

$$\psi(t) = \int_0^{+\infty} f(s)g(1+s)ds$$

for any $t \in (0,+\infty)$, so that, with an uniformization of the notation,

$$\begin{aligned}
\int_0^b\int_0^{+\infty} f(s)h(t)g(s+t)dsdt &\leq \int_0^b tf(t)g(t)h(t)\left(\int_0^{+\infty} f(s)g(1+s)ds\right)dt \\
&= \left(\int_0^{+\infty} f(s)g(1+s)ds\right)\left(\int_0^b tf(t)g(t)h(t)dt\right) \\
&= \left(\int_0^{+\infty} f(t)g(1+t)dt\right)\left(\int_0^b tf(t)g(t)h(t)dt\right).
\end{aligned}$$

This ends the proof of the proposition. □

For example, if we take the sub-multiplicative functions $f(t) = \log(e+t)$ and $g(t) = t^{-\alpha}$ with $\alpha > 1$, we have

$$\begin{aligned}\int_0^b \int_0^{+\infty} \frac{\log(e+s)h(t)}{(s+t)^\alpha} dsdt &= \int_0^b \int_0^{+\infty} f(s)h(t)g(s+t)dsdt \\ &\leq \left(\int_0^{+\infty} f(t)g(1+t)dt\right)\left(\int_0^b tf(t)g(t)h(t)dt\right) \\ &= \left(\int_0^{+\infty} \frac{\log(e+t)}{(1+t)^\alpha}dt\right)\left(\int_0^b t^{1-\alpha}\log(e+t)h(t)dt\right).\end{aligned}$$

In particular, for some values of α, we can determine the exact value of the first integral. For example, for $\alpha = 2$, we have

$$\int_0^{+\infty} \frac{\log(e+t)}{(1+t)^\alpha}dt = \frac{e}{e-1},$$

for $\alpha = 3$, we have

$$\int_0^{+\infty} \frac{\log(e+t)}{(1+t)^\alpha}dt = \frac{(e-1)e-1}{2(e-1)^2},$$

and for $\alpha = 4$, we have

$$\int_0^{+\infty} \frac{\log(e+t)}{(1+t)^\alpha}dt = \frac{3+(e-2)e(2e-1)}{6(e-1)^3}.$$

The proposition below completes Proposition 12 under sub-multiplicative assumptions on different functions and by considering a possible separable domain of integration.

Proposition 13. *Let $a \in (0,+\infty) \cup \{+\infty\}$, $f, g : [0,+\infty) \mapsto [0,+\infty)$ be two sub-multiplicative functions and $h : [0,+\infty) \to [0,+\infty)$ be a function. Then we have*

$$\int\int_{\mathcal{A}} f(s)g(t)h(s+t)dsdt \leq \left(\int_0^a tf(t)g(t)h(t)dt\right)\left(\int_0^1 f(1-t)g(t)dt\right),$$

with $\mathcal{A} = \{(s,t) \in (0,+\infty)^2;\ s+t \leq a\}$, provided that the integrals involved converge. In particular, when $a \to +\infty$, we have

$$\int_0^{+\infty}\int_0^{+\infty} f(s)g(t)h(s+t)dsdt \leq \left(\int_0^{+\infty} tf(t)g(t)h(t)dt\right)\left(\int_0^1 f(1-t)g(t)dt\right),$$

provided that the integrals involved converge.

Proof. Using the change of variables $s = u^2$ and $t = v^2$ with an adaptation of the domain of integration $\mathcal{A}$, we have

$$\int\int_{\mathcal{A}} f(s)g(t)h(s+t)dsdt = 4\int\int_{\mathcal{B}} f(u^2)g(v^2)h(u^2+v^2)uvdudv,$$

with $\mathcal{B} = \{(u,v) \in (0,+\infty)^2;\ u^2+v^2 \leq a\}$. Using the polar change of variables $u = \rho\cos(\theta)$ and $v = \rho\sin(\theta)$ with the associated Jacobian equal to ρ and an adaptation of $\mathcal{B}$, which gives $\rho \in (0,\sqrt{a})$ and $\theta \in (0,\pi/2)$, we get

$$\begin{aligned}&\int\int_{\mathcal{B}} f(u^2)g(v^2)h(u^2+v^2)uvdudv \\ &= \int_0^{\pi/2}\int_0^{\sqrt{a}} f(\rho^2\cos^2(\theta))g(\rho^2\sin^2(\theta))h(\rho^2\cos^2(\theta)+\rho^2\sin^2(\theta))\rho\cos(\theta)\rho\sin(\theta)\rho d\rho d\theta \\ &= \int_0^{\pi/2}\int_0^{\sqrt{a}} f(\rho^2\cos^2(\theta))g(\rho^2\sin^2(\theta))h(\rho^2)\cos(\theta)\sin(\theta)\rho^3 d\rho d\theta.\end{aligned}$$

Using the sub-multiplicative property of f and g combined with the non-negative property of f, g and h, we have

$$\begin{aligned}
&\int_0^{\pi/2}\int_0^{\sqrt{a}} f(\rho^2\cos^2(\theta))g(\rho^2\sin^2(\theta))h(\rho^2)\cos(\theta)\sin(\theta)\rho^3 d\rho d\theta \\
&\leq \int_0^{\pi/2}\int_0^{\sqrt{a}} f(\rho^2)f(\cos^2(\theta))g(\rho^2)g(\sin^2(\theta))h(\rho^2)\cos(\theta)\sin(\theta)\rho^3 d\rho d\theta \\
&= \left(\int_0^{\sqrt{a}} \rho^3 f(\rho^2)g(\rho^2)h(\rho^2)d\rho\right)\left(\int_0^{\pi/2} f(\cos^2(\theta))g(\sin^2(\theta))\cos(\theta)\sin(\theta)d\theta\right).
\end{aligned}$$

Using the changes of variables $w=\rho^2$ and $z=\sin^2(\theta)$ and making an uniformization of the notation, we obtain

$$\begin{aligned}
&\left(\int_0^{\sqrt{a}} \rho^3 f(\rho^2)g(\rho^2)h(\rho^2)d\rho\right)\left(\int_0^{\pi/2} f(\cos^2(\theta))g(\sin^2(\theta))\cos(\theta)\sin(\theta)d\theta\right) \\
&= \left(\int_0^a wf(w)g(w)h(w)\frac{1}{2}dw\right)\left(\int_0^1 f(1-z)g(z)\frac{1}{2}dz\right) \\
&= \frac{1}{4}\left(\int_0^a tf(t)g(t)h(t)dt\right)\left(\int_0^1 f(1-t)g(t)dt\right).
\end{aligned}$$

Putting the equations above together in order, we obtain

$$\int\int_{\mathcal{A}} f(s)g(t)h(s+t)dsdt \leq \left(\int_0^a tf(t)g(t)h(t)dt\right)\left(\int_0^1 f(1-t)g(t)dt\right).$$

When $a\to+\infty$, the domain of integration becomes $\mathcal{A}=(0,+\infty)^2$, and we get

$$\int_0^{+\infty}\int_0^{+\infty} f(s)g(t)h(s+t)dsdt \leq \left(\int_0^{+\infty} tf(t)g(t)h(t)dt\right)\left(\int_0^1 f(1-t)g(t)dt\right).$$

This ends the proof of the proposition. □

In terms of the convolution product, note that

$$\int_0^1 f(1-t)g(t)dt = (f\star g)(1) = (g\star f)(1).$$

For example, if we take the sub-multiplicative function $f(t)=g(t)=\log(e+t)$, we have

$$\int\int_{\mathcal{A}} \log(e+s)\log(e+t)h(s+t)dsdt \leq \left(\int_0^a t\log^2(e+t)h(t)dt\right)\left(\int_0^1 \log(e+1-t)\log(e+t)dt\right),$$

and we can mention that

$$\int_0^1 \log(e+1-t)\log(e+t)dt \approx 1.34864.$$

As another example, if we take $h(t)=(1+t)^{-\alpha}$ with $\alpha\in\mathbb{R}$, we get the following general inequality:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(s)g(t)}{(1+s+t)^{\alpha}}dsdt \leq \left(\int_0^{+\infty} \frac{t}{(1+t)^{\alpha}}f(t)g(t)dt\right)\left(\int_0^1 f(1-t)g(t)dt\right).$$

The double integral inequalities presented under sub-multiplicative assumptions can be applied to estimate bounds in mathematical analysis, particularly in operator theory and functional analysis.

6. Conclusion

In this article, we have addressed a gap in the study of integral inequalities under sub-multiplicative assumptions. Several complementary frameworks were considered, including primitive-type integral inequalities, convolution product-type integral inequalities, integral bounds for functions evaluated at the product of two variables, and double integral inequalities. The statements were formulated rigorously, with detailed proofs employing original techniques that can be reused for other mathematical purposes. Additionally, examples were provided to illustrate key inequalities.

Among possible future directions, one idea is to explore triple integral inequalities under sub-multiplicative assumptions, in particular using spherical change of variables. Another direction is to extend these results to integral inequalities with fractional integrals, which could have applications in harmonic analysis and partial differential equations. Furthermore, the study of connections between sub-multiplicative integral inequalities and operator theory may lead to new developments in spectral analysis and functional spaces.

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