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Article

Hankel determinants with Fekete-Szegö parameter for a subset of Bazilevič functions linked with Ma-Minda function

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Abstract: Consider a unit disk $\Omega = \{z : |z| < 1\}$. A large subset of the set of analytic-univalent functions defined in Ω is examined in this exploration. This new set contains various subsets of the Yamaguchi and starlike functions, both of which have profound properties in the well-known set of Bazilevič functions. The Ma-Minda function and a few mathematical concepts, including subordination, set theory, infinite series formation and product combination of certain geometric expressions, are used in the definition of the new set. The estimates for the coefficient bounds, the Fekete-Szegö functional with real and complex parameters, and the Hankel determinants with a real parameter are some of the accomplishments. In general, when some parameters are changed within their interval of declarations, the set reduces to a number of recognized sets.

Keywords: analytic function, starlike function, Yamaguchi function, Ma-Minda function, coefficient estimate, Fekete-Szegö estimate, Hankel determinant, Bazilevič function, subordination

MSC: 30C45, 30C50

1. Background details

One of the more intriguing areas of complex analysis is Geometric Function Theory (GFT), which has drawn a lot of interest from pure mathematicians. The study of the geometric features of analytic functions is the focus of GFT, which has many applications in different areas of mathematics, including mathematical physics, (q-)calculus, conformal mappings, orthogonal polynomials, and special functions.

In this work, let $\mathcal{A}$ connote the set of functions that are analytic in the unit disc

$$\{z : z \in \mathbb{C} \text{ and } |z| < 1\} = \Omega,$$

and let $\mathcal{S}$ connote the subset of $\mathcal{A}$ of analytic-univalent functions having the Taylor's series kind

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1}$$

satisfying the conditions: $f(0) = 0 = f'(0) - 1$ and $z \in \Omega$.

Bieberbach announced in 1916 that the modulus of the coefficient a_2 of function f in (1) is less than or equals to 2, and generalized this result in a conjecture that $|a_j| \leqq j$, where $j \in \{2, 3, 4, \ldots\}$. This conjecture is a precursor to the study of *coefficient problems*. Nonetheless, Duren stressed in [1] that the *coefficient problem* is the assignment of the points $(a_2, a_3, a_4, \ldots, a_j)$ for function f to the portion of the $(j-1)$-dimensional complex plane. In 1985, Branges [2] confirmed the conjecture to be accurate, hence, this argumented the theory of geometric functions to one of the constantly expanding fields of potential study. Furthermore, the sets of starlike functions, convex functions, close-to-convex functions, close-to-starlike functions, spirallike functions, Yamaguchi functions, and more are some well-known subsets of $\mathcal{S}$. Factually speaking, it can be observed that various (sharp and non-sharp) upper estimates for the coefficient bounds, Fekete-Szegö functional, Hankel determinants, Toeplitz determinants, and many generalizations (for example, see [3–9]), are few of the coefficient properties that have been explored by complex function theorists. In addition, the nature and properties of these subsets which are largely based on the geometries of their image domains are continuously been explored with no end in sight.

For any two functions $f, F \in \mathcal{A}$, function $f \prec F$, if there exists another analytic function

$$s(z) = s_1 z + s_2 z^2 + s_3 z^3 + \cdots \quad (|s(z)| = |z| < 1,\ s(0) = 0, \text{ and } z \in \Omega), \tag{2}$$

such that $f(z) = F(s(z))$. Should F be univalently defined in Ω, then

$$f(z) \prec F(z), \quad \textit{if and only if}, \quad f(0) = F(0) \quad \text{and} \quad f(\Omega) \subset F(\Omega).$$

The notation '$\prec$' connotes subordination.

1.1. Some relevant subsets of analytic functions

An analytic function

$$\wp \in \mathcal{CR} := \left\{ \wp : \wp(z) = 1 + \sum_{j=1}^{\infty} p_j z^j,\ \mathcal{R}e\, \wp(z) > 0,\ \wp(0) = 1, \text{ and } z \in \Omega \right\}, \tag{3}$$

is called a function whose real part is positive in the unit disc Ω or a *Carathéodory function*. Over the decades, this set has proven to play a crucial role in the solution of many problems in GFT. The Möbius function

$$m_0(z) = \frac{1+z}{1-z} = 1 + 2\sum_{j=1}^{\infty} z^j \quad (z \in \Omega), \tag{4}$$

is the extremal function in set $\mathcal{CR}$. It has been well-established that functions $s(z)$ and $\wp(z)$ are interrelated by the relation

$$\wp(z) = \frac{1+s(z)}{1-s(z)} \implies s(z) = \frac{\wp(z)-1}{\wp(z)+1} \quad (z \in \Omega), \tag{5}$$

so that putting (3) into (5) yields

$$s(z) = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right) z^3 + \cdots \right]. \tag{6}$$

In 1994, Ma and Minda [10] amalgamated various kinds of functions in set $\mathcal{CR}$ by introducing the analytic and univalent function $b(z)$ of the Taylor's series kind

$$b(z) = 1 + \beta_1 z + \beta_2 z^2 + \beta_3 z^3 + \cdots \quad (\beta_1 > 0,\ \beta_k \in \mathbb{R},\ z \in \Omega). \tag{7}$$

In this case, $\mathcal{R}e\, b(z) > 0$, $b(0) = 1$, $b'(0) > 0$, $b(z)$ maps Ω onto a starlike domain with respect to 1, and it is symmetric with respect to the real axis. In fact, putting (6) into (7) with some computations gives the infinite series

$$\begin{aligned} b(s(z)) &= 1 + \beta_1 s(z) + \beta_2 (s(z))^2 + \beta_3 (s(z))^3 + \cdots \\ &= 1 + \frac{\beta_1}{2} p_1 z + \left[\frac{\beta_1}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{\beta_2}{4} p_1^2 \right] z^2 \\ &\quad + \left[\frac{\beta_1}{2}\left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right) + \frac{\beta_2}{2} p_1 \left(p_2 - \frac{p_1^2}{2}\right) + \frac{\beta_3}{8} p_1^3 \right] z^3 + \cdots. \end{aligned} \tag{8}$$

A function $f \in \mathcal{S}$ that maps the unit disc Ω onto a starlike domain $\mathcal{ST}$ is known as a starlike function, hence, it is necessary that

$$f \in \mathcal{ST} := \Big\{ f : f \in \mathcal{S},\ \mathcal{R}e(z(f'(z)/f(z))) > 0, \text{ and } z \in \Omega \Big\}. \tag{9}$$

The extremal function in $\mathcal{ST}$ has been proved to be the Koebe function $k(z) = z(1-z)^{-2}$. The set $\mathcal{ST}$ was introduced by Alexandra [11] while its geometric condition (9) was established by Nevanlinda in 1921, see [1,12] for further details. More so, starlike functions have been studied in many forms, for instance, see Lasode and Opoola [7] for some details on the forms and some application areas.

In 1956, Yamaguchi [13] introduced the subset Y of the set $\mathcal{S}$ of functions

$$f \in Y := \Big\{ f : f \in \mathcal{S},\ \mathcal{R}e(f(z)/z) > 0, \text{ and } z \in \Omega \Big\}, \tag{10}$$

where many properties of functions $f \in Y$ such as the univalence, radii, partial sums, growth, distortion, and inclusion properties were established by some GFT researchers in [13–15].

The set $\mathcal{B}(\delta, \gamma, \wp, h)$ introduced by Bazilevič [16] was proved to be the largest known subset of the set $\mathcal{S}$. Functions in $\mathcal{B}(\delta, \gamma, \wp, h)$ have the integral form

$$f(z) = \left\{ (\delta + i\gamma) \int_0^z \wp(t) h(t)^{\gamma} t^{-(1-i\gamma)} dt \right\}^{\frac{1}{\delta + i\gamma}},$$

such that $\delta > 0$, γ has real value, $h \in \mathcal{ST}$, $\wp \in \mathcal{CR}$ and all powers are taken as principal values. Singh [17] studied the subsets $\mathcal{B}(\delta, 0, \wp, h) = \mathcal{B}(\delta)$ and $\mathcal{B}(\delta, 0, \wp, z) = \mathcal{B}_1(\delta)$ such that for $\delta > 0$ and $z \in \Omega$,

$$\mathcal{R}e \frac{zf'(z)f(z)^{\delta-1}}{h(z)^{\delta}} > 0 \quad \text{and} \quad \mathcal{R}e \frac{zf'(z)f(z)^{\delta-1}}{z^{\delta}} > 0, \tag{11}$$

respectively. Furthermore, observe that $\mathcal{B}_1(\delta) \subset \mathcal{B}(\delta) \subset \mathcal{B}(\delta, \gamma, \wp, h)$, $\mathcal{B}_1(0) = \mathcal{B}(0) = \mathcal{ST}$, while $\mathcal{B}(1)$ and $\mathcal{B}_1(1)$ have the conditions $\mathcal{R}e \frac{zf'(z)}{h(z)} > 0$ and $\mathcal{R}e f'(z) > 0$, respectively. These are respectively, the conditions for close-to-convexity and bounded turning functions.

2. Lemmas

The following lemmas with kin interest in (3) shall be needed to proof the results.

Lemma 1. ([12, p. 25]). *Let $\wp \in \mathcal{CR}$. Then*

$$|p_j| \leqq 2 \ \forall j \in \{1, 2, 3, \dots\}.$$

The inequality is sharp for the function (4).

Lemma 2. ([18, Corollary 2.5]). *Let $\wp \in \mathcal{CR}$. Then*

$$\left| p_2 - \lambda \frac{p_1^2}{2} \right| \leqq \begin{cases} 2(1-\lambda) & \text{when} \quad \lambda \leqq 0, \\ 2 & \text{when} \quad 0 \leqq \lambda \leqq 2, \\ 2(\lambda - 1) & \text{when} \quad \lambda \geqq 2, \\ 2\max\{1, |1-\lambda|\} & \text{when} \quad \lambda \in \mathbb{C}. \end{cases}$$

Lemma 3. ([19, Lemma 2.2]). *Let $\wp \in \mathcal{CR}$. Then*

$$|up_1^3 - vp_1p_2 + wp_3| \leqq 2|u| + 2|v - 2u| + 2|u - v + w|.$$

Lemma 4 ([20]). *Let $\wp \in \mathcal{CR}$. Then for $i, j \in \{1, 2, 3, \ldots\}$,*

$$|p_{i+j} - \mu p_i p_j| \leqq \begin{cases} 2 & \text{when} \quad 0 \leqq \mu \leqq 1 \\ 2|2\mu - 1| & \text{elsewhere.} \end{cases}$$

3. Main results

3.1. A new set of analytic-univalent functions

Using the classical definitions of the Bazilevič functions, the other aforementioned functions, and the concept of subordination, we now introduce a new set of analytic-univalent functions as follows.

Definition 1. A function $f \in \mathcal{A}$ is said to be in the set $\Lambda(\delta, b)$ if the subordination condition

$$\left(\frac{zf'(z)}{f(z)}\right)^{\delta} \left(\frac{f(z)}{z}\right)^{1-\delta} \prec b(z), \tag{12}$$

is satisfied for $z \in \Omega$, and $0 \leqq \delta \leqq 1$, where $b(z)$ is the Ma-Minda function in (7).

Remark 1. The following are some well-known subsets of the new set $\Lambda(\delta, b)$.

1. Setting $\delta = 0$ and $b(z) = m_0(z)$ of (4) shows that

$$\frac{f(z)}{z} \prec m_0(z),$$

which is the geometric condition for the Yamaguchi functions presented in (10), and

2. setting $\delta = 1$ and $b(z) = m_0(z)$ of (4) shows that

$$\frac{zf'(z)}{f(z)} \prec m_0(z),$$

which is the geometric condition for the starlike functions presented in (9).

In this exploration, we establish the upper estimates for some early coefficient inequalities, Fekete-Szegö functional (with real and complex parameters) and some Hankel determinants involving a certain real parameter $\tau > 0$.

3.2. Coefficient estimates for Set $\Lambda(\delta, b)$

Theorem 1. *Let $f \in \mathcal{A}$ be a member of the set $\Lambda(\delta, b)$. Then*

$$|a_2| \leqq \beta_1, \tag{13}$$

$$|a_3| \leqq \frac{\beta_1 + \delta(2-\delta)\beta_1^2 + |\beta_2|}{(1+\delta)}, \tag{14}$$

$$\begin{aligned} |a_4| \leqq & \frac{\beta_1}{(1+2\delta)} + \frac{2}{(1+2\delta)} \left\{ \frac{3\delta\beta_1^2}{2(1+\delta)} + |\beta_2| \right\} \\ & + \frac{1}{(1+2\delta)} \left\{ \frac{[3\delta^2(2-\delta) - \delta(1+\delta)]\beta_1^3}{(1+\delta)} + \frac{3\delta\beta_1|\beta_2|}{(1+\delta)} + |\beta_3| \right\}, \end{aligned} \tag{15}$$

and

$$\begin{aligned} |a_5| \leqq & \frac{5\beta_1}{(1+3\delta)} + \frac{9|\beta_2|}{(1+3\delta)} + \frac{3|\beta_3|}{(1+3\delta)} + \frac{|\beta_4|}{(1+3\delta)} + \frac{\delta(64 + 7\delta - 19\delta^2 - 4\delta^3)\beta_1^4}{48\delta(1+3\delta)} \\ & + [2\delta(5 - 4\delta + \delta^2)\beta_1^2] \frac{\beta_1 + [\delta(2-\delta)\beta_1^2 + |\beta_2|]}{(1+\delta)(1+3\delta)} \end{aligned}$$

$$+\frac{2\delta\beta_1}{(1+3\delta)}\left\{\frac{\beta_1}{(1+2\delta)}+\frac{2(\beta_1+|\beta_2|)}{(1+2\delta)}\right.$$
$$\left.+\left(\frac{\beta_1+2|\beta_2|+|\beta_3|+\delta}{(1+2\delta)}+\frac{24\delta\beta_1^2+3\delta(\beta_1|\beta_2|+2\beta_1^2+3\delta^2(2-\delta))\beta_1^3}{(1+\delta)(1+2\delta)}\right)\right\}$$
$$+\frac{4+\delta-\delta^2}{2(1+3\delta)}\left(\frac{\beta_1+[\delta(2-\delta)\beta_1^2+|\beta_2|]}{(1+\delta)}\right)^2. \quad (16)$$

Proof. Let the function $f\in\mathcal{A}$ be in the set $\Lambda(\delta,b)$, then by subordination principle, we can say that

$$\left(\frac{zf'(z)}{f(z)}\right)^{\delta}\left(\frac{f(z)}{z}\right)^{1-\delta}=b(s(z)) \quad (z\in\Omega). \quad (17)$$

Binomially, the series form of the LHS of (17) is

$$\left(\frac{zf'(z)}{f(z)}\right)^{\delta}\left(\frac{f(z)}{z}\right)^{1-\delta}=1+a_2z+\left[(1+\delta)a_3+\delta(\delta-2)a_2^2\right]z^2$$
$$+\left[(1+2\delta)a_4-3\delta a_2a_3+(\delta(\delta-1)-\delta(\delta-2)a_2^3)\right]z^3+\cdots, \quad (18)$$

hence, equating the coefficients in (18) and (8) implies that

$$a_2=\frac{\beta_1}{2}p_1, \quad (19)$$

$$a_3=\frac{2\beta_1\left(p_2-\frac{p_1^2}{2}\right)+\left(\delta(2-\delta)\beta_1^2+\beta_2\right)p_1^2}{4(1+\delta)}, \quad (20)$$

$$a_4=\frac{\beta_1}{2(1+2\delta)}\left(\frac{p_1^3}{4}-p_1p_2+p_3\right)+\frac{1}{2(1+2\delta)}\left(\frac{3\delta\beta_1^2}{2(1+\delta)}+\beta_2\right)p_1\left(p_2-\frac{p_1^2}{2}\right)$$
$$+\frac{1}{8(1+2\delta)}\left(\frac{[3\delta^2(2-\delta)-\delta(1+\delta)]\beta_1^3}{(1+\delta)}+\frac{3\delta\beta_1\beta_2}{(1+\delta)}+\beta_3\right)p_1^3, \quad (21)$$

and

$$a_5=\frac{\beta_1}{2(1+3\delta)}\left(p_4-p_1p_3-\frac{1}{2}p_2^2+\frac{3}{4}p_1^2p_2-\frac{1}{8}p_1^4\right)+\frac{\beta_2}{2(1+3\delta)}\left(\frac{1}{4}p_1^4-\frac{3}{2}p_1^2p_2+p_1p_3+\frac{1}{2}p_2^2\right)$$
$$+\frac{\beta_3}{2(1+3\delta)}\left(\frac{3}{4}p_1^2p_2-\frac{3}{8}p_1^4\right)+\frac{\beta_4}{16(1+3\delta)}p_1^4+\frac{\delta(64+7\delta-19\delta^2-4\delta^3)\beta_1^4}{768\delta(1+3\delta)}p_1^4$$
$$-\frac{\delta(5-4\delta+\delta^2)\beta_1^2}{2}p_1^2\times\frac{2\beta_1(p_2-\frac{p_1^2}{2})+[\delta(2-\delta)\beta_1^2+\beta_2]p_1^2}{4(1+\delta)(1+3\delta)}$$
$$-\frac{2\delta\beta_1}{(1+3\delta)}\left\{\frac{\beta_1}{2(1+2\delta)}p_3+\frac{(\beta_2-\beta_1)}{2(1+2\delta)}p_1p_2\right.$$
$$\left.+\left(\frac{\beta_1-2\beta_2+\beta_3-\delta}{8(1+2\delta)}+\frac{6\delta\beta_1^2p_1p_2+3\delta(\beta_1\beta_2-2\beta_1^2-3\delta^2(2-\delta))\beta_1^3}{8(1+\delta)(1+2\delta)}\right)p_1^3\right\}$$
$$-\frac{4+\delta-\delta^2}{2(1+3\delta)}\left(\frac{2\beta_1(p_2-\frac{p_1^2}{2})+[\delta(2-\delta)\beta_1^2+\beta_2]p_1^2}{4(1+\delta)}\right)^2. \quad (22)$$

Apparently, (19) gives

$$|a_2|=\frac{\beta_1}{2}|p_1|$$

and the application of Lemma 1 gives the result in (13). Also, (20) gives

$$|a_3| \leqq \frac{2\beta_1 \left| p_2 - \frac{p_1^2}{2} \right| + \left\{ \delta(2-\delta)\beta_1^2 + |\beta_2| \right\} |p_1|^2}{4(1+\delta)},$$

and the application of Lemmas 2 (when $\lambda = 1$) and 1 gives the result in (14). From (21) we have

$$\begin{aligned} |a_4| \leqq & \frac{\beta_1}{2(1+2\delta)} \left| \frac{p_1^3}{4} - p_1 p_2 + p_3 \right| + \frac{1}{2(1+2\delta)} \left\{ \frac{3\delta\beta_1^2}{2(1+\delta)} + |\beta_2| \right\} |p_1| \left| p_2 - \frac{p_1^2}{2} \right| \\ & + \frac{1}{8(1+2\delta)} \left\{ \frac{[3\delta^2(2-\delta) - \delta(1+\delta)]\beta_1^3}{(1+\delta)} + \frac{3\delta\beta_1|\beta_2|}{(1+\delta)} + |\beta_3| \right\} |p_1|^3, \end{aligned}$$

and the respective application of Lemmas 3 (when $u = 1/4$, and $v = 1 = w$), 2 (when $\lambda = 1$) and 1 gives the result in (15). Lastly, from (22), we have

$$\begin{aligned} |a_5| \leqq & \frac{\beta_1}{2(1+3\delta)} \left| [p_4 - p_1 p_3] - \frac{1}{2} p_2^2 + \frac{3}{4} p_1^2 \left[p_2 - \frac{1}{3}\frac{p_1^2}{2} \right] \right| + \frac{|\beta_2|}{2(1+3\delta)} \left| -\frac{3}{2} p_1^2 \left[p_2 - \frac{1}{3}\frac{p_1^2}{2} \right] + p_1 p_3 + \frac{1}{2} p_2^2 \right| \\ & + \frac{|\beta_3|}{2(1+3\delta)} \left| \frac{3}{4} p_1^2 \left[p_2 - \frac{p_1^2}{2} \right] \right| + \frac{|\beta_4|}{16(1+3\delta)} |p_1|^4 + \frac{\delta(64 + 7\delta - 19\delta^2 - 4\delta^3)\beta_1^4}{768\delta(1+3\delta)} |p_1|^4 \\ & + \frac{\delta(5 - 4\delta + \delta^2)\beta_1^2}{2} |p_1|^2 \times \frac{2\beta_1 |p_2 - \frac{p_1^2}{2}| + [\delta(2-\delta)\beta_1^2 + |\beta_2|] |p_1|^2}{4(1+\delta)(1+3\delta)} \\ & + \frac{2\delta\beta_1}{(1+3\delta)} \left\{ \frac{\beta_1}{2(1+2\delta)} |p_3| + \frac{(|\beta_2| + \beta_1)}{2(1+2\delta)} |p_1 p_2| \right. \\ & \left. + \left(\frac{\beta_1 + 2|\beta_2| + |\beta_3| + \delta}{8(1+2\delta)} + \frac{6\delta\beta_1^2 |p_1 p_2| + 3\delta(\beta_1|\beta_2| + 2\beta_1^2 + 3\delta^2(2-\delta))\beta_1^3}{8(1+\delta)(1+2\delta)} \right) |p_1|^3 \right\} \\ & + \frac{4 + \delta - \delta^2}{2(1+3\delta)} \left(\frac{2\beta_1 |p_2 - \frac{p_1^2}{2}| + [\delta(2-\delta)\beta_1^2 + |\beta_2|] |p_1|^2}{4(1+\delta)} \right)^2, \end{aligned}$$

which further simplify to

$$\begin{aligned} |a_5| \leqq & \frac{\beta_1}{2(1+3\delta)} \left\{ |p_4 - p_1 p_3| + \frac{1}{2} |p_2|^2 + \frac{3}{4} |p_1|^2 \left| p_2 - \frac{1}{3}\frac{p_1^2}{2} \right| \right\} \\ & + \frac{|\beta_2|}{2(1+3\delta)} \left\{ \frac{3}{2} |p_1|^2 \left| p_2 - \frac{1}{3}\frac{p_1^2}{2} \right| + |p_1 p_3| + \frac{1}{2} |p_2|^2 \right\} \\ & + \frac{|\beta_3|}{2(1+3\delta)} \left\{ \frac{3}{4} |p_1|^2 \left| p_2 - \frac{p_1^2}{2} \right| \right\} + \frac{|\beta_4|}{16(1+3\delta)} |p_1|^4 \\ & + \frac{\delta(64 + 7\delta - 19\delta^2 - 4\delta^3)\beta_1^4}{768\delta(1+3\delta)} |p_1|^4 \\ & + \frac{\delta(5 - 4\delta + \delta^2)\beta_1^2}{2} |p_1|^2 \times \frac{2\beta_1 |p_2 - \frac{p_1^2}{2}| + [\delta(2-\delta)\beta_1^2 + |\beta_2|] |p_1|^2}{4(1+\delta)(1+3\delta)} \\ & + \frac{2\delta\beta_1}{(1+3\delta)} \left\{ \frac{\beta_1}{2(1+2\delta)} |p_3| + \frac{(|\beta_2| + \beta_1)}{2(1+2\delta)} |p_1 p_2| \right. \\ & \left. + \left(\frac{\beta_1 + 2|\beta_2| + |\beta_3| + \delta}{8(1+2\delta)} + \frac{6\delta\beta_1^2 |p_1 p_2| + 3\delta(\beta_1|\beta_2| + 2\beta_1^2 + 3\delta^2(2-\delta))\beta_1^3}{8(1+\delta)(1+2\delta)} \right) |p_1|^3 \right\} \end{aligned}$$

$$+\frac{4+\delta-\delta^2}{2(1+3\delta)}\left(\frac{2\beta_1|p_2-\frac{p_1^2}{2}|+[\delta(2-\delta)\beta_1^2+|\beta_2|]|p_1|^2}{4(1+\delta)}\right)^2,$$

and by systematically applying Lemmas 4 (when $\mu = 1$), 2 (when $\lambda = 1/3$), and 1 we have the result in (16). □

3.3. Fekete-Szegö estimates for set $\Lambda(\delta, b)$

The exploration of the Fekete-Szegö functional

$$\mathcal{FS}(\tau, f) = |a_3 - \tau a_2^2|, \tag{23}$$

is another aspect of coefficient problems of function f in $\mathcal{S}$. It is a mathematical tool in GFT named after two great function theorists Micheal Fekete and Gabor Szegö when they disproved the famous Littlewood-Parley's conjecture. The functional has been explored for several subsets of the set $\mathcal{S}$ as evident in a few works like [3–6,21–23].

Theorem 2. *Let $f \in \Lambda(\delta, b)$. Then*

$$\mathcal{FS}(\tau, f) \leqq \begin{cases} -\frac{\beta_1}{2(1+\delta)}\left\{[\tau(1+\delta)-\delta(2-\delta)]\beta_1-\frac{\beta_2}{\beta_1}-1\right\} & when \quad \tau \leqq \varphi_1, \\ \frac{\beta_1}{(1+\delta)} & when \quad \varphi_1 \leqq \tau \leqq \varphi_2, \\ \frac{\beta_1}{2(1+\delta)}\left\{[\tau(1+\delta)-\delta(2-\delta)]\beta_1-\frac{\beta_2}{\beta_1}-1\right\} & when \quad \tau \geqq \varphi_2, \\ \frac{\beta_1}{(1+\delta)}\max\{1, \varphi_3\} & when \quad \tau \in \mathbb{C}, \end{cases}$$

where

$$\varphi_1 = \frac{1}{(1+\delta)}\left\{\frac{1}{\beta_1}\left(\frac{\beta_2}{\beta_1}-1\right)+\delta(2-\delta)\right\},$$
$$\varphi_2 = \frac{1}{(1+\delta)}\left\{\frac{1}{\beta_1}\left(\frac{\beta_2}{\beta_1}+3\right)+\delta(2-\delta)\right\},$$

and

$$\varphi_3 = |1-\lambda| = 1+\frac{1}{2}\left|1+[\tau(1+\delta)-\delta(2-\delta)]\beta_1-\frac{\beta_2}{\beta_1}\right|. \tag{24}$$

Proof. Considering (19) and (20) in (23) shows that

$$a_3-\tau a_2^2 = \frac{2\beta_1\left(p_2-\frac{p_1^2}{2}\right)+\left(\delta(2-\delta)\beta_1^2+\beta_2\right)p_1^2}{4(1+\delta)}-\tau\left(\frac{\beta_1}{2}p_1\right)^2$$
$$= \frac{\beta_1}{2(1+\delta)}\left\{p_2-\frac{1}{2}\left(1+[\tau(1+\delta)-\delta(2-\delta)]\beta_1-\frac{\beta_2}{\beta_1}\right)\frac{p_1^2}{2}\right\},$$

so that

$$|a_3-\tau a_2^2| \leqq \frac{\beta_1}{2(1+\delta)}\left|p_2-\frac{1}{2}\left(1+[\tau(1+\delta)-\delta(2-\delta)]\beta_1-\frac{\beta_2}{\beta_1}\right)\frac{p_1^2}{2}\right|. \tag{25}$$

If we set

$$\lambda = \frac{1}{2}\left(1+[\tau(1+\delta)-\delta(2-\delta)]\beta_1-\frac{\beta_2}{\beta_1}\right),$$

in (25), then applying Lemma 2 means

$$\left|p_2-\lambda\frac{p_1^2}{2}\right| \leqq 2(1-\lambda) = -\left\{[\tau(1+\delta)-\delta(2-\delta)]\beta_1-\frac{\beta_2}{\beta_1}-1\right\}, \tag{26}$$

so that when $\lambda \leqq 0$, we have

$$\tau \leqq \frac{1}{(1+\delta)}\left\{\frac{1}{\beta_1}\left(\frac{\beta_2}{\beta_1}-1\right)+\delta(2-\delta)\right\}. \tag{27}$$

Secondly,

$$\left|p_2-\lambda\frac{p_1^2}{2}\right| \leqq 2, \tag{28}$$

so that when $0 \leqq \lambda \leqq 2$, we have

$$\frac{1}{(1+\delta)}\left\{\frac{1}{\beta_1}\left(\frac{\beta_2}{\beta_1}-1\right)+\delta(2-\delta)\right\} \leqq \tau \leqq \frac{1}{(1+\delta)}\left\{\frac{1}{\beta_1}\left(\frac{\beta_2}{\beta_1}+3\right)+\delta(2-\delta)\right\}. \tag{29}$$

Thirdly,

$$\left|p_2-\lambda\frac{p_1^2}{2}\right| \leqq 2(\lambda-1) = [\tau(1+\delta)-\delta(2-\delta)]\beta_1 - \frac{\beta_2}{\beta_1} - 1, \tag{30}$$

so that when $\lambda \geqq 2$, we have

$$\tau \geqq \frac{1}{(1+\delta)}\left\{\frac{1}{\beta_1}\left(\frac{\beta_2}{\beta_1}+3\right)+\delta(2-\delta)\right\}. \tag{31}$$

Lastly, if the parameter λ is complex, then

$$1-\lambda = 1-\frac{1}{2}\left(1+[\tau(1+\delta)-\delta(2-\delta)]\beta_1-\frac{\beta_2}{\beta_1}\right), \tag{32}$$

from where φ_3 in (24) holds. Now gathering (26 – 32) into (25) gives the results of the theorem.

□

3.4. Hankel determinant estimates for set $\Lambda(\delta, b)$

At around the mid-nineteenth century, Hermann Hankel (1839-1873), a German mathematician, initiated a square matrix in which each ascending skew-diagonal from left to right is constant. The matrix is known today as the Hankel matrix. Hankel's work was primarily based on the study of the sequence of numbers and their determinants. Since its initiation, the matrix and its determinants have been applied in solving problems involving factorial fractions [24], orthogonal polynomials [25], power series with integral coefficients [26], and in the study of the asymptotic behavior of Hankel determinants [27].

In GFT, Pommerenke [28] initiated the Hankel determinant

$$\mathcal{H}_{i,j}(f) = \begin{vmatrix} a_j & a_{j+1} & \cdots & a_{j+i-1} \\ a_{j+1} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{j+i-1} & \cdots & \cdots & a_{j+2(i-1)} \end{vmatrix}, \tag{33}$$

where $i, j \geqq 1$ and the elements a_j are the coefficients of functions $f \in \mathcal{S}$. In addition, the author used the Hankel determinants to investigate complex function's singularities. Babalola [29] extended the work of Pommerenke [28] by infusing a Fekete-Szegö parameter $\tau > 0$ into the definition of $\mathcal{H}_{i,j}(f)$ and introduced the determinants

$$\mathcal{H}^{\tau}_{i,j}(f) = \begin{vmatrix} a_j & a_{j+1} & \cdots & \tau a_{j+i-1} \\ a_{j+1} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{j+i-1} & \cdots & \cdots & a_{j+2(i-1)} \end{vmatrix}. \tag{34}$$

Observe that (34) simplifies to

$$|\mathcal{H}^{\tau}_{2,1}(f)| = |a_3 - \tau a_2^2|, \tag{35}$$

$$|\mathcal{H}^{\tau}_{2,2}(f)| = |a_2 a_4 - \tau a_3^2|, \tag{36}$$

and

$$|\mathcal{H}_{3,1}^{\tau}(f)| \leqq |a_3||a_2a_4 - \tau a_3^2| + |a_4||a_2a_3 - \tau a_4| + |a_5||a_3 - \tau a_2^2|, \tag{37}$$

which equivalently gives

$$|\mathcal{H}_{3,1}^{\tau}(f)| \leqq |a_3||\mathcal{H}_{2,2}^{\tau}(f)| + |a_4||\mathcal{L}_2^{\tau}(f)| + |a_5||\mathcal{H}_{2,1}^{\tau}(f)|, \tag{38}$$

where

$$|\mathcal{L}_j^{\tau}(f)| = |a_ja_{j+1} - \tau a_{j+2}| \quad (j \in \{2, 3, 4, \dots\}). \tag{39}$$

Remark 2. We remark that

1. setting $\tau = 1$ in (34) makes $\mathcal{H}_{i,j}^{\tau}(f) = \mathcal{H}_{i,j}(f)$ in (33); and
2. $|\mathcal{H}_{2,1}^{\tau}(f)| = \mathcal{FS}(\tau, f)$ in (23).

Interested function theorists can see [3,5,21,23] for more details.

In this work, we establish the upper bounds for the Hankel determinants in (36) and (38) where the parameter τ has a positive real value.

Theorem 3. *Let $f \in \Lambda(\delta, b)$. Then*

$$|\mathcal{H}_{2,2}^{\tau}(f)| \leqq \frac{\beta_1^2}{(1+2\delta)} + \frac{2\beta_1}{(1+2\delta)}\left(\frac{3\delta\beta_1^2}{2(1+\delta)} + |\beta_2|\right) + \frac{\beta_1}{(1+2\delta)}\left\{\frac{[3\delta^2(2-\delta) - \delta(1+\delta)]\beta_1^3}{(1+\delta)} + \frac{3\delta\beta_1|\beta_2|}{(1+\delta)} + |\beta_3|\right\}$$
$$+ \tau\left(\frac{\beta_1 + \delta(2-\delta)\beta_1^2 + |\beta_2|}{(1+\delta)}\right)^2.$$

Proof. Considering (19), (20) and (21) in (36) shows that

$$a_2a_4 - \tau a_3^2 = \left(\frac{\beta_1}{2}p_1\right)\left\{\frac{\beta_1}{2(1+2\delta)}\left(\frac{p_1^3}{4} - p_1p_2 + p_3\right) + \frac{1}{2(1+2\delta)}\left(\frac{3\delta\beta_1^2}{2(1+\delta)} + \beta_2\right)p_1\left(p_2 - \frac{p_1^2}{2}\right)\right.$$
$$\left. + \frac{1}{8(1+2\delta)}\left(\frac{[3\delta^2(2-\delta) - \delta(1+\delta)]\beta_1^3}{(1+\delta)} + \frac{3\delta\beta_1\beta_2}{(1+\delta)} + \beta_3\right)p_1^3\right\}$$
$$- \tau\left(\frac{2\beta_1\left(p_2 - \frac{p_1^2}{2}\right) + \left(\delta(2-\delta)\beta_1^2 + \beta_2\right)p_1^2}{4(1+\delta)}\right)^2,$$

such that

$$|a_2a_4 - \tau a_3^2| \leqq \left(\frac{\beta_1}{2}|p_1|\right)\left\{\frac{\beta_1}{2(1+2\delta)}\left|\frac{p_1^3}{4} - p_1p_2 + p_3\right| + \frac{1}{2(1+2\delta)}\left(\frac{3\delta\beta_1^2}{2(1+\delta)} + |\beta_2|\right)|p_1|\left|p_2 - \frac{p_1^2}{2}\right|\right.$$
$$\left. + \frac{1}{8(1+2\delta)}\left(\frac{[3\delta^2(2-\delta) - \delta(1+\delta)]\beta_1^3}{(1+\delta)} + \frac{3\delta\beta_1|\beta_2|}{(1+\delta)} + |\beta_3|\right)|p_1|^3\right\}$$
$$+ \tau\left(\frac{2\beta_1\left|p_2 - \frac{p_1^2}{2}\right| + \left(\delta(2-\delta)\beta_1^2 + \beta_2\right)|p_1|^2}{4(1+\delta)}\right)^2,$$

and the systematic application of Lemmas 1, 3 (when $u = 1/4$ and $v = 1 = w$), and 2 (when $\lambda = 1$), gives the desired result in the theorem. □

Theorem 4. *Let $f \in \Lambda(\delta, b)$. Then*

$$|\mathcal{L}_2^{\tau}(f)| \leqq \left\{ \frac{\beta_1^2 + \delta(2-\delta)\beta_1^3 + \beta_1|\beta_2|}{(1+\delta)} \right\} + \tau \left\{ \frac{\beta_1}{(1+2\delta)} + \frac{2}{(1+2\delta)} \left(\frac{3\delta\beta_1^2}{2(1+\delta)} + |\beta_2| \right) + \frac{1}{8(1+2\delta)} \left(\frac{[3\delta^2(2-\delta) - \delta(1+\delta)]\beta_1^3}{(1+\delta)} + \frac{3\delta\beta_1|\beta_2|}{(1+\delta)} + |\beta_3| \right) \right\}.$$

Proof. Considering (19), (20) and (21) in (39) shows that

$$a_2a_3 - \tau a_4 = \left(\frac{\beta_1}{2} p_1 \right) \left\{ \frac{2\beta_1 \left(p_2 - \frac{p_1^2}{2} \right) + \left\{ \delta(2-\delta)\beta_1^2 + \beta_2 \right\} p_1^2}{4(1+\delta)} \right\} - \tau \left\{ \frac{\beta_1}{2(1+2\delta)} \left(\frac{p_1^3}{4} - p_1p_2 + p_3 \right) + \frac{1}{2(1+2\delta)} \left(\frac{3\delta\beta_1^2}{2(1+\delta)} + \beta_2 \right) p_1 \left(p_2 - \frac{p_1^2}{2} \right) + \frac{1}{8(1+2\delta)} \left(\frac{[3\delta^2(2-\delta) - \delta(1+\delta)]\beta_1^3}{(1+\delta)} + \frac{3\delta\beta_1\beta_2}{(1+\delta)} + \beta_3 \right) p_1^3 \right\},$$

therefore,

$$|a_2a_3 - \tau a_4| \leqq \frac{\beta_1}{2}|p_1| \left\{ \frac{2\beta_1 \left| p_2 - \frac{p_1^2}{2} \right| + \left\{ \delta(2-\delta)\beta_1^2 + |\beta_2| \right\} |p_1|^2}{4(1+\delta)} \right\} + \tau \left\{ \frac{\beta_1}{2(1+2\delta)} \left| \frac{p_1^3}{4} - p_1p_2 + p_3 \right| + \frac{1}{2(1+2\delta)} \left(\frac{3\delta\beta_1^2}{2(1+\delta)} + |\beta_2| \right) |p_1| \left| p_2 - \frac{p_1^2}{2} \right| + \frac{1}{8(1+2\delta)} \left(\frac{[3\delta^2(2-\delta) - \delta(1+\delta)]\beta_1^3}{(1+\delta)} + \frac{3\delta\beta_1|\beta_2|}{(1+\delta)} + |\beta_3| \right) |p_1|^3 \right\}$$

and applying Lemmas 3 (when $u = 1/4$, and $v = w = 1$), 2 (when $\lambda = 1$) and 1 simplifies to the result in the theorem. □

Theorem 5. *Let $f \in \Lambda(\delta, b)$. Then*

$$|\mathcal{H}_{3,1}^{\tau}(f)| \leqq \frac{\beta_1 + \delta(2-\delta)\beta_1^2 + |\beta_2|}{(1+\delta)} \left\{ \frac{\beta_1^2}{(1+2\delta)} + \frac{2\beta_1}{(1+2\delta)} \left(\frac{3\delta\beta_1^2}{2(1+\delta)} + |\beta_2| \right) + \frac{\beta_1}{(1+2\delta)} \left[\frac{[3\delta^2(2-\delta) - \delta(1+\delta)]\beta_1^3}{(1+\delta)} + \frac{3\delta\beta_1|\beta_2|}{(1+\delta)} + |\beta_3| \right] + \tau \left(\frac{\beta_1 + \delta(2-\delta)\beta_1^2 + |\beta_2|}{(1+\delta)} \right)^2 \right\} + \left\{ \frac{\beta_1}{(1+2\delta)} + \frac{2}{(1+2\delta)} \left[\frac{3\delta\beta_1^2}{2(1+\delta)} + |\beta_2| \right] + \frac{1}{(1+2\delta)} \left[\frac{[3\delta^2(2-\delta) - \delta(1+\delta)]\beta_1^3}{(1+\delta)} + \frac{3\delta\beta_1|\beta_2|}{(1+\delta)} + |\beta_3| \right] \right\} \times \left\{ \left[\frac{\beta_1^2 + \delta(2-\delta)\beta_1^3 + \beta_1|\beta_2|}{(1+\delta)} \right] + \tau \left[\frac{\beta_1}{(1+2\delta)} + \frac{2}{(1+2\delta)} \left(\frac{3\delta\beta_1^2}{2(1+\delta)} + |\beta_2| \right) + \frac{1}{8(1+2\delta)} \left(\frac{[3\delta^2(2-\delta) - \delta(1+\delta)]\beta_1^3}{(1+\delta)} + \frac{3\delta\beta_1|\beta_2|}{(1+\delta)} + |\beta_3| \right) \right] \right\} + \left(\frac{\beta_1}{(1+\delta)} \right) \left\{ \frac{5\beta_1}{(1+3\delta)} + \frac{9|\beta_2|}{(1+3\delta)} + \frac{3|\beta_3|}{(1+3\delta)} + \frac{|\beta_4|}{(1+3\delta)} \right.$$

$$+\frac{\delta(64+7\delta-19\delta^2-4\delta^3)\beta_1^4}{48\delta(1+3\delta)}+[2\delta(5-4\delta+\delta^2)\beta_1^2]\frac{\beta_1+[\delta(2-\delta)\beta_1^2+|\beta_2|]}{(1+\delta)(1+3\delta)}$$
$$+\frac{2\delta\beta_1}{(1+3\delta)}\left[\frac{\beta_1}{(1+2\delta)}+\frac{2(\beta_1+|\beta_2|)}{(1+2\delta)}\right.$$
$$\left.+\left(\frac{\beta_1+2|\beta_2|+|\beta_3|+\delta}{(1+2\delta)}+\frac{24\delta\beta_1^2+3\delta(\beta_1|\beta_2|+2\beta_1^2+3\delta^2(2-\delta))\beta_1^3}{(1+\delta)(1+2\delta)}\right)\right]$$
$$\left.+\frac{4+\delta-\delta^2}{2(1+3\delta)}\left(\frac{\beta_1+[\delta(2-\delta)\beta_1^2+|\beta_2|]}{(1+\delta)}\right)^2\right\}.$$

Proof. The consideration of the results of Theorems 1, 2, 3 and 4 in (38) with some simplifications gives the result in the theorem. □

4. Conclusion

The features of a new collection of Yamaguchi and starlike functions, which both have significant properties in the well-known set of Bazilevič functions, were investigated in this study. The new set is defined using the Ma-Minda function and some mathematical ideas and some well-known principles in GFT. Among the achievements were estimates for the coefficient bounds, the Fekete-Szegö functional with real and complex parameters, and the Hankel determinants with a real parameter. Generally speaking, the set reduces to a handful of recognized sets when certain parameters are altered inside their declaration interval.

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