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Characterizations of Weights in Discrete Hardy's Type inequalities in Cones of Quasi-Nonincreasing Sequences

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Abstract: In this paper, we establish some new characterizations of a weight w such that discrete Hardy operator $\mathcal{H}f(n) := \frac{1}{n}\sum_{s=1}^{n} f(s)$ for quasi-nonincreasing sequence $f(n)$ is bounded in the Banach space $\ell^p_w(\mathbb{Z}_+)$ when $0 < p < \infty$. In particular, we will prove that $\mathcal{H}f$ is bounded in $\ell^p_w(\mathbb{Z}_+)$ if and only if w belongs to the β-discrete Arino and Muckenhoupt class $\mathcal{B}_{p,\beta}$. We prove that the self-improving property for the class $\mathcal{B}_{p,\beta}$ holds, that is we prove that if $w \in \mathcal{B}_{p,\beta}$ then there exists an $\varepsilon > 0$ such that $w \in \mathcal{B}_{p-\varepsilon,\beta}$.

Keywords: new charactrizations, discrete Hardy's type operator, quasi-monotone sequences, self-improving property

MSC: 26D07, 42B25, 42C10

1. Introduction

During the past few years there has been renewed interest in the area of discrete harmonic analysis and then it becomes an active field of research and the studying of regularity and boundedness of discrete operator on $l^p(\mathbb{N})$ analogues for $L^p(\mathbb{R})-$regularity and boundedness has been considered by some authors, see for example [1–10] and the references cited therein. In the unpublished note [11, problem 92.11], Heinig posed the question of characterizing the discrete weights $u(n)$ and $v(n)$ defined on $\mathbb{Z}_+$ for which the discrete weighted Hardy type inequality

$$\left(\sum_{n=1}^{\infty} u(n)\left(\sum_{s=1}^{n} f(s)\right)^{q}\right)^{\frac{1}{q}} \leq C\left(\sum_{n=1}^{\infty} v(n) f^p(n)\right)^{\frac{1}{p}}, \tag{1}$$

holds for a nonnegative sequence $f(n)$, fixed parameters $0 < p, q < \infty$ and a constant $C > 0$. The answer for this question transacting the characterizations of the weights for nonincreasing sequences and for unrestricted nonnegative sequences was presented by many authors, we refer the reader to the papers [12–14] and the references cited therein. Despite of a variety of ideas related to weighted inequalities appeared with the birth of singular integrals, it was only in the 1970s that a better understanding of the subject has been introduced. This was achieved by Muckenhoupt and published in 1972 (see[15]). In this paper the author established a full characterization of the weights w for which the Hardy-Littlewood maximal operator is bounded on $L^p_w(\mathbb{R}_+)$. Muckenhoupt's result became a landmark in the theory of weighted inequalities because most of the previously known results for classical operators had been obtained for special classes of weights (like power weights) and has been extended to cover several operators like Hardy operator, Hilbert operator, Calderón-Zygmund singular integral operators, fractional integral operators, etc. For the boundedness of discrete Hardy operator

$$\mathcal{H}f(n) := \frac{1}{n}\sum_{s=1}^{n} f(s),$$

the authors proved in [16] that the Hardy operator is bounded in the Banach space

$$\ell_v^p(\mathbb{Z}_+) = \left\{ f : \|f\| := \left(\sum_{n=1}^{\infty} |f(n)|^p \, v(n) \right)^{1/p} < \infty \right\}, \tag{2}$$

of nonnegative sequences f defined on $\mathbb{Z}_+$ with a weight v, for $1 < p < \infty$ if and only if $v \in \mathcal{M}_p$, where $\mathcal{M}_p$ is the class of discrete weights v satisfying

$$\left(\sum_{s=n}^{\infty} \frac{v(s)}{s^p} \right)^{1/p} \left(\sum_{s=1}^{n} v^{\frac{-1}{p-1}}(s) \right)^{(p-1)/p} \leq A, \qquad \text{for all } n \geq 1, \tag{3}$$

where A is a positive constant, and there exists a positive constant $C > 0$ such that

$$\sum_{n=1}^{\infty} v(n) \left(\frac{1}{n} \sum_{k=1}^{n} f(k) \right)^p \leq C \sum_{n=1}^{\infty} v(n) f^p(n). \tag{4}$$

In [17], Heinig and Kufner proved that the Hardy operator $\mathcal{H}f$ for decreasing sequences f is bounded in $\ell_v^p(\mathbb{Z}_+)^d$; that is $\mathcal{H} : \ell_v^p(\mathbb{Z}_+)^d \to \ell_v^p(\mathbb{Z}_+)$, and (4) holds for $1 < p < \infty$ if and only if $v \in \mathcal{B}_p$ and $\lim_{n\to\infty}(v(n+1)/v(n)) = c > 0$ and $\sum_{n=1}^{\infty} v(n) = \infty$, where $\mathcal{B}_p$ is the discrete class of weights v defined on the $\mathbb{Z}_+$, for $p > 1$, and satisfying

$$\sum_{k=n}^{\infty} \frac{v(k)}{k^p} \leq \frac{A}{n^p} \sum_{k=1}^{n} v(k), \text{ for all } n \geq 1 \text{ and } A > 0. \tag{5}$$

In [18], Bennett and Grosse-Erdmann improved the result of Heinig and Kufner by excluding the conditions that have been posed on v. The boundedness of discrete Hardy-Littlewood maximal operator

$$\mathcal{M}f(n) := \sup_{n\geq 1} \frac{1}{n} \sum_{s=1}^{n} f(s), \tag{6}$$

was characterized in [19] and it has been proved that $\mathcal{M}$ is bounded in $\ell_v^p(\mathbb{Z}_+)$ that is $\mathcal{M} : \ell_v^p(\mathbb{Z}_+) \to \ell_v^p(\mathbb{Z}_+)$ if and only if $v \in \mathcal{A}^p$, the class of discrete weights v satisfying the inequality

$$\left(\frac{1}{n} \sum_{s=1}^{n} v(s) \right) \left(\frac{1}{n} \sum_{s=1}^{n} v^{\frac{-1}{p-1}}(s) \right)^{p-1} \leq A, \text{ for all } n \geq 1, \tag{7}$$

for $p > 1$ and a positive constant A. In [20], the authors proved that the adjoint Hardy operator $\mathcal{S}f$, which is defined by the form

$$\mathcal{S}f(n) = \sum_{k=n}^{\infty} \frac{f(k)}{k}, \quad \text{for all } n \geq 1, \tag{8}$$

is bounded in $\ell_v^p(\mathbb{Z}_+)$, for $1 < p < \infty$, if and only $v \in \mathcal{B}_p^*$, the class of weights v satisfying

$$\sum_{k=1}^{n} \frac{v(k)}{k^p} \leq \frac{A}{n^p} \sum_{k=1}^{n} v(k), \text{ for all } n \geq 1, \tag{9}$$

where A is a positive constant and there exists a positive $C > 0$ such that

$$\sum_{n=1}^{\infty} (\mathcal{S}f(n))^p \, v(n) \leq C \sum_{n=1}^{\infty} f^p(n) v(n). \tag{10}$$

In [16], the authors proved that $\mathcal{S}f$ is bounded if and only if $v \in \mathcal{M}_p^*$, the class of discrete weights v satisfying

$$\left(\sum_{k=1}^{n} v(k) \right)^{1/p} \left(\sum_{k=n}^{\infty} \frac{v^{-p'/p}(k)}{k^{p'}} \right)^{1/p'} \leq A, \text{ for all } n \geq 1. \tag{11}$$

where $p' = p/(p-1)$ is the conjugate of $p > 1$ and A is a positive constant.

In the literature [21], it is well known that a positive sequence $b(n)$ is said to be an almost increasing sequence if there exists a positive increasing sequence $c(n)$ and two positive numbers A and B such that $Ac(n) \leq b(n) \leq Bc(n)$. A positive sequence $d(n)$ is said to be $\delta-$quasi-monotone, if $d(n) \to 0$, $d(n) > 0$ ultimately and $\Delta d(n) \leq \delta(n)$, where $\Delta d(n) = d(n+1) - d(n)$, and $\delta(n)$ is a sequence of positive numbers (see [22]). For more applications of quasi-monotone sequences, we refer the reader to the papers [23–27] and the references cited therein. We use a notion of such a generalized monotonicity in a more general context.

Definition 1. We say that the positive sequence $f(n)$ is $\alpha-$quasi-nonincreasing if there exists $\alpha \in \mathbb{R}$ such that $n^{-\alpha} f(n)$ is nonincreasing, that is $n^{-\alpha} f(n) \leq m^{-\alpha} f(m)$, for $n > m$, and we shall write $f \in Q_\alpha$.

Definition 2. We say that a positive sequence $f(n)$ is $\alpha-$quasi-nondecreasing, if there exists $\alpha \in \mathbb{R}$ such that $n^{-\alpha} f(n)$ is nondecreasing, that is $n^{-\alpha} f(n) \leq m^{-\alpha} f(m)$, for $n < m$ and we shall write $f \in Q^\alpha$.

Definition 3. The quasi-monotone nonnegative sequence f satisfies both the monotonicity conditions Q_{α_0} and Q^{α_1} for some constants α_0 and α_1 with $-\infty < \alpha_0 < \alpha_1 < \infty$, that is

$$0 \leq f(n) \leq \max\{(n/m)^{\alpha_0}, (n/m)^{\alpha_1}\} f(m), \text{ for all } m,\ n \geq 1,$$

and we shall write $f \in Q_{\alpha_0}^{\alpha_1}$.

In [28] the authors used this concept and proved that if $c(n)$ is a nonnegative sequence and $w(t)$ be a quasi-nonincreasing function defined on $[0, \infty)$, then

$$\sum_{n=1}^{\infty} w(2^n) \left(\sum_{s=1}^{n} c(s) \right)^p \leq K_0 \sum_{n=1}^{\infty} w(2^n) c^p(n),$$

holds for all $p \in \mathbb{R}$ where K_0 does not depend on the sequence $c(n)$, and if $c(n)$ is a nonnegative sequence and $w(t)$ be a quasi-nondecreasing function defined on $[0, \infty)$, then

$$\sum_{n=1}^{\infty} w(2^n) \left(\sum_{s=n}^{\infty} c(s) \right)^p \leq K_1 \sum_{n=1}^{\infty} w(2^n) c^p(n),$$

holds for all $p \in \mathbb{R}$ where K_0 does not depend on the sequence $c(n)$. A discrete weight w defined on $\mathbb{Z}_+$ is said to be belong to the discrete class $\mathcal{B}_{p,\beta}(C)$ for $p > 0$ and $\beta > -1$ if w satisfies the condition

$$\sum_{n=r}^{\infty} \frac{w(n)}{n^p} \leq \frac{C}{r^{p(\beta+1)}} \sum_{n=1}^{r} n^{\beta p} w(n), \quad r > 1. \tag{12}$$

Note that for $\beta = 0$, the class $\mathcal{B}_{p,\beta}$ reduces to the class $\mathcal{B}_p$. The smallest constant $C > 0$ satisfying (12) is called the $\mathcal{B}_{p,\beta}-$constant of the weight w and is denoted by $[w]_{\mathcal{B}_{p,\beta}}$. We define $\mathcal{B}_{p,\beta}-$constant for a weight $w \in \mathcal{B}_{p,\beta}$ as follows

$$[w]_{\mathcal{B}_{p,\beta}} := \inf \left\{ C : \sum_{n=r}^{\infty} \left(\frac{r}{n} \right)^p w(n) \leq (C-1) \sum_{n=1}^{r} \left(\frac{n}{r} \right)^{\beta p} w(n),\ r > 1 \right\}. \tag{13}$$

Note that $[w]_{\mathcal{B}_{p,\beta}} > 1$ and for $-1 < \beta \leq 0$ and $p \leq q$, we have $\mathcal{B}_{p,\beta} \subset \mathcal{B}_{q,\beta}$.

Motivated by the above results of the boundedness of the Hardy operator of monotone sequences, *the following question arises:*

Question 1. Is it possible to prove that the Hardy type inequality (4) holds for quasi-nonincreasing sequences?

Our aim in this paper is to give an affirmative answer to this question. In §2, we prove the boundedness of the Hardy operator for quasi-nonincreasing sequences and obtain new characterizations of the weights $v(n)$.

In §3, we prove that the class of the new weights satisfy the self-improving property, that is we will prove that if $w \in \mathcal{B}_{p,\beta}$, then there exists $\epsilon > 0$ such that $w \in \mathcal{B}_{p-\epsilon,\beta}$.

2. Boundedness in cones of Quasi-nonincreasing weights

In this section, we prove the boundedness of the Hardy operator for quasi-nonincreasing sequences. The sequences in the statements of theorems that follow are assumed to be nonnegative defined on $\mathbb{Z}_+$. In addition, in our proofs, we will use the convention $0.\infty = 0$ and $0/0 = 0$ and $\sum_{k=a}^{b} y(k) = 0$, whenever $a > b$. We need the following lemmas.

Lemma 1. [29] *Assume that ψ, ϕ, g be nonnegative sequences and g is nonincreasing. If*

$$\sum_{k=1}^{n} \psi(k) \leq \sum_{k=1}^{n} \phi(k), \quad \text{for all } n \in \mathbb{Z}_+, \tag{14}$$

then

$$\sum_{k=1}^{\infty} \psi(k) g(n) \leq \sum_{k=1}^{\infty} \phi(k) g(n).$$

Lemma 2. [18] *If $p \geq 1$, then for all $N \in \mathbb{Z}_+$*

$$\sum_{k=1}^{N} a(k) \left(\sum_{s=1}^{k} a(k) \right)^{p-1} \leq \left(\sum_{k=1}^{N} a(k) \right)^{p} \leq p \sum_{k=1}^{N} a(k) \left(\sum_{s=1}^{k} a(s) \right)^{p-1}.$$

The inequalities reverse direction if $0 < p < 1$ and $a(1) > 0$. The constants (1 and p) are best possible.

Lemma 3. [30] *If $p > 0$, then for all $N \in \mathbb{Z}_+$*

$$\left(\sum_{k=1}^{N} a(k) \right)^{-p} \geq \frac{p}{p+1} \sum_{k=N}^{\infty} a(k) \left(\sum_{s=1}^{k} a(s) \right)^{-p-1}.$$

The constant is best possible.

Theorem 1 (Fubini's Theorem [20])**.** *Assume that φ, ψ are nonnegative sequences. Then*

$$\sum_{n=1}^{\infty} \varphi(n) \left(\sum_{k=n}^{\infty} \psi(k) \right) = \sum_{n=1}^{\infty} \psi(n) \left(\sum_{k=1}^{n} \varphi(k) \right).$$

Theorem 2. *Let $0 < p < 1$, $w(n)$ and $f(n)$ be a non-negative sequences such that f is a $\beta-$quasi-nonincreasing sequence, $-1 < \beta \leq 0$. Then the Hardy operator*

$$\mathcal{H}f(n) := \frac{1}{n} \sum_{k=1}^{n} f(k)$$

is bounded in $\ell_w^p(\mathbb{Z}_+)$ if and only if the weight $w \in \mathcal{B}_{p,\beta}(C)$, and there exists a constant $D > 0$ such that

$$\sum_{n=1}^{\infty} w(n) \left(\frac{1}{n} \sum_{k=1}^{n} f(k) \right)^{p} \leq D \sum_{n=1}^{\infty} w(n) f^{p}(n). \tag{15}$$

Moreover, if D and C are chosen best-possible then, we have $D \leq (C+1)/p(\beta+1)$.

Proof. First, we assume that $w \in \mathcal{B}_{p,\beta}(C)$, i.e.,

$$\sum_{n=r}^{\infty} \left(\frac{r}{n}\right)^{p} w(n) \leq C \sum_{n=1}^{r} \left(\frac{n}{r}\right)^{\beta p} w(n), \quad \text{for all } r \in \mathbb{Z}_{+}. \tag{16}$$

Since f is $\beta-$quasi-nonincreasing sequence, choose β such that $h(n) = n^{-\beta} f(n)$ is nonincreasing sequence, then the left hand side of the inequality (15) takes the from

$$\sum_{n=1}^{\infty} \frac{w(n)}{n^{p}} \left(\sum_{k=1}^{n} k^{\beta} h(k)\right)^{p}. \tag{17}$$

Since $0 < p < 1$, then by replacing $a(k)$ by $k^{\beta} h(k)$ in Lemma 2, we see that,

$$\left(\sum_{k=1}^{n} k^{\beta} h(k)\right)^{p} \leq \sum_{k=1}^{n} k^{\beta} h(k) \left(\sum_{s=1}^{k} s^{\beta} h(s)\right)^{p-1}. \tag{18}$$

By combining (17) and (18), we have that

$$\sum_{n=1}^{\infty} \frac{w(n)}{n^{p}} \left(\sum_{k=1}^{n} k^{\beta} h(k)\right)^{p} \leq \sum_{n=1}^{\infty} \frac{w(n)}{n^{p}} \left(\sum_{k=1}^{n} k^{\beta} h(k) \left(\sum_{s=1}^{k} s^{\beta} h(s)\right)^{p-1}\right).$$

By using Fubini's Theorem 1 with

$$\varphi(k) = k^{\beta} h(k) \left(\sum_{s=1}^{k} s^{\beta} h(s)\right)^{p-1}, \quad \text{and} \quad \psi(k) = \frac{w(k)}{k^{p}},$$

it follows that

$$\sum_{n=1}^{\infty} \frac{w(n)}{n^{p}} \left(\sum_{k=1}^{n} k^{\beta} h(k)\right)^{p} \leq \sum_{n=1}^{\infty} n^{\beta} h(n) \left(\sum_{k=1}^{n} k^{\beta} h(k)\right)^{p-1} \left(\sum_{k=n}^{\infty} \frac{w(k)}{k^{p}}\right). \tag{19}$$

Since $f(n)$ is quasi-nonincreasing, we have that $h(n)$ is nonincreasing, then

$$h(n) \leq \frac{1}{n} \sum_{k=1}^{n} h(k).$$

This give us that

$$n^{\beta+1} h(n) \leq n^{\beta} \sum_{k=1}^{n} h(k) \leq \sum_{k=1}^{n} k^{\beta} h(k).$$

Since $p - 1 < 0$, we have that

$$\left(\sum_{k=1}^{n} k^{\beta} h(k)\right)^{p-1} \leq \left(n^{\beta+1} h(n)\right)^{p-1}. \tag{20}$$

By combining (19) and (20), we have that

$$\sum_{n=1}^{\infty} \frac{w(n)}{n^{p}} \left(\sum_{k=1}^{n} k^{\beta} h(k)\right)^{p} \leq \sum_{n=1}^{\infty} n^{p(\beta+1)-1} \left(\sum_{k=n}^{\infty} \frac{w(k)}{k^{p}}\right) h^{p}(n). \tag{21}$$

Now, we give an estimate for the term $\sum\limits_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum\limits_{k=n}^{\infty}\frac{w(k)}{k^p}\right)$, as follows

$$\begin{aligned}\sum_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{k^p}\right) &= \sum_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum_{k=n}^{N-1}\frac{w(k)}{k^p}+\sum_{k=N}^{\infty}\frac{w(k)}{k^p}\right)\\ &= \sum_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum_{k=n}^{N-1}\frac{w(k)}{k^p}\right)+\left(\sum_{k=N}^{\infty}\frac{w(k)}{k^p}\right)\left(\sum_{n=1}^{N} n^{p(\beta+1)-1}\right). \qquad (22)\end{aligned}$$

Applying summation by parts formula

$$\sum_{r=m}^{\infty} u(r)\,\Delta v(r) = u(r)\,\Delta v(r)\big|_m^{\infty} - \sum_{r=m}^{\infty}\Delta u(r)\,v(r+1), \qquad (23)$$

on the term

$$\sum_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum_{k=n}^{N-1}\frac{w(k)}{k^p}\right),$$

with

$$u(n)=\sum_{k=n}^{N-1}\frac{w(k)}{k^p}, \text{ and } \Delta v(n)=n^{p(\beta+1)-1},$$

we get that

$$\sum_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum_{k=n}^{N-1}\frac{w(k)}{k^p}\right) = u(n)\,v(n)\big|_1^{N+1} - \sum_{n=1}^{N}\Delta u(n)\,v(n+1),$$

where $v(n)=\sum\limits_{k=1}^{n-1} k^{p(\beta+1)-1}$. Since $u(N+1)=v(1)=0$, then we obtain

$$\sum_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum_{k=n}^{N-1}\frac{w(k)}{k^p}\right) = -\sum_{n=1}^{N}\Delta u(n)\,v(n+1). \qquad (24)$$

Since

$$\Delta(k-1)^{p(\beta+1)} = k^{p(\beta+1)}-(k-1)^{p(\beta+1)},$$

then, by employing the inequality [31, (2.15.2)]

$$cx^{c-1}(x-y)\le x^c-y^c\le cy^{c-1}(x-y), \text{ for } x\ge y>0, \qquad (25)$$

when $0<c<1$, we obtain for $c=p(\beta+1)<1$ that

$$(p(\beta+1))\,k^{p(\beta+1)-1}\le\Delta(k-1)^{p(\beta+1)}.$$

That is

$$v(n)\le\frac{1}{p(\beta+1)}\sum_{k=1}^{n-1}\Delta(k-1)^{p(\beta+1)}=\frac{1}{p(\beta+1)}(n-1)^{p(\beta+1)}.$$

By substituting the last inequality into (24), we get that

$$\begin{aligned}\sum_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum_{k=n}^{N-1}\frac{w(k)}{k^p}\right) &\le \frac{1}{p(\beta+1)}\sum_{n=1}^{N}\frac{w(n)}{n^p}n^{p(\beta+1)}\\ &= \frac{1}{p(\beta+1)}\sum_{n=1}^{N} n^{p\beta}w(n). \qquad (26)\end{aligned}$$

By combining (22) and (26), we have that

$$\sum_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{k^{p}}\right) \leq \frac{1}{p(\beta+1)}\sum_{n=1}^{N} n^{p\beta}w(n) + \frac{N^{p(\beta+1)}}{p(\beta+1)}\left(\sum_{k=N}^{\infty}\frac{w(k)}{k^{p}}\right).$$

Applying the condition (16) for the second term, we get that

$$\begin{aligned}\sum_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{k^{p}}\right) &\leq \frac{1}{p(\beta+1)}\sum_{n=1}^{N} n^{p\beta}w(n) + \frac{C}{p(\beta+1)}\sum_{n=1}^{N} n^{\beta p}w(n)\\ &= \frac{C+1}{p(\beta+1)}\sum_{n=1}^{N} n^{p\beta}w(n). \end{aligned} \tag{27}$$

Since h is nonincreasing, we see that $h^{p}(n)$ is also nonincreasing. So, by applying Lemma 1 with

$$\psi = n^{p(\beta+1)-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{k^{p}}\right), \text{ and } \phi = \frac{C+1}{p(\beta+1)} n^{p\beta}w(n),$$

and $g = h^{p}$, we obtain from (27) that

$$\sum_{n=1}^{\infty} n^{p(\beta+1)-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{k^{p}}\right) h^{p}(n) \leq \frac{C+1}{p(\beta+1)}\sum_{n=1}^{\infty} n^{p\beta}w(n)h^{p}(n). \tag{28}$$

Substituting (28) into (21), we have that

$$\sum_{n=1}^{\infty}\frac{w(n)}{n^{p}}\left(\sum_{k=1}^{n} k^{\beta}h(k)\right)^{p} \leq \frac{C+1}{p(\beta+1)}\sum_{n=1}^{\infty} n^{p\beta}w(n)h^{p}(n).$$

By replacing $h(n)$ by $n^{-\beta}f(n)$, we get the desired inequality (15). Now, we consider the reverse and suppose that

$$\sum_{n=1}^{\infty} w(n)\left(\frac{1}{n}\sum_{k=1}^{n} f(k)\right)^{p} \leq B\sum_{n=1}^{\infty} w(n)f^{p}(n), \tag{29}$$

holds for some constant $B > 0$. Then (29) holds when

$$f(k) = k^{\beta}\chi_{[1,r]}(k) = \begin{cases} k^{\beta}, & k \in [1,r], \\ 0, & k \notin [1,r]. \end{cases}$$

For this f in (29), we obtain

$$\sum_{n=1}^{\infty} w(n)\left(\frac{1}{n}\sum_{k=1}^{r} k^{\beta}\right)^{p} \leq B\sum_{n=1}^{r} w(n)n^{p\beta}. \tag{30}$$

By noting that

$$\sum_{n=1}^{\infty}\frac{w(n)}{n^{p}}\left(\sum_{k=1}^{r} k^{\beta}\right)^{p} \geq \sum_{n=r}^{\infty}\frac{w(n)}{n^{p}}\left(\sum_{k=1}^{r} k^{\beta}\right)^{p} = \left(\sum_{k=1}^{r} k^{\beta}\right)^{p}\sum_{n=r}^{\infty}\frac{w(n)}{n^{p}}, \tag{31}$$

and applying Lemma 2 with $a(s) = 1$, and $p-1 = \beta$, we obtain that $\sum_{k=1}^{r} k^{\beta} \geq r^{\beta+1}$. By substituting the last inequality into (31), we get that

$$\sum_{n=1}^{\infty}\frac{w(n)}{n^{p}}\left(\sum_{k=1}^{r} k^{\beta}\right)^{p} \geq r^{p(\beta+1)}\sum_{n=r}^{\infty}\frac{w(n)}{n^{p}}.$$

Then, we have from (30) that

$$\sum_{n=r}^{\infty}\frac{w(n)}{n^{p}}\leq\frac{B}{r^{p(\beta+1)}}\sum_{n=1}^{r}w(n)\,n^{p\beta},$$

which implies that $w\in\mathcal{B}_{p,\beta}(C)$, with a constant $[w]_{\mathcal{B}_{p,\beta}}\leq B$. This proves the necessary condition. The proof is complete. □

Theorem 3. *Let $p\geq 1$, $w(n)$ and $f(n)$ be a non-negative sequences such that f is a $\beta-$quasi-nonincreasing sequence, $-1<\beta\leq 0$. Then the Hardy operator*

$$\mathcal{H}f(n):=\frac{1}{n}\sum_{k=1}^{n}f(k),$$

is bounded in $\ell_{w}^{p}(\mathbb{Z}_{+})$ if and only if the weight $w\in\mathcal{B}_{p,\beta}(C)$, and there exists a constant $D>0$ such that

$$\sum_{n=1}^{\infty}w(n)\left(\frac{1}{n}\sum_{k=1}^{n}f(k)\right)^{p}\leq D\sum_{n=1}^{\infty}w(n)\,f^{p}(n). \tag{32}$$

Moreover, if D and C are chosen best-possible then, we have

$$D\leq\begin{cases}p^{p}(1+C)^{p}, & \text{for}\quad p(\beta+1)>1,\\ \left(\frac{1+C}{1+\beta}\right)^{p}, & \text{for}\quad 0<p(\beta+1)<1.\end{cases}$$

Proof. First, we assume that $w\in\mathcal{B}_{p,\beta}(C)$, i.e.,

$$\sum_{n=r}^{\infty}\frac{w(n)}{n^{p}}\leq\frac{C}{r^{p(\beta+1)}}\sum_{n=1}^{r}n^{\beta p}w(n),\ \text{ for all } n\in\mathbb{Z}_{+}. \tag{33}$$

Since f is $\beta-$quasi-nonincreasing sequence, choose β such that $h(n)=n^{-\beta}f(n)$ is nonincreasing sequence, then the left hand side of the inequality (32) takes the from

$$\sum_{n=1}^{\infty}\frac{w(n)}{n^{p}}\left(\sum_{k=1}^{n}k^{\beta}h(k)\right)^{p}.$$

From Lemma 2, we see that

$$\sum_{n=1}^{\infty}\frac{w(n)}{n^{p}}\left(\sum_{k=1}^{n}k^{\beta}h(k)\right)^{p}\leq p\sum_{n=1}^{\infty}\frac{w(n)}{n^{p}}\left(\sum_{k=1}^{n}k^{\beta}h(k)\left(\sum_{s=1}^{k}s^{\beta}h(s)\right)^{p-1}\right). \tag{34}$$

Moreover, applying Fubini's Theorem 1, with

$$\varphi(k)=k^{\beta}h(k)\left(\sum_{s=1}^{k}s^{\beta}h(s)\right)^{p-1},\quad\text{and}\quad\psi(n)=\frac{w(n)}{n^{p}},$$

it follows that

$$\sum_{n=1}^{\infty}\frac{w(n)}{n^{p}}\left(\sum_{k=1}^{n}k^{\beta}h(k)\left(\sum_{s=1}^{k}s^{\beta}h(s)\right)^{p-1}\right)=\sum_{n=1}^{\infty}n^{\beta}h(n)\left(\sum_{k=1}^{n}k^{\beta}h(k)\right)^{p-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{k^{p}}\right).$$

Finally, by combining (34) and the last identity, we get the inequality

$$\sum_{n=1}^{\infty}\frac{w(n)}{n^{p}}\left(\sum_{k=1}^{n}k^{\beta}h(k)\right)^{p}\quad\leq\quad p\sum_{n=1}^{\infty}n^{\beta}h(n)\left(\sum_{k=1}^{n}k^{\beta}h(k)\right)^{p-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{k^{p}}\right)$$

$$\leq \quad p\sum_{n=1}^{\infty} n^{p(\beta+1)-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{k^{p}}\right)\left(\frac{\sum_{k=1}^{n}k^{\beta}h(k)}{n^{\beta+1}}\right)^{p-1} h(n). \tag{35}$$

On the other hand, let $N \geq 1$ be fixed. Then, it is easy to see that

$$\begin{aligned}\sum_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{k^{p}}\right) &= \sum_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum_{k=n}^{N-1}\frac{w(k)}{k^{p}}+\sum_{k=N}^{\infty}\frac{w(k)}{k^{p}}\right)\\ &= \sum_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum_{k=n}^{N-1}\frac{w(k)}{k^{p}}\right)+\left(\sum_{k=N}^{\infty}\frac{w(k)}{k^{p}}\right)\left(\sum_{n=1}^{N} n^{p(\beta+1)-1}\right)\\ &\leq \sum_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum_{k=n}^{N}\frac{w(k)}{k^{p}}\right)+\left(\sum_{k=N}^{\infty}\frac{w(k)}{k^{p}}\right)\left(\sum_{n=1}^{N} n^{p(\beta+1)-1}\right).\end{aligned}$$

Furthermore, by applying Fubini's Theorem on the term

$$\sum_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum_{k=n}^{N}\frac{w(k)}{k^{p}}\right),$$

we have that

$$\sum_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum_{k=n}^{N}\frac{w(k)}{k^{p}}\right)=\sum_{n=1}^{N}\frac{w(n)}{n^{p}}\left(\sum_{k=1}^{n}k^{p(\beta+1)-1}\right),$$

so the previous identity can be rewritten as

$$\sum_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{k^{p}}\right)\leq\sum_{n=1}^{N}\frac{w(n)}{n^{p}}\left(\sum_{k=1}^{n}k^{p(\beta+1)-1}\right)+\left(\sum_{k=N}^{\infty}\frac{w(k)}{k^{p}}\right)\left(\sum_{n=1}^{N} n^{p(\beta+1)-1}\right). \tag{36}$$

Case 1. If $p(\beta+1)>1$, we obtain by applying Lemma 2 with $a(k)=1,$ and $p=p(\beta+1)>1$, that

$$\sum_{k=1}^{n}k^{p(\beta+1)-1}\leq n^{p(\beta+1)},$$

which together with (36) yields the inequality

$$\sum_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{k^{p}}\right)\leq\sum_{n=1}^{N}n^{p\beta}w(n)+\left(\sum_{k=N}^{\infty}\frac{w(k)}{k^{p}}\right)N^{p(\beta+1)}. \tag{37}$$

Further, substituting (33) in (37), we have that

$$\sum_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{k^{p}}\right)\leq\sum_{n=1}^{N}n^{p\beta}w(n)+C\sum_{n=1}^{N}n^{p\beta}w(n)=(1+C)\sum_{n=1}^{N}n^{p\beta}w(n). \tag{38}$$

Case 2. If $0<p(\beta+1)<1$, we obtain by applying Lemma 2 with $a(k)=1,$ and $p=p(\beta+1)$, that

$$\sum_{k=1}^{n}k^{p(\beta+1)-1}\leq\frac{n^{p(\beta+1)}}{p(\beta+1)},$$

which together with (36) yields the inequality

$$\sum_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{k^{p}}\right)\leq\frac{1}{p(\beta+1)}\left[\sum_{n=1}^{N}n^{p\beta}w(n)+\left(\sum_{k=N}^{\infty}\frac{w(k)}{k^{p}}\right)N^{p(\beta+1)}\right]. \tag{39}$$

Further, substituting (33) in (39), we have that

$$\begin{aligned}\sum_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{k^{p}}\right) &\leq \frac{1}{p(\beta+1)}\left[\sum_{n=1}^{N} n^{p\beta}w(n)+C\sum_{n=1}^{N} n^{p\beta}w(n)\right] \\ &= \frac{(1+C)}{p(\beta+1)}\sum_{n=1}^{N} n^{p\beta}w(n). \end{aligned} \tag{40}$$

Now, we have from (38) and (40) that

$$\sum_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{k^{p}}\right)\leq \mathcal{K}\sum_{n=1}^{N} n^{p\beta}w(n),$$

with

$$\mathcal{K}=\begin{cases}(1+C), & \text{for} \quad p(\beta+1)>1,\\ \frac{(1+C)}{p(\beta+1)}, & \text{for} \quad 0<p(\beta+1)<1,\end{cases}$$

then, we have

$$n^{-\beta(p-1)}\sum_{n=1}^{N} n^{p(\beta+1)-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{k^{p}}\right)\leq \mathcal{K}n^{-\beta(p-1)}\sum_{n=1}^{N} n^{p\beta}w(n).$$

Consequently, since the sequences k^{β} and $h(k)$ are nonincreasing, then, the sequence

$$\left(\frac{\sum_{k=1}^{n} k^{\beta}h(k)}{n}\right)^{p-1} h(n)$$

is nonincreasing, utilizing Lemma 1 with

$$\begin{aligned}\psi(n) &= n^{\beta+p-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{k^{p}}\right),\ \phi(n)=\mathcal{K}n^{\beta}w(n),\\ g(n) &= \left(\frac{\sum_{k=1}^{n} k^{\beta}h(k)}{n}\right)^{p-1} h(n),\end{aligned}$$

we obtain the inequality

$$\begin{aligned}\sum_{n=1}^{\infty} n^{p(\beta+1)-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{k^{p}}\right)\left(\frac{\sum_{k=1}^{n} k^{\beta}h(k)}{n^{\beta+1}}\right)^{p-1} h(n) &= \sum_{n=1}^{\infty} n^{\beta+p-1}\left(\sum_{k=n}^{\infty}\frac{w(k)}{k^{p}}\right)\left(\frac{\sum_{k=1}^{n} k^{\beta}h(k)}{n}\right)^{p-1} h(n)\\ &\leq \mathcal{K}\sum_{n=1}^{\infty} n^{\beta}w(n)\left(\frac{\sum_{k=1}^{n} k^{\beta}h(k)}{n}\right)^{p-1} h(n).\end{aligned}$$

Substituting the last inequality into the right-hand side of relation (35), it follows that

$$\sum_{n=1}^{\infty}\frac{w(n)}{n^{p}}\left(\sum_{k=1}^{n} k^{\beta}h(k)\right)^{p}\leq p\mathcal{K}\sum_{n=1}^{\infty} n^{\beta}w(n)\left(\frac{\sum_{k=1}^{n} k^{\beta}h(k)}{n}\right)^{p-1} h(n).$$

Using that $n^{\beta}h(n) = f(n)$, we have that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{w(n)}{n^p}\left(\sum_{k=1}^{n}f(k)\right)^p &\leq p\mathcal{K}\sum_{n=1}^{\infty}w(n)\left(\frac{\sum_{k=1}^{n}f(k)}{n}\right)^{p-1}f(n)\\ &= p\mathcal{K}\sum_{n=1}^{\infty}w^{\frac{p-1}{p}}(n)\left(\frac{\sum_{k=1}^{n}f(k)}{n}\right)^{p-1}w^{\frac{1}{p}}(n)f(n).\end{aligned}$$

By applying the Hölder inequality on the term

$$\sum_{n=1}^{\infty}\left[w^{\frac{p-1}{p}}(n)\left(\frac{\sum_{k=1}^{n}f(k)}{n}\right)^{p-1}\right]\left[w^{\frac{1}{p}}(n)f(n)\right]$$

with indices p and $p/(p-1)$, we have that

$$\sum_{n=1}^{\infty}\frac{w(n)}{n^p}\left(\sum_{k=1}^{n}f(k)\right)^p \leq p\mathcal{K}\left[\sum_{n=1}^{\infty}w(n)\left(\frac{\sum_{k=1}^{n}f(k)}{n}\right)^{p}\right]^{\frac{p-1}{p}}\left[\sum_{n=1}^{\infty}w(n)f^p(n)\right]^{\frac{1}{p}},$$

and consequently,

$$\sum_{n=1}^{\infty}\frac{w(n)}{n^p}\left(\sum_{k=1}^{n}f(k)\right)^p \leq p^p\mathcal{K}^p\sum_{n=1}^{\infty}w(n)f^p(n),$$

which is the inequality (32).

It remains to prove the opposite direction. More precisely, suppose that (32) holds, i.e.,

$$\sum_{n=1}^{\infty}w(n)\left(\frac{1}{n}\sum_{k=1}^{n}f(k)\right)^p \leq B^*\sum_{n=1}^{\infty}w(n)f^p(n). \tag{41}$$

holds for some constant $B^* > 0$. Then (41) holds for the sequence

$$f(k) = k^{\beta}\chi_{[1,r]}(k) = \begin{cases} 1, & k\in[1,r], \\ 0, & k\notin[1,r]. \end{cases}$$

Clearly, in this particular setting inequality (41) reduces to

$$\sum_{n=1}^{\infty}w(n)\left(\frac{1}{n}\sum_{k=1}^{r}k^{\beta}\right)^p \leq B^*\sum_{n=1}^{r}w(n)n^{p\beta}. \tag{42}$$

On the other hand, since

$$\sum_{n=1}^{\infty}\frac{w(n)}{n^p}\left(\sum_{k=1}^{r}k^{\beta}\right)^p \geq \sum_{n=r}^{\infty}\frac{w(n)}{n^p}\left(\sum_{k=1}^{r}k^{\beta}\right)^p = \left(\sum_{k=1}^{r}k^{\beta}\right)^p\sum_{n=r}^{\infty}\frac{w(n)}{n^p}. \tag{43}$$

Applying Lemma 2 with $a(r) = 1$, and $p-1=\beta$, we obtain that

$$\sum_{k=1}^{r}k^{\beta} \geq r^{\beta+1}.$$

Substituting the last inequality into (43), we get that

$$\sum_{n=1}^{\infty}\frac{w(n)}{n^{p}}\left(\sum_{k=1}^{r}k^{\beta}\right)^{p}\geq r^{p(\beta+1)}\sum_{n=r}^{\infty}\frac{w(n)}{n^{p}}.$$

Then, we have from (42) that

$$\sum_{n=r}^{\infty}\frac{w(n)}{n^{p}}\leq\frac{B^{*}}{r^{p(\beta+1)}}\sum_{n=1}^{r}w(n)\,n^{p\beta},$$

which means that $w\in\mathcal{B}_{p,\beta}(C)$, with a constant $[w]_{\mathcal{B}_{p,\beta}}\leq B^{*}$. This proves the necessary part of the theorem. The proof is complete. □

By replacing f by g^{p_0} and p by p/p_0 in Theorems 2 and 3 we have the following theorems.

Theorem 4. *Let p_0, $p>0$ such that $p/p_0<1$ and g be a nonnegative and $\beta-$quasi-nonincreasing sequence, $-1<\beta\leq 0$, and define the Hardy operator*

$$\mathcal{H}g^{p_0}(n):=\frac{1}{n}\sum_{k=1}^{n}g^{p_0}(k).$$

Then

$$\sum_{n=1}^{\infty}w(n)\left(\frac{1}{n}\sum_{k=1}^{n}g^{p_0}(k)\right)^{p/p_0}\leq D\sum_{n=1}^{\infty}w(n)\,g^{p}(n),\quad p/p_0<1, \tag{44}$$

if and only if the weight $w\in\mathcal{B}^{Q}_{\frac{p}{p_0},\beta}(C)$, with

$$\sum_{n=r}^{\infty}\left(\frac{r}{n}\right)^{\frac{p}{p_0}}w(n)\leq C\sum_{n=1}^{r}\left(\frac{n}{r}\right)^{\frac{\beta p}{p_0}}w(n),\ \textit{for all } n\in\mathbb{Z}_{+},\ \textit{and } C>0. \tag{45}$$

Moreover, if D and C are chosen best-possible then, we have $D\leq p_0(C+1)/p(\beta+1)$.

Theorem 5. *Let p_0, $p>0$ such that $p/p_0>1$ and g be a nonnegative and $\beta-$quasi-nonincreasing sequence, $-1<\beta\leq 0$, and define the Hardy operator*

$$\mathcal{H}g^{p_0}(n):=\frac{1}{n}\sum_{k=1}^{n}g^{p_0}(k).$$

Then

$$\sum_{n=1}^{\infty}w(n)\left(\frac{1}{n}\sum_{k=1}^{n}g^{p_0}(k)\right)^{p/p_0}\leq D\sum_{n=1}^{\infty}w(n)\,g^{p}(n),\quad p/p_0>1, \tag{46}$$

if and only if the weight $w\in\mathcal{B}^{Q}_{\frac{p}{p_0},\beta}(C)$, with

$$\sum_{n=r}^{\infty}\left(\frac{r}{n}\right)^{\frac{p}{p_0}}w(n)\leq C\sum_{n=1}^{r}\left(\frac{n}{r}\right)^{\frac{\beta p}{p_0}}w(n),\ \textit{for all } n\in\mathbb{Z}_{+},\ \textit{and } C>0. \tag{47}$$

Moreover, if D and C are chosen best-possible then, we have

$$D\leq\begin{cases}\left(\frac{p(1+C)}{p_0}\right)^{p/p_0}, & \textit{for}\quad \frac{p(\beta+1)}{p_0}>1,\\ \left(\frac{1+C}{1+\beta}\right)^{p/p_0}, & \textit{for}\quad 0<\frac{p(\beta+1)}{p_0}<1.\end{cases}$$

3. Self-Improving Property of $\mathcal{B}_{p,\beta}$

In this section, we study the cases for which the self improving property of the class $\mathcal{B}_{p,\beta}$ holds. That, we will prove that the self-improving property of the class $\mathcal{B}_{p,\beta}$ holds for any weight w when $\beta\geq 0$. While for

$-1 < \beta < 0$, it is not completely true but we were able to prove for a special case which is the power weights $w(n) = n^{\alpha}$.

Theorem 6. *If $p > 0$, $\beta \geq 0$ and $w \in \mathcal{B}_{p,\beta}(C)$, then there exists $\epsilon > 0$ such that $w \in \mathcal{B}_{p-\epsilon,\beta}$. Moreover, $0 < \epsilon < p(\beta+1)/(C+1)$.*

Proof. Since $w \in \mathcal{B}_{p,\beta}(C)$ for $p > 0$ and $\beta \geq 0$, then

$$\sum_{n=r}^{\infty} \frac{w(n)}{n^p} \leq Cr^{-p} \sum_{n=1}^{r} \left(\frac{n}{r}\right)^{\beta p} w(n) = Cr^{-p(\beta+1)} \sum_{n=1}^{r} n^{\beta p} w(n), \quad \text{for all } r \in \mathbb{Z}_+. \tag{48}$$

Multiplying (48) by $r^{\epsilon-1}$ and summing from m to ∞, we have that

$$\sum_{r=m}^{\infty} r^{\epsilon-1} \sum_{n=r}^{\infty} \frac{w(n)}{n^p} \leq C \sum_{r=m}^{\infty} r^{\epsilon-p(\beta+1)-1} \sum_{n=1}^{r} n^{\beta p} w(n). \tag{49}$$

Using the fact that $\frac{r+1}{2} < r$ for $r \geq 1$, and since $\epsilon - p(\beta+1) - 1 < 0$, then (49) becomes

$$\sum_{r=m}^{\infty} r^{\epsilon-1} \sum_{n=r}^{\infty} \frac{w(n)}{n^p} \leq 2^{p(\beta+1)-\epsilon+1} C \sum_{r=m}^{\infty} (r+1)^{\epsilon-p(\beta+1)-1} \sum_{n=1}^{r} n^{\beta p} w(n). \tag{50}$$

Applying the summation by parts formula

$$\sum_{r=m}^{\infty} u(r) \Delta v(r) = u(r) \Delta v(r)|_m^{\infty} - \sum_{r=m}^{\infty} \Delta u(r) v(r+1),$$

with $u(r) = \sum_{n=r}^{\infty} \frac{w(n)}{n^p}$ and $\Delta v(r) = r^{\epsilon-1}$, the left side of (50) becomes

$$\sum_{r=m}^{\infty} r^{\epsilon-1} \sum_{n=r}^{\infty} \frac{w(n)}{n^p} = \left(\sum_{n=r}^{\infty} \frac{w(n)}{n^p}\right)\left(\sum_{n=m}^{r-1} n^{\epsilon-1}\right)\Bigg|_{r=m}^{\infty} - \sum_{r=m}^{\infty}\left(\sum_{n=m}^{r} n^{\epsilon-1}\right) \Delta \left(\sum_{n=r}^{\infty} \frac{w(n)}{n^p}\right)$$

where $v(r) = \sum_{n=m}^{r-1} n^{\epsilon-1}$. Using $v(m) = u(\infty) = 0$ (recall all summations are assumed to be convergent), we obtain that

$$\begin{aligned}
\sum_{r=m}^{\infty} r^{\epsilon-1} \sum_{n=r}^{\infty} \frac{w(n)}{n^p} &= -\sum_{r=m}^{\infty}\left(\sum_{n=m}^{r} n^{\epsilon-1}\right) \Delta \left(\sum_{n=r}^{\infty} \frac{w(n)}{n^p}\right) \\
&= \sum_{r=m}^{\infty}\left(\sum_{n=m}^{r} n^{\epsilon-1}\right) \frac{w(r)}{r^p} \\
&\geq \sum_{r=m}^{\infty}\left(\sum_{n=m}^{r-1} n^{\epsilon-1}\right) \frac{w(r)}{r^p}.
\end{aligned}$$

By employing the inequality (25) with $0 < \epsilon < 1$, we have that

$$(n+1)^{\epsilon} - n^{\epsilon} = \Delta n^{\epsilon} \leq \epsilon n^{\epsilon-1},$$

and then

$$\sum_{n=m}^{r-1} n^{\epsilon-1} \geq \frac{1}{\epsilon} \sum_{n=m}^{r-1} \Delta n^{\epsilon} = \frac{r^{\epsilon}}{\epsilon} - \frac{m^{\epsilon}}{\epsilon}.$$

This implies that

$$\sum_{r=m}^{\infty} r^{\epsilon-1} \sum_{n=r}^{\infty} \frac{w(n)}{n^p} \geq \frac{1}{\epsilon} \sum_{r=m}^{\infty} \frac{w(r)}{r^p} \left[r^{\epsilon} - m^{\epsilon}\right]. \tag{51}$$

Next, we consider the right hand side of (50) and applying summation by parts, by setting

$$u(r+1)=\sum_{n=1}^{r}n^{\beta p}w(n), \text{ and } \Delta v(r)=(r+1)^{\epsilon-p(\beta+1)-1},$$

we get that

$$\begin{aligned}&\sum_{r=m}^{\infty}(r+1)^{\epsilon-p(\beta+1)-1}\sum_{n=1}^{r}n^{\beta p}w(n)\\&=-\left(\sum_{n=r+1}^{\infty}n^{\epsilon-p(\beta+1)-1}\right)\left(\sum_{n=1}^{r-1}n^{\beta p}w(n)\right)\Bigg|_{m}^{\infty}+\sum_{r=m}^{\infty}\left(\sum_{n=r+1}^{\infty}n^{\epsilon-p(\beta+1)-1}\right)r^{\beta p}w(r)\\&\leq\left(\sum_{n=m+1}^{\infty}n^{\epsilon-p(\beta+1)-1}\right)\left(\sum_{n=1}^{m}n^{\beta p}w(n)\right)+\sum_{r=m}^{\infty}\left(\sum_{n=r+1}^{\infty}n^{\epsilon-p(\beta+1)-1}\right)r^{\beta p}w(r).\end{aligned}\tag{52}$$

By applying the inequality

$$cy^{c-1}(x-y)\leq x^{c}-y^{c}\leq cx^{c-1}(x-y), \text{ for } x\geq y>0,\ c<0 \text{ or } c>1,\tag{53}$$

with $c=\epsilon-p(\beta+1)<0$, we see that

$$n^{\epsilon-p(\beta+1)}-(n-1)^{\epsilon-p(\beta+1)}=\Delta(n-1)^{\epsilon-p(\beta+1)}\leq(\epsilon-p(\beta+1))\,n^{\epsilon-p(\beta+1)-1},$$

and then

$$\sum_{n=r+1}^{\infty}n^{\epsilon-p(\beta+1)-1}\leq\frac{1}{\epsilon-p(\beta+1)}\sum_{n=r+1}^{\infty}\Delta(n-1)^{\epsilon-p(\beta+1)}=\frac{r^{\epsilon-p(\beta+1)}}{p(\beta+1)-\epsilon}.\tag{54}$$

Substituting (54) into (52), we have that

$$\sum_{r=m}^{\infty}(r+1)^{\epsilon-p(\beta+1)-1}\sum_{n=1}^{r}n^{\beta p}w(n)\leq\frac{m^{\epsilon-p(\beta+1)}}{p(\beta+1)-\epsilon}\left(\sum_{r=1}^{m}r^{\beta p}w(r)\right)+\frac{1}{p(\beta+1)-\epsilon}\sum_{r=m}^{\infty}r^{\epsilon-p}w(r).\tag{55}$$

By combining (51) and (55) into (50), we have that

$$\frac{1}{\epsilon}\sum_{r=m}^{\infty}\frac{w(r)}{r^{p}}\left[r^{\epsilon}-m^{\epsilon}\right]\leq\frac{2^{p(\beta+1)-\epsilon+1}Cm^{\epsilon-p(\beta+1)}}{p(\beta+1)-\epsilon}\left(\sum_{r=1}^{m}r^{\beta p}w(r)\right)+\frac{2^{p(\beta+1)-\epsilon+1}C}{p(\beta+1)-\epsilon}\sum_{r=m}^{\infty}r^{\epsilon-p}w(r).$$

This gives us after once more using (48), that

$$\begin{aligned}&\left(\frac{1}{\epsilon}-\frac{2^{p(\beta+1)-\epsilon+1}C}{p(\beta+1)-\epsilon}\right)\sum_{r=m}^{\infty}\frac{w(r)}{r^{p-\epsilon}}\\&\leq\frac{m^{\epsilon}}{\epsilon}\sum_{r=m}^{\infty}\frac{w(r)}{r^{p}}+\left(\frac{2^{p(\beta+1)-\epsilon+1}C}{p(\beta+1)-\epsilon}\right)m^{\epsilon-p(\beta+1)}\left(\sum_{r=1}^{m}r^{\beta p}w(r)\right)\\&\leq\frac{m^{\epsilon}}{\epsilon}\left(Cm^{-p(\beta+1)}\sum_{r=1}^{m}r^{\beta p}w(r)\right)+\left(\frac{2^{p(\beta+1)-\epsilon+1}C}{p(\beta+1)-\epsilon}\right)m^{\epsilon-p(\beta+1)}\left(\sum_{r=1}^{m}r^{\beta p}w(r)\right)\\&=C\left(\frac{1}{\epsilon}+\frac{2^{p(\beta+1)-\epsilon+1}}{p(\beta+1)-\epsilon}\right)m^{\epsilon-p(\beta+1)}\left(\sum_{r=1}^{m}r^{\beta p}w(r)\right)\\&\leq 2^{p(\beta+1)+1}C\left(\frac{1}{\epsilon}+\frac{1}{p(\beta+1)-\epsilon}\right)m^{\epsilon-p(\beta+1)}\left(\sum_{r=1}^{m}r^{\beta p}w(r)\right),\end{aligned}$$

that is

$$\left(\frac{p(\beta+1)-\epsilon-\epsilon C}{\epsilon(p(\beta+1)-\epsilon)}\right)\sum_{r=m}^{\infty}\frac{w(r)}{r^{p-\epsilon}}\leq\left(\frac{2^{p(\beta+1)+1}Cp(\beta+1)}{\epsilon(p(\beta+1)-\epsilon)}\right)m^{\epsilon-p(\beta+1)}\left(\sum_{r=1}^{m}r^{\beta p}w(r)\right), \tag{56}$$

and thus

$$\begin{aligned}\sum_{r=m}^{\infty}\left(\frac{m}{r}\right)^{p-\epsilon}w(r) &\leq 2^{p(\beta+1)+1}\left(\frac{Cp(\beta+1)}{p(\beta+1)-\epsilon(C+1)}\right)\left(\sum_{r=1}^{m}\left(\frac{r}{m}\right)^{\beta p}w(r)\right)\\ &\leq 2^{p(\beta+1)+1}\left(\frac{Cp(\beta+1)}{p(\beta+1)-\epsilon(C+1)}\right)\left(\sum_{r=1}^{m}\left(\frac{r}{m}\right)^{\beta(p-\epsilon)}w(r)\right),\end{aligned}$$

which implies that $w\in\mathcal{B}_{p-\epsilon,\beta}$ for $0<\epsilon<\frac{p(\beta+1)}{C+1}$ and with a constant

$$\bar{C}\leq\left(\frac{2^{p(\beta+1)+1}Cp(\beta+1)}{p(\beta+1)-\epsilon(C+1)}\right),$$

which is the desired result. The proof is complete. □

We study below the self improving property for the power weights for $-1<\beta<0$.

Lemma 4. *Let $1\leq p<\infty$, $-1<\beta<0$, and $-\beta p-1<\alpha<p-1$. If $x^{\alpha}\in\mathcal{B}_{p,\beta}(C)$, then there exists $0<\varepsilon<p(\beta+1)$ such that $x^{\alpha}\in\mathcal{B}_{p-\varepsilon,\beta}$.*

Proof. Since $x^{\alpha}\in\mathcal{B}_{p,\beta}(C)$, we have that

$$\sum_{n=r}^{\infty}\left(\frac{r}{n}\right)^{p}n^{\alpha}\leq C\sum_{n=1}^{r}\left(\frac{n}{r}\right)^{\beta p}n^{\alpha},\quad r>1, \tag{57}$$

which holds if and only if

$$-\beta p-1<\alpha<p-1. \tag{58}$$

Choose $\varepsilon>0$ such that $0<\varepsilon<p-\alpha-1$, clearly $0<\varepsilon<p(\beta+1)$. We need to prove that $x^{\alpha}\in\mathcal{B}_{p-\varepsilon,\beta}$, *i.e.*,

$$\sum_{n=r}^{\infty}\left(\frac{r}{n}\right)^{p-\varepsilon}n^{\alpha}\leq C\sum_{n=1}^{r}\left(\frac{n}{r}\right)^{\beta(p-\varepsilon)}n^{\alpha},\quad r>1.$$

Consider the left side, we obtain

$$\sum_{n=r}^{\infty}\left(\frac{r}{n}\right)^{p-\varepsilon}n^{\alpha}=r^{p-\varepsilon}\sum_{n=r}^{\infty}n^{\alpha-p+\varepsilon}. \tag{59}$$

By applying Lemma 3 with

$$a(k)=1,\text{ and } p=p-\alpha-\varepsilon-1>0,$$

we obtain that

$$\sum_{n=r}^{\infty}n^{\alpha-p+\varepsilon}\leq\frac{(p-\alpha-\varepsilon)}{(p-\alpha-\varepsilon-1)}r^{\alpha-p+\varepsilon+1}.$$

By substituting last inequality into (59), we obtain

$$\sum_{n=r}^{\infty}\left(\frac{r}{n}\right)^{p-\varepsilon}n^{\alpha}\leq\frac{(p-\alpha-\varepsilon)}{(p-\alpha-\varepsilon-1)}r^{\alpha+1}. \tag{60}$$

Again, by applying the inequality (53) for $c = \alpha - p + 1 < 0$, with $x = n + 1$ and $y = n$, we see that

$$\Delta n^{\alpha-p+1} \geq (\alpha - p + 1)\, n^{\alpha-p}.$$

Now, by summating from r to ∞, we have

$$\sum_{n=r}^{\infty} \Delta n^{\alpha-p+1} \geq (\alpha - p + 1) \sum_{n=r}^{\infty} n^{\alpha-p},$$

and then

$$-r^{\alpha-p+1} \geq (\alpha - p + 1) \sum_{n=r}^{\infty} n^{\alpha-p},$$

this implies that

$$r^{\alpha+1} \leq (p - \alpha - 1)\, r^{p} \sum_{n=r}^{\infty} n^{\alpha-p} = (p - \alpha - 1) \sum_{n=r}^{\infty} \left(\frac{r}{n}\right)^{p} n^{\alpha}.$$

By substituting last inequality into (60), we obtain

$$\begin{aligned} \sum_{n=r}^{\infty} \left(\frac{r}{n}\right)^{p-\varepsilon} n^{\alpha} &\leq \frac{(p-\alpha-\varepsilon)(p-\alpha-1)}{(p-\alpha-\varepsilon-1)} \sum_{n=r}^{\infty} \left(\frac{r}{n}\right)^{p} n^{\alpha} \\ &\leq C\left(\frac{(p-\alpha-\varepsilon)(p-\alpha-1)}{p-\alpha-\varepsilon-1}\right) \sum_{n=1}^{r} \left(\frac{n}{r}\right)^{\beta p} n^{\alpha} \\ &= K r^{-\beta p} \sum_{n=1}^{r} n^{\alpha+\beta p}, \end{aligned} \tag{61}$$

for $K = C\left(\frac{(p-\alpha-\varepsilon)(p-\alpha-1)}{p-\alpha-\varepsilon-1}\right)$.

Case 1. If $\alpha + \beta p < 0$, we obtain by applying the inequality (25) for $c := 0 < \alpha + \beta p + 1 < 1$, with $x = n$ and $y = n - 1$, that

$$\Delta\left((n-1)^{\alpha+\beta p+1}\right) \geq (\alpha + \beta p + 1)\, n^{\alpha+\beta p}.$$

Now, by summating from 1 to r, we have

$$\begin{aligned} \sum_{n=1}^{r} n^{\alpha+\beta p} &\leq \frac{r^{\alpha+\beta p+1}}{(\alpha+\beta p+1)} - \frac{1}{(\alpha+\beta p+1)} \\ &\leq \frac{r^{\alpha+\beta p+1}}{(\alpha+\beta p+1)}. \end{aligned}$$

By substituting last inequality into (61), we obtain

$$\begin{aligned} \sum_{n=r}^{\infty} \left(\frac{r}{n}\right)^{p-\varepsilon} n^{\alpha} &\leq \frac{K}{(\alpha+\beta p+1)\, r^{\beta p}} r^{\alpha+\beta p+1} = \frac{K}{(\alpha+\beta p+1)} r^{\alpha+1} \\ &= \frac{K}{(\alpha+\beta p+1)\, r^{\beta(p-\varepsilon)}} r^{\alpha+\beta(p-\varepsilon)+1}. \end{aligned} \tag{62}$$

Case 2. If $\alpha + \beta p > 0$, we obtain by applying Lemma 2 with

$$a(n) = 1, \text{ and } p = \alpha + \beta p + 1 > 1,$$

that

$$\sum_{n=1}^{r} n^{\alpha+\beta p} \leq r^{\alpha+\beta p+1}.$$

By substituting last inequality into (61), we obtain

$$\sum_{n=r}^{\infty} \left(\frac{r}{n}\right)^{p-\varepsilon} n^{\alpha} \leq \frac{K}{r^{\beta p}} r^{\alpha+\beta p+1} = K r^{\alpha+1} = \frac{K}{r^{\beta(p-\varepsilon)}} r^{\alpha+\beta(p-\varepsilon)+1}. \tag{63}$$

Now, we have from (62) and (63) that

$$\sum_{n=r}^{\infty}\left(\frac{r}{n}\right)^{p-\varepsilon} n^{\alpha} \leq \mathcal{R}\frac{1}{r^{\beta(p-\varepsilon)}} r^{\alpha+\beta(p-\varepsilon)+1}, \tag{64}$$

with

$$\mathcal{R}=\begin{cases} \frac{K}{(\alpha+\beta p+1)}, & \text{for} \quad \alpha+\beta p<0, \\ K, & \text{for} \quad \alpha+\beta p>0. \end{cases}$$

It's clear that $\beta(p-\varepsilon)>\beta p$, which mean that $\alpha+\beta(p-\varepsilon)+1>0$, then we have two cases:
Case 1. If $\alpha+\beta(p-\varepsilon)+1>1$, we obtain by applying Lemma 2 with

$$a(n)=1, \text{ and } p=\alpha+\beta(p-\varepsilon)+1>1,$$

we obtain that

$$r^{\alpha+\beta(p-\varepsilon)+1} \leq (\alpha+\beta(p-\varepsilon)+1)\sum_{n=1}^{r} n^{\alpha+\beta(p-\varepsilon)}. \tag{65}$$

Case 2. If $\alpha+\beta(p-\varepsilon)+1<1$, we obtain by applying Lemma 2 with

$$a(n)=1, \text{ and } 0<\alpha+\beta(p-\varepsilon)+1<1,$$

we obtain that

$$r^{\alpha+\beta(p-\varepsilon)+1} \leq \sum_{n=1}^{r} n^{\alpha+\beta(p-\varepsilon)}. \tag{66}$$

Now, we have from (65) and (66) that

$$r^{\alpha+\beta(p-\varepsilon)+1} \leq \hat{\mathcal{R}}\sum_{n=1}^{r} n^{\alpha+\beta(p-\varepsilon)}, \tag{67}$$

with

$$\hat{\mathcal{R}}=\begin{cases} 1, & \text{for} \quad \alpha+\beta(p-\varepsilon)<0, \\ (\alpha+\beta(p-\varepsilon)+1), & \text{for} \quad \alpha+\beta(p-\varepsilon)>0. \end{cases}$$

By substituting inequality(67) into (64), we get that

$$\sum_{n=r}^{\infty}\left(\frac{r}{n}\right)^{p-\varepsilon} n^{\alpha} \leq \mathcal{R}\hat{\mathcal{R}}\sum_{n=1}^{r}\left(\frac{n}{r}\right)^{\beta(p-\varepsilon)} n^{\alpha},$$

i.e., $x^{\alpha}\in\mathcal{B}_{p-\varepsilon,\beta}$ with the constant $C^{*}:=\mathcal{R}\hat{\mathcal{R}}$, which is the desired result. The proof is complete. □

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