Open Journal of
Mathematical Analysis

PSR Press

Article

Localization operators and Shapiro-type inequality for the modified Whittaker-Stockwell transform

Fethi Soltani[1,2]

[1] Faculté des Sciences de Tunis, Laboratoire d'Analyse Mathématique et Applications, LR11ES11, Université de Tunis El Manar, Tunis 2092, Tunisia

[2] Ecole Nationale d'Ingénieurs de Carthage, Université de Carthage, Tunis 2035, Tunisia

* Correspondence: fethi.soltani@fst.utm.tn

Received: 23 February 2025; Accepted: 19 March 2025; Published: 29 March 2025.

Abstract: We define and study the Stockwell transform $\mathscr{S}_g$ associated with the Whittaker operator

$$\Delta_\alpha := -\frac{1}{4}\left[x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (x^{-1} + (3-4\alpha)x)\frac{\mathrm{d}}{\mathrm{d}x}\right],$$

and prove a Plancherel theorem. Moreover, we define the localization operators $L_{g,\xi}$ associated to this transform. We study the boundedness and compactness of these operators and establish a trace formula. Finally, we give a Shapiro-type uncertainty inequality for the modified Whittaker-Stockwell transform $\mathscr{S}_g$.

Keywords: modified Whittaker-Stockwell transform, localization operators, Shapiro-type uncertainty inequality

MSC: 42B10, 44A05, 44A20.

---

1. Introduction

Time-frequency analysis has emerged as an important field in signal processing as it can be used to represent time-varying signals in the time-frequency plane. Usually, the time-frequency resolution is associated with the Stockwell transform also known as the windowed Fourier transform [1,2]. A considerable attention has been devoted recently to discovering new mathematical formulations of the uncertainty principle for the the Stockwell transform, see for examples [1,2] and the references therein. The most famous of them is the sharp Heisenberg-type uncertainty inequality (see [1], Theorem 5.1). Recently this transform is the goal of many generalized works; and another fundamental tool in time-frequency analysis is the modified Whittaker-Stockwell transform which is the aim of the study of this paper. Precisely, let $\alpha < 1/2$ and we denote by $L^p(m) := L^p((0,\infty), m(x)\mathrm{d}x)$, $p \in [1,\infty]$, the Lebesgue space with respect to the weight

$$m(x) := x^{1-4\alpha}e^{-\frac{1}{2x^2}}, \tag{1}$$

equipped with the norm $\|.\|_{L^p(m)}$. We denote also by $L^p(\rho) := L^p((0,\infty), \rho(\lambda)\mathrm{d}\lambda)$, $p \in [1,\infty]$, the Lebesgue space with respect to the weight

$$\rho(\lambda) := 2^{1-2\alpha}\pi^{-2}\sinh(-2\pi i\nu_\lambda)|\Gamma(\frac{1}{2}-\alpha+\nu_\lambda)|^2\chi_\Lambda(\lambda), \tag{2}$$

equipped with the norm $\|.\|_{L^p(\rho)}$; here χ_Λ is the characteristic function of the interval $\Lambda := ((\frac{1}{2}-\alpha)^2, \infty)$ and

$$\nu_\lambda := \sqrt{(\frac{1}{2}-\alpha)^2 - \lambda}. \tag{3}$$

We define the modified Whittaker function $\mathbf{W}_{\alpha,\nu}(x)$ by

$$\mathbf{W}_{\alpha,\nu}(x) := (2x^2)^{-\frac{1}{2}+\alpha+\nu}\Psi\left(\frac{1}{2}-\alpha-\nu, 1-2\nu; \frac{1}{2x^2}\right),$$

where $\Psi(a,b;z)$ is the confluent hypergeometric function of the second kind ([3], Chapter VI), and $\alpha < 1/2$, $\nu \in \mathbb{C}$ are parameters.

The modified Whittaker function $\mathbf{W}_{\alpha,\nu}(x)$ satisfies the product formula (see [4,5])

$$\mathbf{W}_{\alpha,\nu}(x)\mathbf{W}_{\alpha,\nu}(y) = \int_{\mathbb{R}_+} \mathbf{W}_{\alpha,\nu}(\xi) q(x,y,z) m(z) \mathrm{d}z,$$

where

$$q(x,y,z) = \frac{(xyz)^{-1+2\alpha}}{(2\pi)^{\frac{1}{2}}} \exp\left(\frac{1}{2x^2} + \frac{1}{2y^2} + \frac{1}{2z^2} - \left(\frac{x^2+y^2+z^2}{4xyz}\right)^2\right) D_{2\alpha}\left(\frac{x^2+y^2+z^2}{2xyz}\right),$$

where $D_{2\alpha}$ is the parabolic cylinder function ([3], Chapter VIII).

For $f \in L^1(m)$ the modified Whittaker transform $\mathscr{F}_W$ is defined (see [5]) by

$$\mathscr{F}_W(f)(\lambda) := \int_{\mathbb{R}_+} f(x)\Phi_\lambda^\alpha(x) m(x) \mathrm{d}x, \quad \lambda \in \mathbb{R}_+,$$

where

$$\Phi_\lambda^\alpha(x) := \mathbf{W}_{\alpha,\nu_\lambda}(x), \quad x \in \mathbb{R}_+.$$

The modified Whittaker transform $\mathscr{F}_W$ is a Sturm-Liouville type integral transform [6] associated with the eigenfunction expansion of the differential operator

$$\Delta_\alpha := -\frac{1}{4}\left[x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + (x^{-1} + (3-4\alpha)x)\frac{\mathrm{d}}{\mathrm{d}x}\right].$$

Let $f, g \in L^2(\rho)$. We define the convolution product $f\sharp g$ of f and g by

$$f\sharp g(\lambda) := \mathscr{F}_W(\mathscr{F}_W^{-1}(f)\mathscr{F}_W^{-1}(g))(\lambda), \quad \lambda \in \mathbb{R}_+.$$

We introduce the Whittaker translation operators [4,5] for $f \in L^2(m)$ by

$$\tau_y f(x) := \int_{\mathbb{R}_+} f(z) q(x,y,z) m(z) \mathrm{d}z, \quad x, y > 0.$$

Let $g \in L^2(\rho)$. The modified Whittaker-Stockwell transform is the mapping $\mathscr{S}_g$ defined for $f \in L^2(\rho)$ by

$$\mathscr{S}_g(f)(\lambda, y) := f\sharp g_y(\lambda), \quad \lambda, y > 0,$$

where g_y is the modulation of g by y defined by

$$g_y := \mathscr{F}_W\left(\sqrt{\tau_y|\mathscr{F}_W^{-1}(g)|^2}\right).$$

We give many harmonic analysis results related to the modified Whittaker-Stockwell transform $\mathscr{S}_g$. In particular, we establish the following Plancherel theorem. Let $g \in L^2(\rho)$ be a non-zero function. Then, for all $f \in L^2(\rho)$, we have

$$\|\mathscr{S}_g(f)\|_{L^2(\rho\otimes m)} = \|g\|_{L^2(\rho)}\|f\|_{L^2(\rho)}.$$

Next, we we give the following integral representation for the transform $\mathscr{S}_g$. Let $f, g \in L^2(\nu)$. Then

$$\mathscr{S}_g(f)(\lambda, y) = \int_{\mathbb{R}_+} f(t)\sigma_\lambda g_y(t)\rho(t) \mathrm{d}t, \quad \lambda, y \in \mathbb{R}_+,$$

where σ_λ is the operator defined by

$$\mathscr{F}_W^{-1}(\sigma_\lambda f)(x) = \Phi_\lambda^\alpha(x)\mathscr{F}_W^{-1}(f)(x).$$

Let $\xi \in L^2(\rho \otimes \mu)$ and $g \in L^2(\rho)$, we define the localization operators associated with the Sturm-Liouville-Stockwell transform $\mathscr{S}_g$, for $f \in L^2(\rho)$ and $\lambda \in \mathbb{R}_+$ by

$$L_{g,\xi}(f)(\lambda) := \int_{\mathbb{R}_+^2} \xi(t,y)\mathscr{S}_g(f)(t,y)\sigma_\lambda g_y(t)\rho(t)m(y)\mathrm{d}t\mathrm{d}y.$$

The localization operators were introduced firstly by Daubechies et al. [7,8], Ramanathan and Topiwala [9], and extensively investigated by Wong in [10]. This class of operators occurs in various branches of pure and applied mathematics and has been studied by many authors. In this paper, we study the boundedness and the compactness of the localization operators $L_{g,\xi}$.

Finally, we give a Shapiro-type uncertainty inequality for the Sturm-Liouville-Stockwell transform that is, for $\{\phi_n\}_{n=1}^N$ be an orthonormal sequence in $L^2(\rho)$ and U be measurable subset of $\mathbb{R}_+^2$ such that $0 < \rho \otimes m(U) < \infty$, we have

$$\sum_{n=1}^{N} \left(1 - \|\chi_{U^c}\mathscr{S}_g(\phi_n)\|_{L^2(\rho\otimes m)}\right) \leq \rho \otimes m(U),$$

where χ_{U^c} is the characteristic function of the set U^c.

The paper is organized as follows: In Section 2, we recall some results about the modified Whittaker-Stockwell transform $\mathscr{S}_g$. Section 3 is devoted to study the boundedness and the compactness of the localization operators $L_{g,\xi}$, and we give a trace formula. Finally, in Section 4, we establish a Shapiro-type uncertainty inequality for the transform $\mathscr{S}_g$.

2. The modified Whittaker-Stockwell transform

Let us describe some results about the Whittaker-harmonic analysis. We consider the Whittaker operator Δ_α defined by

$$\Delta_\alpha := -\frac{1}{4}\left[x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (x^{-1} + (3-4\alpha)x)\frac{\mathrm{d}}{\mathrm{d}x}\right].$$

This operator has the form of Sturm-Liouville operator

$$\Delta_\alpha = -\frac{1}{4}\left[x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{[x^2m(x)]'}{m(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right],$$

where m is the function given by (1).

We define the modified Whittaker function $\mathbf{W}_{\alpha,\nu}(x)$ as the following function of confluent hypergeometric type

$$\mathbf{W}_{\alpha,\nu}(x) := 2^\alpha x^{2\alpha} e^{\frac{1}{4x^2}} W_{\alpha,\nu}\left(\frac{1}{2x^2}\right) = (2x^2)^{-\frac{1}{2}+\alpha+\nu}\Psi\left(\frac{1}{2}-\alpha-\nu, 1-2\nu; \frac{1}{2x^2}\right),$$

where $W_{\alpha,\nu}(z)$ is the Whittaker function of the second kind, $\Psi(a,b;z)$ is the confluent hypergeometric function of the second kind ([3], Chapter VI), and $\alpha < 1/2$, $\nu \in \mathbb{C}$ are parameters. Unless stated otherwise, the parameter $\alpha < 1/2$ is held fixed throughout the discussion.

By transformation of the Whittaker differential equation [11] the function $\mathbf{W}_{\alpha,\nu}(x)$ is a solution of the differential equation

$$\Delta_\alpha w = \left(\left(\frac{1}{2}-\alpha\right)^2 - \nu^2\right) w.$$

The function $\mathbf{W}_{\alpha,\nu}(x)$ extends continuously to $x = 0$ by setting $\mathbf{W}_{\alpha,\nu}(0) = 1$. In addition, we have ([11], Equation 13.18.2)

$$\mathbf{W}_{\alpha,\frac{1}{2}-\alpha}(x) = 1, \quad x > 0.$$

The modified Whittaker function $\mathbf{W}_{\alpha,\nu}(x)$ admits the integral representation [5],

$$\mathbf{W}_{\alpha,\nu}(x) = \int_{\mathbb{R}_+} \cosh(\nu s)\eta_x(s)\mathrm{d}s, \quad \alpha,\nu \in \mathbb{C},\ x > 0,$$

where

$$\eta_x(s) := (2\pi)^{-\frac{1}{2}}x^{-1+2\alpha}\exp\left(\frac{1}{2x^2} - \frac{1}{4x^2}\cosh^2\left(\frac{s}{2}\right)\right)D_{2\alpha}\left(\frac{1}{x}\cosh\left(\frac{s}{2}\right)\right),$$

being $D_\mu(z)$ the parabolic cylinder function ([3], Chapter VIII), given by

$$D_\mu(z) := \frac{z^\mu e^{-\frac{z^2}{4}}}{\Gamma\left(\frac{1}{2}(1-\mu)\right)} \int_0^\infty e^{-s} s^{-\frac{1}{2}(1+\mu)} \left(1+\frac{2s}{z^2}\right)^{\frac{\mu}{2}} \mathrm{d}s, \quad \operatorname{Re} z > 0, \operatorname{Re}\mu < 1. \tag{4}$$

Moreover, the following inequality holds

$$|\mathbf{W}_{\alpha,\nu}(x)| \le 1, \quad x \in \mathbb{R}_+, \quad |\operatorname{Re}\nu| \le \frac{1}{2} - \alpha.$$

For $\alpha, \nu \in \mathbb{C}$ and $x, y > 0$, the product $\mathbf{W}_{\alpha,\nu}(x)\mathbf{W}_{\alpha,\nu}(y)$ admits the following integral representation [4,5],

$$\mathbf{W}_{\alpha,\nu}(x)\mathbf{W}_{\alpha,\nu}(y) = \int_{\mathbb{R}_+} \mathbf{W}_{\alpha,\nu}(\xi) q(x,y,z) m(z) \mathrm{d}z, \tag{5}$$

where

$$q(x,y,z) = \frac{(xyz)^{-1+2\alpha}}{(2\pi)^{\frac{1}{2}}} \exp\left(\frac{1}{2x^2} + \frac{1}{2y^2} + \frac{1}{2z^2} - \left(\frac{x^2+y^2+z^2}{4xyz}\right)^2\right) D_{2\alpha}\left(\frac{x^2+y^2+z^2}{2xyz}\right),$$

$D_{2\alpha}$ is the parabolic cylinder function given by (4).

In particular, for $x, y, z > 0$, we have

$$q(x,y,z) = q(y,x,z) = q(x,z,y) = q(z,y,x), \tag{6}$$

and

$$\int_{\mathbb{R}_+} q(x,y,z) m(z) \mathrm{d}z = 1. \tag{7}$$

In addition, if $\alpha < 1/2$, we have the positivity condition

$$q(x,y,z) > 0, \quad x, y, z > 0.$$

We now define the generalized translation operator induced by (5), for $\alpha < 1/2$.

We denote by $C_b(0,\infty)$, the space of bounded continuous functions f on $(0,\infty)$. For $f \in C_b(0,\infty)$, the linear operator

$$\tau_y f(x) := \int_{\mathbb{R}_+} f(z) q(x,y,z) m(z) \mathrm{d}z, \quad x, y > 0,$$

is called the Whittaker translation.

As a first remark, we note that the relations (6) and (7) mean that

$$\tau_y f(x) = \tau_x f(y), \quad f \in C_b(0,\infty), \quad x, y > 0,$$

$$\int_{\mathbb{R}_+} \tau_y f(x) m(x) \mathrm{d}x = \int_{\mathbb{R}_+} f(x) m(x) \mathrm{d}x, \quad f \in L^1(m). \tag{8}$$

Theorem 1. *For all $y > 0$ and $f \in L^p(m)$, $p \in [1,\infty]$,*

$$\|\tau_y f\|_{L^p(m)} \le \|f\|_{L^p(m)}.$$

Proof. If $p = 1, \infty$, the result follows from (6) and (7). Assume therefore that $p \in (1,\infty)$ and let p' be the conjugate exponent of p, i.e. $1/p + 1/p' = 1$. We write

$$|f(z)| q(x,y,z) = |f(z)| [q(x,y,z)]^{1/p} [q(x,y,z)]^{1/p'}.$$

Applying Hölder's inequality and (7), we obtain

$$|\tau_y f(x)|^p \le \int_{\mathbb{R}_+} |f(z)|^p q(x,y,z) m(z) \mathrm{d}z, \quad x, y > 0.$$

This gives the result. □

We set

$$\Phi_\lambda^\alpha(x) := \mathbf{W}_{\alpha,\nu_\lambda}(x), \quad x \in \mathbb{R}_+,$$

where ν_λ is the parameter given by (3). The function Φ_λ^α possesses the following property

$$-1 \leq \Phi_\lambda^\alpha(x) \leq 1, \quad x \in \mathbb{R}_+.$$

This kernel gives rise to an integral transform, which is called the modified Whittaker transform, and was introduced by Sousa et al. in [4,5], where already many basic properties were established.

The modified form $\mathscr{F}_W$ of the Whittaker transform is defined for $f \in L^1(m)$ by

$$\mathscr{F}_W(f)(\lambda) := \int_{\mathbb{R}_+} f(x)\Phi_\lambda^\alpha(x)m(x)\mathrm{d}x, \quad \lambda \in \mathbb{R}_+, \tag{9}$$

where $m(x)$ is the weight defined by (1).

The modified Whittaker transform $\mathscr{F}_W$ satisfies [12] the inequality

$$|\mathscr{F}_W(f)(\lambda)| \leq \|f\|_{L^1(m)}, \quad f \in L^1(m),\ \lambda \in \mathbb{R}_+.$$

The basic L^2-property of the modified Whittaker transform is given in the next theorem [5].

Theorem 2. *For $\alpha < 1/2$, the modified Whittaker transform (9) defines an isometric isomorphism $\mathscr{F}_W : L^2(m) \longrightarrow L^2(\rho)$, whose inverse is given by*

$$\mathscr{F}_W^{-1}(\phi)(x) = \int_{\mathbb{R}_+} \phi(\lambda)\Phi_\lambda^\alpha(x)\rho(\lambda)d\lambda, \tag{10}$$

where $\rho(\lambda)$ is the weight defined by (2). The convergence of the integral (10) is understood with respect to the $L^2(m)$-norm.

The following inversion theorem for the modified Whittaker transform $\mathscr{F}_W$ is proved in [5].

Theorem 3. *Let $f \in L^1(m)$ be such $\mathscr{F}_W(f) \in L^1(\rho)$. Then for $\alpha < 1/2$,*

$$f(x) = \int_{\mathbb{R}_+} \mathscr{F}_W(f)(\lambda)\Phi_\lambda^\alpha(x)\rho(\lambda)d\lambda, \quad a.e.\ x > 0.$$

The Whittaker translation operators are connected with the modified Whittaker transform $\mathscr{F}_W$ (see [12]) via the following formula.

Theorem 4. *For $f \in L^2(m)$ and $y > 0$, we have*

$$\mathscr{F}_W(\tau_y f)(\lambda) = \Phi_\lambda^\alpha(y)\mathscr{F}_W(f)(\lambda), \quad \lambda \in \mathbb{R}_+.$$

Let $f, g \in L^2(\rho)$. We define the convolution product $f\sharp g$ of f and g (see [12]) by

$$f\sharp g(\lambda) := \mathscr{F}_W(\mathscr{F}_W^{-1}(f)\mathscr{F}_W^{-1}(g))(\lambda). \tag{11}$$

The following assertion is proved in ([12], Lemma 3.1 (iii)). Let $f, g \in L^2(\rho)$. Then

$$\int_{\mathbb{R}_+} |f\sharp g(\lambda)|^2\rho(\lambda)\mathrm{d}\lambda = \int_{\mathbb{R}_+} |\mathscr{F}_W^{-1}(f)(x)|^2|\mathscr{F}_W^{-1}(g)(x)|^2 m(x)\mathrm{d}x, \tag{12}$$

where both sides are finite or infinite.

We assume that $g \in L^2(\rho)$ and $y > 0$. The modulation of g by y is the function

$$g_y := \mathscr{F}_W\left(\sqrt{\tau_y|\mathscr{F}_W^{-1}(g)|^2}\right).$$

From (8) and Theorem 2 we have

$$\|g_y\|_{L^2(\rho)} = \|g\|_{L^2(\rho)}. \tag{13}$$

Let $g \in L^2(\rho)$. The modified Whittaker-Stockwell transform (see [12]) is the mapping $\mathscr{S}_g$ defined for $f \in L^2(\rho)$ by

$$\mathscr{S}_g(f)(\lambda, y) := f\sharp g_y(\lambda), \quad \lambda, y > 0.$$

From (11) and (13) we have

$$\|\mathscr{S}_g(f)\|_{L^\infty(\rho\otimes m)} \leq \|g\|_{L^2(\rho)}\|f\|_{L^2(\rho)}. \tag{14}$$

Using formula (12), the following Plancherel formula for$\mathscr{S}_g$ is proved in ([12], Theorem 3.3).

Theorem 5. *(Plancherel formula). Let $g \in L^2(\rho)$ be a non-zero function. Then, for all $f \in L^2(\rho)$,*

$$\|\mathscr{S}_g(f)\|_{L^2(\rho\otimes m)} = \|g\|_{L^2(\rho)}\|f\|_{L^2(\rho)}.$$

Let $f \in L^2(\nu)$ and $\lambda \in \mathbb{R}_+$. We define the operator σ_λ by

$$\mathscr{F}_W^{-1}(\sigma_\lambda f)(x) = \Phi_\lambda^\alpha(x)\mathscr{F}_W^{-1}(f)(x). \tag{15}$$

The operator σ_λ satisfies

$$\sigma_\lambda f(y) = \sigma_y f(\lambda), \quad \|\sigma_\lambda f\|_{L^2(\rho)} \leq \|f\|_{L^2(\rho)}. \tag{16}$$

Theorem 6. *Let $f, g \in L^2(\rho)$. Then*

$$\mathscr{S}_g(f)(\lambda, y) = \int_{\mathbb{R}_+} f(t)\sigma_\lambda g_y(t)\rho(t)dt, \quad \lambda, y \in \mathbb{R}_+.$$

Proof. Let $f, g \in L^2(\rho)$. From (11) and (15) we have

$$\begin{aligned}\mathscr{S}_g(f)(\lambda, y) &= \int_{\mathbb{R}_+} \mathscr{F}_W^{-1}(f)(x)\Phi_\lambda^\alpha(x)\mathscr{F}_W^{-1}(g_y)(x)m(x)\mathrm{d}x \\ &= \int_{\mathbb{R}_+} \mathscr{F}_W^{-1}(f)(x)\mathscr{F}^{-1}(\sigma_\lambda g_y)(x)m(x)\mathrm{d}x.\end{aligned}$$

Then by Theorem 2 we obtain

$$\mathscr{S}_g(f)(\lambda, y) = \int_{\mathbb{R}_+} f(t)\sigma_\lambda g_y(t)\rho(t)\mathrm{d}t.$$

The theorem is proved. □

3. Localization operators

In this section, we define the localization operators for the modified Whittaker-Stockwell transform and we prove that they are bounded and in the so-called Schatten-von Neumann classes, the proofs are inspired from related results in [13].

We denote by $B(L^2(\rho))$ the space of all bounded operators Ψ from $L^2(\rho)$ into itself, equipped with the norm

$$\|\Psi\| := \sup_{\|f\|_{L^2(\rho)}=1} \|\Psi(f)\|_{L^2(\rho)}.$$

For a compact operator $\Psi \in B(L^2(\rho))$, the eigenvalues of the positive self-adjoint operator $|\Psi| := \sqrt{\Psi^*\Psi}$ are called the singular values of Ψ and denoted by $\{s_n(\Psi)\}_{n\in\mathbb{N}}$.

The Schatten-von Neumann class S_p, $p \in [1, \infty)$ is the space of all compact operators Ψ whose singular values $s_n(\Psi)$ lie in $l^p(\mathbb{N})$. The class S_p is provided with the norm

$$\|\Psi\|_{S_p} := \left[\sum_{n=1}^{\infty} (s_n(\Psi))^p\right]^{\frac{1}{p}}.$$

The Schatten-von Neumann class S_∞ is the class of all compact operators with the norm $\|\Psi\|_{S_\infty} := \|\Psi\|$.

We note that the space S_1 is the space of trace class operators. We define the trace of an operator Ψ in S_1 by

$$\mathrm{Tr}(\Psi) := \sum_{n=1}^{\infty} \langle \Psi(v_n), v_n \rangle_{L^2(\rho)}, \tag{17}$$

where $\{v_n\}_{n\in\mathbb{N}}$ is any orthonormal basis of $L^2(\rho)$. Moreover, if Ψ is positive, then

$$\mathrm{Tr}(\Psi) = \|\Psi\|_{S_1}.$$

We note that the space S_2 is the space of Hilbert-Schmidt operators. A compact operator Ψ on the Hilbert space $L^2(\rho)$ is called the Hilbert-Schmidt operator, if the positive operator $\Psi^*\Psi$ is in the trace class S_1. Then for any orthonormal basis $\{v_n\}_{n\in\mathbb{N}}$ of $L^2(\rho)$, we have

$$\|\Psi\|_{HS}^2 = \|\Psi\|_{S_2}^2 = \|\Psi^*\Psi\|_{S_1} = \mathrm{Tr}(\Psi^*\Psi) = \sum_{n=1}^{\infty} \|\Psi(v_n)\|_{L^2(\rho)}^2.$$

Let $\xi \in L^1 \cup L^\infty(\rho \otimes m)$ and $g \in L^2(\rho)$. We define the localization operators associated with the modified Whittaker-Stockwell transform $\mathscr{S}_g$, for $f \in L^2(\rho)$ and $\lambda \in \mathbb{R}_+$ by

$$L_{g,\xi}(f)(\lambda) := \int_{\mathbb{R}_+^2} \xi(t,y)\mathscr{S}_g(f)(t,y)\sigma_\lambda g_y(t)\rho(t)m(y)\mathrm{d}t\mathrm{d}y.$$

For all $f, h \in L^2(\rho)$, we have

$$\langle L_{g,\xi}(f), h\rangle_{L^2(\rho)} := \int_{\mathbb{R}_+^2} \xi(t,y)\mathscr{S}_g(f)(t,y)\overline{\mathscr{S}_g(h)(t,y)}\rho(t)m(y)\mathrm{d}t\mathrm{d}y. \tag{18}$$

Therefore the adjoint of $L_{g,\xi}$ is the operator $L_{g,\xi}^*$ given by

$$L_{g,\xi}^* = L_{g,\overline{\xi}} : L^2(\rho) \to L^2(\rho).$$

Lemma 1. *Let $g \in L^2(\rho)$ and let $\xi \in L^1(\rho \otimes m)$, then the localization operator $L_{g,\xi}$ is bounded from $L^2(\rho)$ into itself, and*

$$\|L_{g,\xi}\|_{S_\infty} \le \|g\|_{L^2(\rho)}^2 \|\xi\|_{L^1(\rho\otimes m)}.$$

Proof. Let $f, h \in L^2(\rho)$. From relations (14) and (18) we get

$$\begin{aligned}|\langle L_{g,\xi}(f), h\rangle_{L^2(m)}| &\le \int_{\mathbb{R}_+^2} |\xi(t,y)||\mathscr{S}_g(f)(t,y)||\mathscr{S}_g(h)(t,y)|\rho(t)m(y)\mathrm{d}t\mathrm{d}y \\ &\le \|\mathscr{S}_g(f)\|_{L^\infty(\rho\otimes m)}\|\mathscr{S}_g(h)\|_{L^\infty(\rho\otimes m)}\|\xi\|_{L^1(\rho\otimes m)} \\ &\le \|g\|_{L^2(\rho)}^2\|f\|_{L^2(\rho)}\|h\|_{L^2(\rho)}\|\xi\|_{L^1(\rho\otimes m)}.\end{aligned}$$

Therefore

$$\|L_{g,\xi}\|_{S_\infty} \le \|g\|_{L^2(\rho)}^2 \|\xi\|_{L^1(m\otimes\rho)}.$$

We obtain the desired inequality. □

Lemma 2. *Let $g \in L^2(\rho)$ and let $\xi \in L^\infty(\rho \otimes m)$, then the localization operator $L_{g,\xi}$ is bounded from $L^2(\rho)$ into itself, and*

$$\|L_{g,\xi}\|_{S_\infty} \le \|g\|_{L^2(\rho)}^2 \|\xi\|_{L^\infty(\rho\otimes m)}.$$

Proof. Let $f, h \in L^2(\rho)$ and let $g \in L^2(\rho)$. From (18) and Hölder's inequality, we get

$$|\langle L_{g,\xi}(f), h\rangle_{L^2(\rho)}| \le \|\xi\|_{L^\infty(\rho\otimes m)}\|\mathscr{S}_g(f)\|_{L^2(\rho\otimes m)}\|\mathscr{S}_g(h)\|_{L^2(\rho\otimes m)}.$$

Using Theorem 5, we obtain

$$|\langle L_{g,\xi}(f),h\rangle_{L^2(\rho)}| \leq \|\xi\|_{L^\infty(\rho\otimes m)}\|g\|^2_{L^2(\rho)}\|f\|_{L^2(\rho)}\|h\|_{L^2(\rho)}.$$

Thus

$$\|L_{g,\xi}\|_{S_\infty} \leq \|g\|^2_{L^2(\rho)}\|\xi\|_{L^\infty(\rho\otimes m)}.$$

This proves the desired result. □

Theorem 7. *Let $g \in L^2(\rho)$ and let $\xi \in L^p(\rho\otimes m)$, $1 \leq p \leq \infty$. Then the localization operator $L_{g,\xi}$ is bounded from $L^2(\rho)$ into itself, and*

$$\|L_{g,\xi}\|_{S_\infty} \leq \|g\|^2_{L^2(\rho)}\|\xi\|_{L^p(\rho\otimes m)}.$$

Proof. Let $g \in L^2(\rho)$ and let $\xi \in L^1 \cap L^\infty(\rho\otimes m)$. By Lemma 1, Lemma 2 and the Riesz-Thorin's theorem [14], for every $p \in [1,\infty]$, we have

$$\|L_{g,\xi}\|_{S_\infty} \leq \|g\|^2_{L^2(\rho)}\|\xi\|_{L^p(\rho\otimes m)}.$$

Let $\xi \in L^p(\rho\otimes m)$ and $\{\xi_n\}_{n\geq 1}$ be a sequence of functions in $L^1 \cap L^\infty(\rho\otimes m)$ such that $\xi_n \to \xi$ in $L^p(\rho\otimes m)$ as $n \to \infty$. Hence for every $n,k \in \mathbb{N}$, we have

$$\|L_{g,\xi_n} - L_{g,\xi_k}\|_{S_\infty} \leq \|g\|^2_{L^2(\rho)}\|\xi_n - \xi_k\|_{L^p(\rho\otimes m)}.$$

Therefore $\{L_{g,\xi_n}\}_{n\geq 1}$ is a Cauchy sequence in S_∞. Hence, $L_{g,\xi_n} \to L_{g,\xi}$ in S_∞ as $n \to \infty$. Then the limit $L_{g,\xi}$ is independent of the choice of $\{\xi_n\}_{n\geq 1}$ and we obtain

$$\|L_{g,\xi}\|_{S_\infty} = \lim_{n\to\infty}\|L_{g,\xi_n}\|_{S_\infty} \leq \lim_{n\to\infty}\|\xi_n\|_{L^p(\rho\otimes m)}\|g\|^2_{L^2(\rho)} = \|\xi\|_{L^p(\rho\otimes m)}\|g\|^2_{L^2(m)}.$$

Which gives the desired result. □

Theorem 8. *Let $g \in L^2(\rho)$ and let $\xi \in L^1(\rho\otimes m)$. Then the localization operator $L_{g,\xi} : L^2(\rho) \to L^2(\rho)$ is in S_1 with*

$$Tr(L_{g,\xi}) = \int_{\mathbb{R}^2_+}\xi(t,y)\|\sigma_t g_y\|^2_{L^2(\rho)}\rho(t)m(y)dtdy \leq \|g\|^2_{L^2(\rho)}\|\xi\|_{L^1(\rho\otimes m)}.$$

Proof. Let $\xi \in L^1(\rho\otimes m)$, and let $\{v_n\}_{n\geq 1}$ be an orthonormal basis of $L^2(\rho)$. Using (18), Fubini's theorem and Theorem 9, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\langle L_{g,\xi}(v_n),v_n\rangle_{L^2(\rho)} &= \sum_{n=1}^{\infty}\int_{\mathbb{R}^2_+}\xi(t,y)|\mathscr{S}_g(v_n)(t,y)|^2\rho(t)m(y)\mathrm{d}t\mathrm{d}y\\ &= \int_{\mathbb{R}^2_+}\xi(t,y)\sum_{n=1}^{\infty}|\langle v_n,\sigma_t g_y\rangle_{L^2(\rho)}|^2\rho(t)m(y)\mathrm{d}t\mathrm{d}y\\ &= \int_{\mathbb{R}^2_+}\xi(t,y)\|\sigma_t g_y\|^2_{L^2(\rho)}\rho(t)m(y)\mathrm{d}t\mathrm{d}y.\end{aligned}$$

Thus from (13) and (16) we get

$$\sum_{n=1}^{\infty}\langle L_{g,\xi}(v_n),v_n\rangle_{L^2(m)} \leq \|g\|^2_{L^2(\rho)}\|\xi\|_{L^1(\rho\otimes m)}.$$

Then, the operator $L_{g,\xi}$ is in S_1 and by relation (3.1) we have

$$\mathrm{Tr}(L_{g,\xi}) = \int_{\mathbb{R}^2_+}\xi(t,y)\|\sigma_t g_y\|^2_{L^2(\rho)}\rho(t)m(y)\mathrm{d}t\mathrm{d}y \leq \|g\|^2_{L^2(\rho)}\|\xi\|_{L^1(\rho\otimes m)}.$$

The theorem is proved. □

Theorem 9. *Let $g \in L^2(\rho)$ and let $\xi \in L^p(\rho\otimes m)$, $p \in [1,\infty)$, then the localization operator $L_{g,\xi} : L^2(\rho) \to L^2(\rho)$ is compact.*

Proof. Let $\xi \in L^p(\rho \otimes m)$, $p \in [1, \infty)$. We consider a sequence of functions $\{\xi_n\}_{n\geq 1}$ in $L^1 \cap L^p(\rho \otimes m)$ such that $\xi_n \to \xi$ in $L^p(\rho \otimes m)$ as $n \to \infty$. Then, using Theorem 7, we get

$$\|L_{g,\xi_n} - L_{g,\xi}\|_{S_\infty} \leq \|g\|^2_{L^2(\rho)} \|\xi_n - \xi\|_{L^p(\rho\otimes m)} \to 0, \quad \text{as} \quad n \to \infty.$$

Hence, $L_{g,\xi_n} \to L_{g,\xi}$ in S_∞ as $n \to \infty$. On the other hand, from Theorem 8, we obtain that $\{L_{g,\xi_n}\}_{n\geq 1}$ is a sequence of linear operators in S_1 and hence compact, so $L_{g,\xi}$ is compact. □

4. Shapiro-type inequality

In this section we shall prove the above mentioned of Shapiro-type inequality for the modified Whittaker-Stockwell transform $\mathscr{S}_g$. Here we need the theory of reproducing kernels [15].

Lemma 3. *Let $g \in L^2(\rho)$ be a non-zero function. Then $\mathscr{S}_g(L^2(\rho))$ is a reproducing kernel Hilbert space in $L^2(\rho \otimes m)$ with kernel function*

$$W_g((\lambda, y); (t, x)) := \frac{1}{\|g\|^2_{L^2(\rho)}} \overline{\mathscr{S}_g(\sigma_\lambda g_y)(t, x)}.$$

Moreover, the kernel W_g satisfies for $\lambda, y), (t, x) \in \mathbb{R}^2_+$,

$$|W_g((\lambda, y); (t, x))| \leq 1.$$

Proof. We have

$$\mathscr{S}_g(f)(\lambda, y) = \int_{\mathbb{R}_+} f(t)\sigma_\lambda g_y(t)\rho(t)\mathrm{d}t = \langle f, \overline{\sigma_\lambda g_y}\rangle_{L^2(\rho)}.$$

Then from Theorem 5, we get

$$\mathscr{S}_g(f)(\lambda, y) = \frac{1}{\|g\|^2_{L^2(\rho)}} \langle \mathscr{S}_g(f), \overline{\mathscr{S}_g(\sigma_\lambda g_y)}\rangle_{L^2(\rho\otimes m)}.$$

Moreover, from (13) and (16) for every $(\lambda, y) \in \mathbb{R}^2_+$, the function $\sigma_\lambda g_y$ belongs to $L^2(\rho)$ and therefore the function $W_g((\lambda, y); (.,.)) = \frac{1}{\|g\|^2_{L^2(\rho)}} \overline{\mathscr{S}_g(\sigma_\lambda g_y)(.,.)}$ belongs to $\mathscr{S}_g(L^2(\rho))$. We conclude that

$$W_g((\lambda, y); (t, x)) = \frac{1}{\|g\|^2_{L^2(\rho)}} \overline{\mathscr{S}_g(\sigma_\lambda g_y)(t, x)},$$

is a reproducing kernel of the Hilbert space $\mathscr{S}_g(L^2(\rho))$.

Finally, from (13), (14) and (16), for $(\lambda, y), (t, x) \in \mathbb{R}^2_+$, we get

$$|W_g((\lambda, y); (t, x))| \leq 1.$$

The lemma is proved. □

In order to prove the Shapiro-type inequality for $\mathscr{S}_g$, we introduce a pair of orthogonal projection on $L^2(\rho \otimes m)$. We need the following notations.

• Let $g \in L^2(\rho)$ be a nonzero function such that $\|g\|_{L^2(\rho)} = 1$. We define the orthogonal projection $\mathcal{P}_g : L^2(\rho \otimes m) \longrightarrow L^2(\rho \otimes m)$, by

$$\mathcal{P}_g F(\lambda, y) := \int_{\mathbb{R}^2_+} F(t, x) W_g((\lambda, y); (t, x))\rho(t)m(x)\mathrm{d}t\mathrm{d}x,$$

where W_g is the kernel given by Lemma 3.

• Let $U \subset \mathbb{R}^2_+$ with $\rho \otimes m(U) < \infty$. We define the orthogonal projection $\mathcal{P}_U : L^2(\rho \otimes m) \longrightarrow L^2(\rho \otimes m)$, by

$$\mathcal{P}_U F(\lambda, y) := \chi_U(\lambda, y)F(\lambda, y),$$

where χ_U is the characteristic function of the set U.

Lemma 4. *Let $g \in L^2(\rho)$ and U be a measurable subset of $\mathbb{R}_+^2$ such that $0 < \rho \otimes m(U) < \infty$, then*

$$\|\mathcal{P}_U \mathcal{P}_g\|_{HS}^2 \leq \rho \otimes m(U).$$

Proof. For $F \in L^2(\rho \otimes m)$ arbitrary, we have

$$\mathcal{P}_U \mathcal{P}_g F(\lambda, y) = \int_{\mathbb{R}_+^2} \chi_U(\lambda, y) F(t, x) W_g((\lambda, y); (t, x)) \rho(t) m(x) \mathrm{d}t \mathrm{d}x.$$

On the other hand, the Hilbert-Schmidt norm

$$\|\mathcal{P}_U \mathcal{P}_g\|_{HS}^2 = \frac{1}{\|g\|_{L^2(\rho)}^4} \int_{\mathbb{R}_+^4} \chi_U(\lambda, y) |\mathscr{S}_g(\sigma_\lambda g_y)(t, x)|^2 \rho(\lambda) m(y) \rho(t) m(x) \mathrm{d}\lambda \mathrm{d}y \mathrm{d}t \mathrm{d}x.$$

Thus, by Fubini's theorem and Theorem 5, we obtain

$$\|\mathcal{P}_U \mathcal{P}_g\|_{HS}^2 \leq \rho \otimes m(U).$$

This completes the proof of the lemma. □

The following theorem is the similar to ([16], Theorem 2).

Theorem 10. *(Shapiro-type theorem). Let $\{\phi_n\}_{n=1}^N$ be an orthonormal system of $L^2(\rho)$ and U be measurable subset of $\mathbb{R}_+^2$ such that $0 < \rho \otimes m(U) < \infty$. Then*

$$\sum_{n=1}^{N} \left(1 - \|\chi_{U^c} \mathscr{S}_g(\phi_n)\|_{L^2(\rho \otimes m)}\right) \leq \rho \otimes m(U).$$

Proof. We define the trace of an operator Ψ in $L^2(\rho \otimes m)$ by

$$\mathrm{Tr}(\Psi) := \sum_{n=1}^{\infty} \langle \Psi(v_n), v_n \rangle_{L^2(\rho \otimes m)},$$

where $\{v_n\}_{n \in \mathbb{N}}$ is any orthonormal basis of $L^2(\rho \otimes m)$. From Lemma 4, $\mathcal{P}_U \mathcal{P}_g$ is a Hilbert-Schmidt operator. Then from ([17], Theorem 5.6) we have

$$\mathrm{Tr}(\mathcal{P}_U \mathcal{P}_g) = \|\mathcal{P}_U \mathcal{P}_g\|_{HS}^2.$$

Thus, for all $N \geq 1$, we get

$$\begin{aligned} \sum_{n=1}^{N} \langle \mathcal{P}_U \mathscr{S}_g(\phi_n), \mathscr{S}_g(\phi_n) \rangle_{L^2(\rho \otimes m)} &= \sum_{n=1}^{N} \langle \mathcal{P}_U \mathcal{P}_g \mathscr{S}_g(\phi_n), \mathscr{S}_g(\phi_n) \rangle_{L^2(\rho \otimes m)} \\ &\leq \mathrm{Tr}(\mathcal{P}_U \mathcal{P}_g) \leq \rho \otimes m(U). \end{aligned}$$

Then by Cauchy-Schwartz inequality we deduce that

$$\begin{aligned} \langle \mathcal{P}_U \mathscr{S}_g(\phi_n), \mathscr{S}_g(\phi_n) \rangle_{L^2(\rho \otimes m)} &= 1 - \langle \mathcal{P}_{U^c} \mathscr{S}_g(\phi_n), \mathscr{S}_g(\phi_n) \rangle_{L^2(\rho \otimes m)} \\ &\geq 1 - \|\chi_{U^c} \mathscr{S}_g(\phi_n)\|_{L^2(\rho \otimes m)}. \end{aligned}$$

The theorem is proved. □

References

[1] Bonami, A., Demange, B., & Jaming, P. (2003). Hermite functions and uncertainty principles for the Fourier and the windowed Fourier transforms. *Revista Matemática Iberoamericana, 19*(1), 23-55.

[2] Demange, B. (2005). Uncertainty principles for the ambiguity function. *Journal of the London Mathematical Society, 72*(3), 717-730.

[3] Bateman, H., & Erdélyi, A. (1953). *Higher transcendental Functions*, volume II. Bateman Manuscript Project) Mc Graw-Hill Book Company, 410.

[4] Sousa, R., Guerra, M., & Yakubovich, S. (2019). On the product formula and convolution associated with the index Whittaker transform. *Journal of Mathematical Analysis and Applications, 475*(1), 939-965.

[5] Sousa, R., Guerra, M., & Yakubovich, S. (2021). Lévy processes with respect to the Whittaker convolution. *Transactions of the American Mathematical Society, 374*(4), 2383-2419.

[6] Sousa, R., & Yakubovich, S. (2018). The spectral expansion approach to index transforms and connections with the theory of diffusion processes. *Communications on Pure and Applied Analysis, 17*(6), 2351-2378.

[7] Daubechies, I. (1988). Time-frequency localization operators: a geometric phase space approach. *IEEE Transactions on Information Theory, 34*(4), 605-612.

[8] Daubechies, I., & Paul, T. (1988). Time-frequency localisation operators-a geometric phase space approach: II. The use of dilations. *Inverse problems, 4*(3), 661-680.

[9] Ramanathan, J., & Topiwala, P. (1993). Time-frequency localization via the Weyl correspondence. *SIAM Journal on Mathematical Analysis, 24*(5), 1378-1393.

[10] Wong, M. W. (1999). *Localization Operators*, Seoul National University, Research Institute of Mathematics.

[11] Olver, F. W. (Ed.). (2010). *NIST Handbook of Mathematical Functions Hardback and CD-ROM*. Cambridge university press.

[12] Soltani, F. (2024). Reconstruction and best approximate inversion formulas for the modified Whittaker–Stockwell transform. *The Ramanujan Journal, 65*(1), 313-331.

[13] Wong, M. W. (2012). *Wavelet Transforms and Localization Operators* (Vol. 136). Birkhäuser.

[14] Stein, E. M. (1956). Interpolation of linear operators. *Transactions of the American Mathematical Society, 83*(2), 482-492.

[15] Saitoh, S. (2005). Best approximation, Tikhonov regularization and reproducing kernels. *Kodai Mathematical Journal, 28*(2), 359-367.

[16] Malinnikova, E. (2010). Orthonormal sequences in $L^2(R^d)$ and time frequency localization. Journal of Fourier Analysis and Applications, 16(6), 983-1006.

[17] Gohberg, I., Goldberg, S., & Krupnik, N. (2012). *Traces and Determinants of Linear Operators* (Vol. 116). Birkhäuser.