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Localization operators for Wigner transform on the Chébli-Trimèche hypergroups of exponential growths

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Abstract: The main focus of this paper is to define the Wigner transform on Chébli-Trimèche hypergroups of exponential growth and to present several related results. Next, we introduce a new class of pseudo-differential operators $\mathscr{L}_{\psi_1,\psi_2}(\sigma)$, called localization operators, which depend on a symbol σ and two admissible functions ψ_1 and ψ_2. We provide criteria, in terms of the symbol σ, for their boundedness and compactness. We also show that these operators belong to the Schatten–von Neumann class S^p for all $p \in [1, +\infty]$, and we derive a trace formula.

Keywords: Fourier-Wigner transform, localization operators, Chébli-Trimèche Hypergroups, Schatten-von Neumann classes

MSC: 47G30, 35J05, 47B10, 42B10.

1. Introduction

The Wigner transform has a long history, originating in 1932 with Eugene Wigner's introduction of a probability quasi-distribution that allows quantum mechanical expectation values to be expressed similarly to classical statistical averages. It is also used in signal processing as a tool in time-frequency analysis; for more details, see [1,2].

Significant attention has been devoted to various generalizations of the classical Fourier transform. This paper focuses on the generalized Fourier transform on Chébli-Trimèche hypergroups of exponential growth. More precisely, we consider the following second-order differential operator:

$$\Delta_A = \frac{\partial^2}{\partial x^2} + \frac{A'(x)}{A(x)}\frac{\partial}{\partial x} + \rho^2, \quad \rho > 0, \tag{1}$$

where A is a nonnegative function satisfying certain regularity conditions.

The operator (1) plays an important role in analysis. It generalizes several classical operators, such as the Bessel operator [3], the Jacobi operator [4], and it corresponds to the radial part of the Laplace–Beltrami operator in symmetric spaces (see [5] for more information).

The eigenfunctions of Δ_A satisfy a product formula that enables the development of a harmonic analysis on the Chébli-Trimèche hypergroups, denoted by $(\mathbb{R}_+, *_A)$. This hypergroup is commutative, with neutral element 0 and the identity map as the involution. The Haar measure μ_A on $(\mathbb{R}_+, *_A)$ is given by

$$d\mu_A(x) := A(x)\,dx. \tag{2}$$

For more information on Chébli-Trimèche hypergroups, see [6,7].

One important application of the Fourier transform is in the theory of localization operators, also known as Gabor multipliers, Toeplitz operators, or Anti-Wick operators. This theory was initiated by Daubechies in [8], and further developed in detail by Wong in [9]. Wong was the first to define localization operators on the Weyl–Heisenberg group [10], and later, Boggiatto and Wong extended these results to $L^p(\mathbb{R}^d)$ spaces in [11].

The theory of localization operators associated with the Fourier–Wigner transform on hypergroups has seen significant development, including in the spherical mean hypergroups [12], the Heckman–Opdam hypergroups [13], and the Laguerre hypergroups [14]. In a previous work, we also extended this theory to the Laguerre–Bessel hypergroups [15]. However, to the best of our knowledge, localization operators have not yet been studied on the Chébli–Trimèche hypergroups, which generalize the Bessel–Kingman hypergroups [3] and the Jacobi hypergroups [4].

The main purpose of this paper is twofold: First, we define the Fourier–Wigner transform on the Chébli–Trimèche hypergroups and provide results related to this transform. Second, we introduce a class of localization operators $\mathscr{L}_{\psi_1,\psi_2}(\sigma)$ associated with this transform and establish criteria, in terms of the symbol σ, for their boundedness and compactness. We also show that these operators belong to the Schatten–von Neumann classes S^p for all $p \in [1,+\infty]$, and we derive a trace formula.

The remainder of this paper is organized as follows: In Section 2, we recall the main results concerning harmonic analysis on Chébli–Trimèche hypergroups and Schatten–von Neumann classes. In Section 3, we study the boundedness, compactness, and Schatten properties of localization operators associated with the Wigner transform on Chébli–Trimèche hypergroups.

2. Harmonic Analysis on the Chébli–Trimèche Hypergroups

In this section, we introduce some notations and recall key results related to harmonic analysis on Chébli–Trimèche hypergroups, as well as essential properties of the Schatten–von Neumann classes. For more detailed discussions, we refer the reader to [6,7,9,16].

We denote:

- $\mathcal{D}_*(\mathbb{R})$ — the space of even, differentiable functions on $\mathbb{R}$ with compact support;
- $L^p_A(\mathbb{R}_+)$ for $p \geq 1$ — the space of measurable functions f on $\mathbb{R}_+$ such that

$$\|f\|_{p,A} = \begin{cases} \left(\int_{\mathbb{R}_+} |f(x)|^p \, d\mu_A(x)\right)^{1/p} < +\infty & \text{if } 1 \leq p < +\infty, \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}_+} |f(x)| < +\infty & \text{if } p = +\infty, \end{cases}$$

where μ_A is the measure defined in (2), and A is a nonnegative function on $\mathbb{R}_+$ called the Chébli–Trimèche function, satisfying the following conditions [6]:

(i) There exists a positive, even, and infinitely differentiable function B on $\mathbb{R}$ with $B(x) \geq 1$ for all $x \in \mathbb{R}_+$ such that

$$A(x) = x^{2\alpha+1}B(x), \quad \alpha > -\tfrac{1}{2}.$$

(ii) The function A is increasing on $\mathbb{R}_+$, and $\lim_{x\to\infty} A(x) = \infty$.

(iii) The function $\frac{A'}{A}$ is decreasing on $(0,\infty)$ and

$$\lim_{x\to\infty} \frac{A'(x)}{A(x)} = 2\rho.$$

(iv) There exists a constant $\sigma > 0$ such that for all $x \in [x_0,\infty)$, with $x_0 > 0$, we have

$$\frac{A'(x)}{A(x)} = \begin{cases} 2\rho + e^{-\sigma x}F(x), & \text{if } \rho > 0, \\ \frac{2\alpha+1}{x} + e^{-\sigma x}F(x), & \text{if } \rho = 0, \end{cases} \tag{3}$$

where F is a C^∞ function on $(0,\infty)$ that is bounded along with all its derivatives.

For $p = 2$, the space $L^2_A(\mathbb{R}_+)$ becomes a Hilbert space with inner product defined, for $f, g \in L^2_A(\mathbb{R}_+)$, by

$$\langle f, g\rangle_{\mu_A} := \int_{\mathbb{R}_+} f(x)\overline{g(x)} \, d\mu_A(x).$$

Now consider the space $L^p_\sigma(\mathbb{R}_+)$, $p \geq 1$, consisting of measurable functions f on $\mathbb{R}_+$ such that

$$\|f\|_{p,\sigma} = \begin{cases} \left(\int_{\mathbb{R}_+} |f(\lambda)|^p \, d\sigma(\lambda)\right)^{1/p} < +\infty & \text{if } 1 \leq p < +\infty, \\ \operatorname*{ess\,sup}_{\lambda \in \mathbb{R}_+} |f(\lambda)| < +\infty & \text{if } p = +\infty, \end{cases}$$

where σ is a measure on $\mathbb{R}_+$ given by

$$d\sigma(\lambda) = \frac{d\lambda}{2\pi |C(\lambda)|^2},$$

and $C(\lambda)$ is the Harish–Chandra function, whose explicit form can be found in [5].

2.1. The Characters of the Chébli–Trimèche Hypergroups

For each $\lambda \in \mathbb{C}$, consider the following Cauchy problem:

$$\begin{cases} \Delta_A u(x) = -\lambda^2 u(x), \\ u(0) = 1, \quad u'(0) = 0. \end{cases}$$

According to [4], this problem admits a unique solution denoted by φ_λ, called the **character** of the Chébli–Trimèche hypergroup $(\mathbb{R}_+, *_A)$.

This function is infinitely differentiable on $\mathbb{R}$, even, and satisfies the following important estimate:

$$\forall \lambda \geq 0, \ \forall x \in \mathbb{R}, \quad |\varphi_\lambda(x)| \leq 1. \tag{4}$$

2.2. The Generalized Fourier Transform on the Chébli–Trimèche Hypergroups

Definition 1 ([17])**.** The generalized Fourier transform $\mathscr{F}_A$ is defined on $L^1_A(\mathbb{R}_+)$ by

$$\mathscr{F}_A(f)(\lambda) := \int_{\mathbb{R}_+} \varphi_\lambda(x) f(x) \, d\mu_A(x), \quad \lambda \in \mathbb{R}. \tag{5}$$

Special Cases:

- If $A(x) = x^{2\alpha+1}$ with $\alpha \geq -\frac{1}{2}$ and $\rho = 0$, then $(\mathbb{R}_+, *_A)$ is the Bessel–Kingman hypergroup, and $\mathscr{F}_A$ coincides with the Fourier–Bessel transform (see [3]).
- If $A(x) = \sinh^{2\alpha+1}(x) \cosh^{2\beta+1}(x)$ with $\alpha \geq \beta \geq -\frac{1}{2}$ and $\alpha \neq -\frac{1}{2}$, and $\rho = \alpha + \beta + 1$, then we recover the Jacobi operator

$$\Delta_{\alpha,\beta}(u) = \frac{d^2u}{dx^2} + [(2\alpha+1)\coth x + (2\beta+1)\tanh x] \frac{du}{dx} + \rho^2 u.$$

In this case, $(\mathbb{R}_+, *_A)$ is the Jacobi hypergroup and $\mathscr{F}_A$ corresponds to the Jacobi transform (see [4]).

Some basic properties of the generalized Fourier transform (5) are as follows (for proofs, see [6,7,16]):

Proposition 1. *1. **(Riemann–Lebesgue Lemma)** For all $f \in L^1_A(\mathbb{R}_+)$, the function $\mathscr{F}_A(f)$ is continuous, and*

$$\|\mathscr{F}_A(f)\|_{\infty,\sigma} \leq \|f\|_{1,A}. \tag{6}$$

*2. **(Inversion Formula)** If $f \in L^1_A(\mathbb{R}_+)$ and $\mathscr{F}_A(f) \in L^1_\sigma(\mathbb{R}_+)$, then*

$$f(x) = \int_{\mathbb{R}_+} \varphi_\lambda(x) \mathscr{F}_A(f)(\lambda) \, d\sigma(\lambda), \quad a.e. \ x \in \mathbb{R}_+. \tag{7}$$

*3. **(Plancherel Theorem)** The generalized Fourier transform extends uniquely to a unitary isomorphism from $L^2_A(\mathbb{R}_+)$ onto $L^2_\sigma(\mathbb{R}_+)$, and for all $f \in L^2_A(\mathbb{R}_+)$, we have*

$$\int_{\mathbb{R}_+} |f(x)|^2 \, d\mu_A(x) = \int_{\mathbb{R}_+} |\mathscr{F}_A(f)(\lambda)|^2 \, d\sigma(\lambda). \tag{8}$$

2.3. Generalized Translation Operator on the Chébli–Trimèche Hypergroups

The character φ_λ is multiplicative on $\mathbb{R}_+$ in the sense of [7,16], meaning that

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int_{\mathbb{R}_+} \varphi_\lambda(z)\, K(x,y,z) A(z)\, dz, \tag{9}$$

where $K(x,y,\cdot)$ is a measurable positive function given explicitly in [7].

The product formula (9) permits the definition of a **generalized translation operator**, a convolution product, and hence the development of a harmonic analysis on the Chébli–Trimèche hypergroups.

Definition 2. Let $x, y \in \mathbb{R}_+$ and f be a measurable function on $\mathbb{R}_+$. The **translation operator** associated with the Chébli–Trimèche hypergroup $(\mathbb{R}_+, *_A)$ is defined by

$$\mathcal{T}_A^x(f)(y) := \int_{\mathbb{R}_+} f(z)\, K(x,y,z) A(z)\, dz.$$

The following proposition summarizes key properties of the generalized translation operator. For details and proofs, see [7,16].

Proposition 2. *Let $x, y, z \in \mathbb{R}_+$ and f a measurable function on $\mathbb{R}_+$. Then:*

1. ***Symmetry:***
$$\mathcal{T}_A^x(f)(y) = \mathcal{T}_A^y(f)(x). \tag{10}$$
2. ***Preservation of Integration:***
$$\int_{\mathbb{R}_+} \mathcal{T}_A^x(f)(y)\, d\mu_A(x) = \int_{\mathbb{R}_+} f(y)\, d\mu_A(x). \tag{11}$$
3. ***Norm Inequality:*** *For $f \in L_A^p(\mathbb{R}_+)$ with $p \in [1, \infty]$, the function $\mathcal{T}_A^x(f)$ also belongs to $L_A^p(\mathbb{R}_+)$ and satisfies*
$$\|\mathcal{T}_A^x(f)\|_{p,A} \le \|f\|_{p,A}. \tag{12}$$

Using the generalized translation, we define the **generalized convolution product** of functions $f, g \in L_A^p(\mathbb{R}_+)$ at $x \in \mathbb{R}_+$ as follows:

$$(f *_A g)(x) := \int_{\mathbb{R}_+} \mathcal{T}_A^x(f)(y)\, g(y)\, d\mu_A(y).$$

This convolution is **commutative**, **associative**, and satisfies the following key properties:

Proposition 3. 1. ***(Young's Inequality)*** *Let $p, q, r \in [1, \infty]$ satisfy $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. For all $f \in L_A^p(\mathbb{R}_+)$ and $g \in L_A^q(\mathbb{R}_+)$, the convolution $f *_A g$ belongs to $L_A^r(\mathbb{R}_+)$ and*
$$\|f *_A g\|_{r,A} \le \|f\|_{p,A} \|g\|_{q,A}. \tag{13}$$

2. ***(Fourier Transform of the Convolution)*** *For $f, g \in L_A^2(\mathbb{R}_+)$, the convolution $f *_A g$ belongs to $L_A^2(\mathbb{R}_+)$ if and only if the product $\mathscr{F}_A(f)\mathscr{F}_A(g)$ belongs to $L_\sigma^2(\mathbb{R})$, in which case:*
$$\mathscr{F}_A(f *_A g)(\lambda) = \mathscr{F}_A(f)(\lambda) \cdot \mathscr{F}_A(g)(\lambda), \tag{14}$$
and
$$\int_{\mathbb{R}_+} |(f *_A g)(x)|^2\, d\mu_A(x) = \int_{\mathbb{R}_+} |\mathscr{F}_A(f)(\lambda)|^2\, |\mathscr{F}_A(g)(\lambda)|^2\, d\sigma(\lambda). \tag{15}$$

2.4. The Schatten–von Neumann Classes

In this subsection, we recall essential concepts regarding the Schatten–von Neumann classes of compact operators on the Hilbert space $L_A^2(\mathbb{R}_+)$.

Notation 1.
- $l^p(\mathbb{N})$, $1 \le p \le \infty$: the space of all infinite sequences of real or complex numbers $u = (u_j)_{j\in\mathbb{N}}$ such that
$$\|u\|_p := \begin{cases} \left(\sum_{j=1}^{\infty} |u_j|^p\right)^{\frac{1}{p}} < \infty, & \text{if } 1 \le p < \infty, \\ \sup_{j\in\mathbb{N}} |u_j| < \infty, & \text{if } p = \infty. \end{cases}$$
- $B(L^p_A(\mathbb{R}_+))$: the space of bounded linear operators from $L^p_A(\mathbb{R}_+)$ into itself.
- For $p = 2$, we define the space $S_\infty := B(L^2_A(\mathbb{R}_+))$, equipped with the operator norm
$$\|A\|_{S_\infty} := \sup_{\substack{v\in L^2_A(\mathbb{R}_+) \\ \|v\|_{2,A}=1}} \|Av\|_{2,A}. \tag{16}$$

Definition 3.

1. The *singular values* $(s_n(A))_{n\in\mathbb{N}}$ of a compact operator $A \in B(L^2_A(\mathbb{R}_+))$ are the eigenvalues of the positive self-adjoint operator $|A| := \sqrt{A^*A}$.
2. For $1 \le p < \infty$, the *Schatten class* S_p consists of all compact operators A such that $(s_n(A)) \in l^p(\mathbb{N})$. The norm in S_p is given by
$$\|A\|_{S_p} := \left(\sum_{n=1}^{\infty} s_n(A)^p\right)^{1/p}.$$

Remark 1. The class S_2 corresponds to the space of Hilbert–Schmidt operators, while S_1 is the space of trace-class operators.

Definition 4. The *trace* of an operator $A \in S_1$ is defined as
$$\mathrm{tr}(A) = \sum_{n=1}^{\infty} \langle A\phi_n, \phi_n\rangle_{\mu_A}, \tag{17}$$
where $(\phi_n)_{n\in\mathbb{N}}$ is any orthonormal basis of $L^2_A(\mathbb{R}_+)$.

Remark 2. If A is a positive operator, then
$$\mathrm{tr}(A) = \|A\|_{S_1}. \tag{18}$$
Moreover, a compact operator A on the Hilbert space $L^2_A(\mathbb{R}_+)$ is Hilbert–Schmidt if and only if $A^*A \in S_1$. In this case, we have
$$\|A\|^2_{HS} := \|A\|^2_{S_2} = \|A^*A\|_{S_1} = \mathrm{tr}(A^*A) = \sum_{n=1}^{\infty} \|A\phi_n\|^2_{2,A}, \tag{19}$$
for any orthonormal basis $(\phi_n)_{n\in\mathbb{N}}$ of $L^2_A(\mathbb{R}_+)$.

For more information about the Schatten–von Neumann classes, we refer the reader to [9].

2.5. Generalized Wigner Transform on the Chébli–Trimèche Hypergroups

The main objective of this subsection is to define the generalized Fourier–Wigner transform on the Chébli–Trimèche hypergroups, as introduced in [17], and to present some associated results.

Notation 2. We introduce the following notations:

- $\mathcal{S}_*(\mathbb{R}^2)$: the Schwartz space on $\mathbb{R}^2$, equipped with its usual topology.
- $L^p_{\theta_A}(\mathbb{R}^2_+)$, for $1 \le p \le \infty$: the space of measurable functions on $\mathbb{R}^2_+$ such that
$$\|f\|_{p,\theta_A} := \begin{cases} \left(\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} |f(x,\lambda)|^p \, d\theta_A(x,\lambda)\right)^{1/p}, & \text{if } 1 \le p < \infty, \\ \operatorname{ess\,sup}_{(x,\lambda)\in\mathbb{R}^2_+} |f(x,\lambda)|, & \text{if } p = \infty, \end{cases}$$

where θ_A is the product measure on $\mathbb{R}_+^2$ defined by

$$d\theta_A(x,\lambda) := d\sigma(\lambda) \otimes d\mu_A(x).$$

Definition 5. The Fourier–Wigner transform associated with the operator Δ_A is defined on $\mathcal{D}_*(\mathbb{R}) \times \mathcal{D}_*(\mathbb{R})$ by

$$\mathcal{W}(f,g)(x,\lambda) := \int_{\mathbb{R}_+} f(y)\, \mathcal{T}_A^x(g)(y)\, \varphi_\lambda(y)\, d\mu_A(y). \tag{20}$$

Particular Case:

- If $A(x) = x^{2\alpha+1}$ with $\alpha \geq -\frac{1}{2}$ and $\rho = 0$, then $(\mathbb{R}_+, *_A)$ is the Bessel–Kingman hypergroup, and $\mathcal{W}$ coincides with the Bessel–Wigner transform (see [18]).

Remark 3. The transform $\mathcal{W}$ is a bilinear mapping from $\mathcal{D}_*(\mathbb{R}) \times \mathcal{D}_*(\mathbb{R})$ into $\mathcal{S}(\mathbb{R}^2)$, and can be expressed in the following equivalent forms:

$$\mathcal{W}(f,g)(x,\lambda) = \mathscr{F}_A\left(f \cdot \mathcal{T}_A^x(g)\right)(\lambda), \tag{21}$$

$$= \left(g *_A f\varphi_\lambda\right)(x). \tag{22}$$

We now state some fundamental results related to this transform.

Proposition 4 ([17])**.** *Let $f, g \in L_A^2(\mathbb{R}_+)$, then $\mathcal{W}(f,g)$ is well-defined and belongs to both $L^2_{\theta_A}(\mathbb{R}_+^2)$ and $L^\infty_{\theta_A}(\mathbb{R}_+^2)$. Moreover, the following estimates hold:*

$$\|\mathcal{W}(f,g)\|_{2,\theta_A} \leq \|f\|_{2,A} \cdot \|g\|_{2,A}, \tag{23}$$

$$\|\mathcal{W}(f,g)\|_{\infty,\theta_A} \leq \|f\|_{2,A} \cdot \|g\|_{2,A}. \tag{24}$$

3. Localization Operators Associated with the Generalized Fourier–Wigner Transform

In this section, we introduce localization operators $\mathscr{L}_{\psi_1,\psi_2}(\sigma)$ associated with the generalized Fourier–Wigner transform, and provide sufficient conditions for their boundedness, compactness, and Schatten class membership in terms of the symbol σ and the window functions ψ_1 and ψ_2.

Definition 6. Let ψ_1 and ψ_2 be measurable functions on $\mathbb{R}_+$, and let σ be a measurable function on $\mathbb{R}_+^2$. The localization operator $\mathscr{L}_{\psi_1,\psi_2}(\sigma)$ associated with the generalized Fourier–Wigner transform is defined by

$$\mathscr{L}_{\psi_1,\psi_2}(\sigma)(f)(y) := \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \sigma(x,\lambda)\, \mathcal{W}(f,\psi_1)(x,\lambda)\, \varphi_\lambda(y)\, \overline{\mathcal{T}_A^x(\psi_2)(y)}\, d\theta_A(x,\lambda), \tag{25}$$

where $f \in L_A^p(\mathbb{R}_+)$ for some $1 \leq p \leq \infty$.

Remark 4. Depending on the properties of the symbol σ and the required continuity of the operator, different regularity and integrability assumptions may be imposed on ψ_1 and ψ_2. Under appropriate conditions, $\mathscr{L}_{\psi_1,\psi_2}(\sigma)$ defines a bounded operator on $L_A^p(\mathbb{R}_+)$ for all $1 \leq p \leq \infty$.

It is often more convenient to interpret the localization operator in a weak sense. That is, for all $f \in L_A^p(\mathbb{R}_+)$ and $g \in L_A^{p'}(\mathbb{R}_+)$ (where p' is the Hölder conjugate of p), we have

$$\left\langle \mathscr{L}_{\psi_1,\psi_2}(\sigma)(f), g\right\rangle_{\mu_A} = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \sigma(x,\lambda)\, \mathcal{W}(f,\psi_1)(x,\lambda)\, \overline{\mathcal{W}(g,\psi_2)(x,\lambda)}\, d\theta_A(x,\lambda). \tag{26}$$

We now state a result concerning the adjoint of the localization operator.

Proposition 5. *Let $1 \leq p \leq \infty$. Then the adjoint of the operator*

$$\mathscr{L}_{\psi_1,\psi_2}(\sigma) : L_A^p(\mathbb{R}_+) \to L_A^p(\mathbb{R}_+)$$

is given by

$$\mathscr{L}_{\psi_1,\psi_2}(\sigma)^* = \mathscr{L}_{\psi_2,\psi_1}(\overline{\sigma}) : L_A^{p'}(\mathbb{R}_+) \to L_A^{p'}(\mathbb{R}_+),$$

where p' is the Hölder conjugate of p.

Proof. Let $f \in L_A^p(\mathbb{R}_+)$ and $g \in L_A^{p'}(\mathbb{R}_+)$. From equation (26), we compute:

$$\begin{aligned}\langle \mathscr{L}_{\psi_1,\psi_2}(\sigma)(f), g\rangle_{\mu_A} &= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \sigma(x,\lambda)\, \mathcal{W}(f,\psi_1)(x,\lambda)\, \overline{\mathcal{W}(g,\psi_2)(x,\lambda)}\, d\theta_A(x,\lambda) \\ &= \overline{\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \overline{\sigma(x,\lambda)}\, \mathcal{W}(g,\psi_2)(x,\lambda)\, \overline{\mathcal{W}(f,\psi_1)(x,\lambda)}\, d\theta_A(x,\lambda)} \\ &= \overline{\langle \mathscr{L}_{\psi_2,\psi_1}(\overline{\sigma})(g), f\rangle_{\mu_A}} = \langle f, \mathscr{L}_{\psi_2,\psi_1}(\overline{\sigma})(g)\rangle_{\mu_A}.\end{aligned}$$

Hence, $\mathscr{L}_{\psi_1,\psi_2}(\sigma)^* = \mathscr{L}_{\psi_2,\psi_1}(\overline{\sigma})$. □

In what follows, we assume that $\psi_1, \psi_2 \in L_A^2(\mathbb{R}_+)$ satisfy $\|\psi_1\|_{2,A} = \|\psi_2\|_{2,A} = 1$. This normalization is not essential and all results remain valid up to a constant depending on $\|\psi_1\|_{2,A}$ and $\|\psi_2\|_{2,A}$.

3.1. Boundedness of $\mathscr{L}_{\psi_1,\psi_2}(\sigma)$ in S_∞

The main goal of this subsection is to prove that the localization operator

$$\mathscr{L}_{\psi_1,\psi_2}(\sigma) : L_A^2(\mathbb{R}_+) \longrightarrow L_A^2(\mathbb{R}_+)$$

is bounded for every symbol $\sigma \in L_{\theta_A}^p(\mathbb{R}_+^2)$ with $1 \le p \le +\infty$. We first treat the cases $\sigma \in L_{\theta_A}^1(\mathbb{R}_+^2)$ and $\sigma \in L_{\theta_A}^\infty(\mathbb{R}_+^2)$ separately, and then generalize the result using interpolation theory.

Proposition 6. *Let $\sigma \in L_{\theta_A}^1(\mathbb{R}_+^2)$. Then the localization operator $\mathscr{L}_{\psi_1,\psi_2}(\sigma)$ belongs to the class S_∞ and satisfies*

$$\|\mathscr{L}_{\psi_1,\psi_2}(\sigma)\|_{S_\infty} \le \|\sigma\|_{1,\theta_A}. \tag{27}$$

Proof. Let $f, g \in L_A^2(\mathbb{R}_+)$. Using the weak formulation in (26), we obtain

$$\left|\langle \mathscr{L}_{\psi_1,\psi_2}(\sigma)(f), g\rangle_{\mu_A}\right| \le \|\mathcal{W}(f,\psi_1)\|_{\infty,\theta_A}\|\mathcal{W}(g,\psi_2)\|_{\infty,\theta_A}\|\sigma\|_{1,\theta_A}.$$

From inequality (24), we have

$$\|\mathcal{W}(f,\psi_1)\|_{\infty,\theta_A} \le \|f\|_{2,A}, \quad \|\mathcal{W}(g,\psi_2)\|_{\infty,\theta_A} \le \|g\|_{2,A},$$

and hence,

$$\left|\langle \mathscr{L}_{\psi_1,\psi_2}(\sigma)(f), g\rangle_{\mu_A}\right| \le \|f\|_{2,A}\|g\|_{2,A}\|\sigma\|_{1,\theta_A}.$$

By the operator norm characterization in (16), the desired estimate follows. □

Proposition 7. *Let $\sigma \in L_{\theta_A}^\infty(\mathbb{R}_+^2)$. Then the localization operator $\mathscr{L}_{\psi_1,\psi_2}(\sigma)$ belongs to S_∞ and satisfies*

$$\|\mathscr{L}_{\psi_1,\psi_2}(\sigma)\|_{S_\infty} \le \|\sigma\|_{\infty,\theta_A}. \tag{28}$$

Proof. Let $f, g \in L_A^2(\mathbb{R}_+)$, and apply Hölder's inequality to the weak formulation (26):

$$\left|\langle \mathscr{L}_{\psi_1,\psi_2}(\sigma)(f), g\rangle_{\mu_A}\right| \le \|\sigma\|_{\infty,\theta_A}\|\mathcal{W}(f,\psi_1)\|_{2,\theta_A}\|\mathcal{W}(g,\psi_2)\|_{2,\theta_A}.$$

Using inequality (23), we get

$$\|\mathcal{W}(f,\psi_1)\|_{2,\theta_A} \le \|f\|_{2,A}, \quad \|\mathcal{W}(g,\psi_2)\|_{2,\theta_A} \le \|g\|_{2,A},$$

which implies

$$\left|\left\langle \mathscr{L}_{\psi_1,\psi_2}(\sigma)(f), g\right\rangle_{\mu_A}\right| \leq \|\sigma\|_{\infty,\theta_A}\|f\|_{2,A}\|g\|_{2,A}.$$

Hence, the operator norm estimate follows. □

We now generalize to all $1 \leq p \leq \infty$.

Theorem 1. *Let $\sigma \in L^p_{\theta_A}(\mathbb{R}^2_+)$ for some $1 \leq p \leq \infty$. Then there exists a unique bounded linear operator*

$$\mathscr{L}_{\psi_1,\psi_2}(\sigma) : L^2_A(\mathbb{R}_+) \longrightarrow L^2_A(\mathbb{R}_+),$$

such that

$$\|\mathscr{L}_{\psi_1,\psi_2}(\sigma)\|_{S_\infty} \leq \|\sigma\|_{p,\theta_A}. \tag{29}$$

Proof. Let $\sigma \in L^p_{\theta_A}(\mathbb{R}^2_+)$ with $1 \leq p \leq \infty$, and fix $f \in L^2_A(\mathbb{R}_+)$. Define the operator

$$T : L^1_{\theta_A}(\mathbb{R}^2_+) \cap L^\infty_{\theta_A}(\mathbb{R}^2_+) \to L^2_A(\mathbb{R}_+), \quad T(\sigma) := \mathscr{L}_{\psi_1,\psi_2}(\sigma)(f).$$

From Propositions (27) and (28), we have

$$\|T(\sigma)\|_{2,A} \leq \|f\|_{2,A}\|\sigma\|_{1,\theta_A}, \tag{30}$$

$$\|T(\sigma)\|_{2,A} \leq \|f\|_{2,A}\|\sigma\|_{\infty,\theta_A}. \tag{31}$$

By the Riesz–Thorin interpolation theorem (see [9,19]), the operator T extends uniquely to a bounded linear operator on $L^p_{\theta_A}(\mathbb{R}^2_+)$ for all $1 \leq p \leq \infty$, satisfying

$$\left\|\mathscr{L}_{\psi_1,\psi_2}(\sigma)(f)\right\|_{2,A} \leq \|f\|_{2,A}\|\sigma\|_{p,\theta_A}. \tag{32}$$

Since inequality (32) holds for all $f \in L^2_A(\mathbb{R}_+)$, we conclude the proof. □

3.2. L^p_A-Boundedness of the Localization Operator $\mathscr{L}_{\psi_1,\psi_2}(\sigma)$

In this subsection, we use Schur's test [20] to establish the boundedness of the localization operator

$$\mathscr{L}_{\psi_1,\psi_2}(\sigma) : L^p_A(\mathbb{R}_+) \longrightarrow L^p_A(\mathbb{R}_+)$$

for all $1 \leq p \leq +\infty$.

Theorem 2. *Let $\sigma \in L^1_{\theta_A}(\mathbb{R}^2_+)$, and let $\psi_1, \psi_2 \in L^1_A(\mathbb{R}_+) \cap L^\infty_A(\mathbb{R}_+)$. Then the localization operator $\mathscr{L}_{\psi_1,\psi_2}(\sigma)$ extends to a unique bounded linear operator on $L^p_A(\mathbb{R}_+)$ for all $1 \leq p \leq +\infty$. Moreover, we have the estimate:*

$$\|\mathscr{L}_{\psi_1,\psi_2}(\sigma)\|_{B\left(L^p_A(\mathbb{R}_+)\right)} \leq \max\left\{\|\psi_1\|_{\infty,A}\|\psi_2\|_{1,A}, \|\psi_1\|_{1,A}\|\psi_2\|_{\infty,A}\right\} \cdot \|\sigma\|_{1,\theta_A}. \tag{33}$$

Proof. Define the kernel function $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{C}$ by

$$F(y,s) := \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \sigma(x,\lambda)\varphi_\lambda(y)\overline{\mathcal{T}^x_A(\psi_2)(y)}\varphi_\lambda(s)\mathcal{T}^x_A(\psi_1)(s)\,\mathrm{d}\theta_A(x,\lambda).$$

Using Fubini's theorem, we write the localization operator as an integral operator:

$$\mathscr{L}_{\psi_1,\psi_2}(\sigma)(f)(y) = \int_{\mathbb{R}_+} F(y,s)f(s)\,\mathrm{d}\mu_A(s).$$

We estimate the integral of $|F(y,s)|$ with respect to y:

$$\int_{\mathbb{R}_+} |F(y,s)|\,\mathrm{d}\mu_A(y) \leq \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} |\sigma(x,\lambda)| \cdot \|\psi_2\|_{\infty,A} \cdot |\varphi_\lambda(s)\mathcal{T}^x_A(\psi_1)(s)|\,\mathrm{d}\theta_A(x,\lambda) \tag{34}$$

$$\leq \|\psi_2\|_{\infty,A} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} |\sigma(x,\lambda)| \cdot |\varphi_\lambda(s) \mathcal{T}_A^x(\psi_1)(s)| \, \mathrm{d}\theta_A(x,\lambda).$$

Then integrating in s, we obtain

$$\int_{\mathbb{R}_+} |F(y,s)| \, \mathrm{d}\mu_A(s) \leq \|\psi_1\|_{\infty,A} \|\psi_2\|_{1,A} \|\sigma\|_{1,\theta_A}, \quad \int_{\mathbb{R}_+} |F(y,s)| \, \mathrm{d}\mu_A(y) \leq \|\psi_1\|_{1,A} \|\psi_2\|_{\infty,A} \|\sigma\|_{1,\theta_A}. \tag{35}$$

By applying Schur's lemma (see [20]) to the integral operator with kernel $F(y,s)$, the operator $\mathscr{L}_{\psi_1,\psi_2}(\sigma)$ extends to a bounded linear operator on $L^p_A(\mathbb{R}_+)$ for all $1 \leq p \leq +\infty$, and satisfies the bound in (36). □

3.3. Trace and Compactness of Localization Operators $\mathscr{L}_{\psi_1,\psi_2}(\sigma)$

The main purpose of this subsection is to prove that the localization operator

$$\mathscr{L}_{\psi_1,\psi_2}(\sigma) : L^2_A(\mathbb{R}_+) \longrightarrow L^2_A(\mathbb{R}_+)$$

belongs to the Schatten–von Neumann class S^p for all $1 \leq p \leq +\infty$. We begin with the case $p = 2$, i.e., Hilbert–Schmidt operators.

Theorem 3. *Let $\sigma \in L^1_{\theta_A}(\mathbb{R}^2_+)$. Then the localization operator*

$$\mathscr{L}_{\psi_1,\psi_2}(\sigma) : L^2_A(\mathbb{R}_+) \to L^2_A(\mathbb{R}_+)$$

is a Hilbert–Schmidt operator and, in particular, compact. Moreover, its Hilbert–Schmidt norm satisfies

$$\|\mathscr{L}_{\psi_1,\psi_2}(\sigma)\|_{HS} \leq 1 + \|\sigma\|^2_{1,\theta_A}.$$

Proof. Let $\{\phi_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2_A(\mathbb{R}_+)$. Using Fubini's theorem and relations (21) and (26), we compute

$$\|\mathscr{L}_{\psi_1,\psi_2}(\sigma)(\phi_k)\|^2_{2,A} = \int_{\mathbb{R}^2_+} \sigma(x,\lambda) \, \mathscr{F}_A\left(\phi_k \, \mathcal{T}_A^x(\psi_1)\right)(\lambda) \, \overline{\mathscr{F}_A\left(\mathscr{L}_{\psi_1,\psi_2}(\sigma)(\phi_k) \, \mathcal{T}_A^x(\psi_2)\right)(\lambda)} \, \mathrm{d}\theta_A(x,\lambda).$$

By the identity,

$$\mathscr{F}_A\left(\mathscr{L}_{\psi_1,\psi_2}(\sigma)(\phi_k) \, \mathcal{T}_A^x(\psi_2)\right)(\lambda) = \left\langle \phi_k, \mathscr{L}_{\psi_2,\psi_1}(\overline{\sigma}) \left(\overline{\mathcal{T}_A^x(\psi_2)}\right) \right\rangle_{\mu_A},$$

and by Parseval's identity and Fubini's theorem, we obtain:

$$\begin{aligned}
\|\mathscr{L}_{\psi_1,\psi_2}(\sigma)\|^2_{HS} &= \sum_{k=1}^{\infty} \|\mathscr{L}_{\psi_1,\psi_2}(\sigma)(\phi_k)\|^2_{2,A} \\
&\leq \frac{1}{2} \int_{\mathbb{R}^2_+} |\sigma(x,\lambda)| \left[\sum_k |\langle \varphi_\lambda \mathcal{T}_A^x(\psi_1), \phi_k \rangle_{\mu_A}|^2 + |\langle \mathscr{L}_{\psi_2,\psi_1}(\overline{\sigma})(\overline{\mathcal{T}_A^x(\psi_2)} \varphi_\lambda), \phi_k \rangle_{\mu_A}|^2 \right] \mathrm{d}\theta_A(x,\lambda).
\end{aligned}$$

Using Parseval's identity and the boundedness result from (27), and assuming $\|\psi_1\|_{2,A} = \|\psi_2\|_{2,A} = 1$, we get:

$$\|\mathscr{L}_{\psi_1,\psi_2}(\sigma)\|^2_{HS} \leq \frac{1}{2} \|\sigma\|_{1,\theta_A} (1 + \|\sigma\|^2_{1,\theta_A}) \leq \left(1 + \|\sigma\|^2_{1,\theta_A}\right)^2 < \infty.$$

This proves the operator is Hilbert–Schmidt and therefore compact. □

Next, we extend compactness to all $\sigma \in L^p_{\theta_A}(\mathbb{R}^2_+)$ with $1 \leq p < \infty$.

Proposition 8. *Let $\sigma \in L^p_{\theta_A}(\mathbb{R}^2_+)$ with $1 \leq p < \infty$. Then the localization operator*

$$\mathscr{L}_{\psi_1,\psi_2}(\sigma) : L^2_A(\mathbb{R}_+) \longrightarrow L^2_A(\mathbb{R}_+)$$

is compact.

Proof. Let $\{\sigma_n\}_{n\in\mathbb{N}} \subset L^1_{\theta_A}(\mathbb{R}^2_+) \cap L^\infty_{\theta_A}(\mathbb{R}^2_+)$ be a sequence such that $\sigma_n \to \sigma$ in $L^p_{\theta_A}(\mathbb{R}^2_+)$ as $n \to \infty$.
By Theorem 1, the localization operators satisfy

$$\|\mathscr{L}_{\psi_1,\psi_2}(\sigma_n) - \mathscr{L}_{\psi_1,\psi_2}(\sigma)\|_{S_\infty} \leq \|\sigma_n - \sigma\|_{p,\theta_A} \to 0.$$

Moreover, each $\mathscr{L}_{\psi_1,\psi_2}(\sigma_n)$ is Hilbert–Schmidt by Theorem 3, and thus compact. Since the limit of compact operators in the norm topology is compact, it follows that $\mathscr{L}_{\psi_1,\psi_2}(\sigma)$ is compact. □

In the next result, we prove that the localization operator is compact on the weighted space $L^1_A(\mathbb{R}_+)$.

Theorem 4. *Let $\sigma \in L^1_{\theta_A}(\mathbb{R}^2_+)$ and $\psi_1, \psi_2 \in L^1_A(\mathbb{R}_+) \cap L^\infty_A(\mathbb{R}_+)$, then the localization operator*

$$\mathscr{L}_{\psi_1,\psi_2}(\sigma) : L^1_A(\mathbb{R}_+) \longrightarrow L^1_A(\mathbb{R}_+)$$

is compact.

Proof. By Theorem 2, the operator $\mathscr{L}_{\psi_1,\psi_2}(\sigma)$ is bounded on $L^1_A(\mathbb{R}_+)$. Let $\{f_n\} \subset L^1_A(\mathbb{R}_+)$ be a sequence such that $f_n \rightharpoonup 0$ weakly in $L^1_A(\mathbb{R}_+)$ as $n \to \infty$. We aim to prove

$$\lim_{n\to\infty} \|\mathscr{L}_{\psi_1,\psi_2}(\sigma)(f_n)\|_{1,A} = 0.$$

From the integral representation of the localization operator (cf. equation (25)), we have

$$\|\mathscr{L}_{\psi_1,\psi_2}(\sigma)(f_n)\|_{1,A} \leq \int_{\mathbb{R}_+} \left[\int_{\mathbb{R}^2_+} |\sigma(x,\lambda)|\, |\mathcal{W}(f_n,\psi_1)(x,\lambda)|\, |\mathcal{T}^x_A(\psi_2)(y)|\, \mathrm{d}\theta_A(x,\lambda)\right] \mathrm{d}\mu_A(y). \tag{36}$$

Since $f_n \rightharpoonup 0$ weakly in $L^1_A(\mathbb{R}_+)$, it follows that

$$\lim_{n\to\infty} |\mathcal{W}(f_n,\psi_1)(x,\lambda)| \cdot |\mathcal{T}^x_A(\psi_2)(y)| = 0 \quad \text{for all } x, y, \lambda \in \mathbb{R}_+. \tag{37}$$

Moreover, since $\{f_n\}$ is bounded in L^1_A, there exists a constant $C > 0$ such that $\|f_n\|_{1,A} \leq C$ for all n. Hence, using the boundedness of ψ_1 and ψ_2, we get

$$|\mathcal{W}(f_n,\psi_1)(x,\lambda)| \cdot |\mathcal{T}^x_A(\psi_2)(y)| \leq C\|\psi_1\|_{\infty,A}\|\psi_2\|_{\infty,A}|\sigma(x,\lambda)|. \tag{38}$$

Applying Fubini's theorem, we obtain

$$\int_{\mathbb{R}_+} \left[\int_{\mathbb{R}^2_+} |\sigma(x,\lambda)| \cdot |\mathcal{W}(f_n,\psi_1)(x,\lambda)| \cdot |\mathcal{T}^x_A(\psi_2)(y)|\, \mathrm{d}\theta_A(x,\lambda)\right] \mathrm{d}\mu_A(y) < \infty. \tag{39}$$

From the pointwise convergence in (37), the uniform bound in (38), and the integrability condition in (39), we may apply the Lebesgue Dominated Convergence Theorem to conclude:

$$\lim_{n\to\infty} \|\mathscr{L}_{\psi_1,\psi_2}(\sigma)(f_n)\|_{1,A} = 0,$$

which proves that the operator is compact. □

In the following theorem, we prove that the localization operator belongs to the trace class S^1.

Theorem 5. *Let $\sigma \in L^1_{\theta_A}(\mathbb{R}^2_+)$. Then the localization operator*

$$\mathscr{L}_{\psi_1,\psi_2}(\sigma) : L^2_A(\mathbb{R}_+) \longrightarrow L^2_A(\mathbb{R}_+)$$

is a trace-class operator, i.e., $\mathscr{L}_{\psi_1,\psi_2}(\sigma) \in S^1$*, and satisfies*

$$\|\tilde{\sigma}\|_{1,\theta_A} \leq \|\mathscr{L}_{\psi_1,\psi_2}(\sigma)\|_{S_1} \leq \|\sigma\|_{1,\theta_A}, \tag{40}$$

where $\tilde{\sigma}$ *is defined by*

$$\tilde{\sigma}(x,\lambda) = \left\langle \mathscr{L}_{\psi_1,\psi_2}(\sigma)\left(\varphi_\lambda \mathcal{T}_A^x(\psi_1)\right) \mid \varphi_\lambda \mathcal{T}_A^x(\psi_2)\right\rangle_{\mu_A}.$$

Proof. Since $\sigma \in L^1_{\theta_A}(\mathbb{R}^2_+)$, Theorem 4 implies that $\mathscr{L}_{\psi_1,\psi_2}(\sigma)$ is compact. By standard results (see [9]), there exist orthonormal sequences (ϕ_j) and (h_j) in $L^2_A(\mathbb{R}_+)$, and singular values (s_j) such that the operator has the Schmidt decomposition

$$\mathscr{L}_{\psi_1,\psi_2}(\sigma)(f) = \sum_{j=1}^{\infty} s_j \left\langle f, \phi_j\right\rangle_{\mu_A} h_j. \tag{41}$$

Thus, the trace norm is

$$\|\mathscr{L}_{\psi_1,\psi_2}(\sigma)\|_{S^1} = \sum_{j=1}^{\infty} s_j = \sum_{j=1}^{\infty} \left\langle \mathscr{L}_{\psi_1,\psi_2}(\sigma)(\phi_j), h_j\right\rangle_{\mu_A}.$$

Using the integral representation of the operator (cf. (25), (26)) and Fubini's theorem, we obtain:

$$\|\mathscr{L}_{\psi_1,\psi_2}(\sigma)\|_{S^1} \leq \frac{1}{2}\int_{\mathbb{R}^2_+} |\sigma(x,\lambda)| \left[\sum_{j=1}^{\infty}\left|\left\langle \varphi_\lambda \mathcal{T}_A^x(\psi_1), \phi_j\right\rangle_{\mu_A}\right|^2 + \sum_{j=1}^{\infty}\left|\left\langle \varphi_\lambda \mathcal{T}_A^x(\psi_2), h_j\right\rangle_{\mu_A}\right|^2\right] \mathrm{d}\theta_A(x,\lambda).$$

By Parseval's identity and the fact that $\|\psi_1\|_{2,A} = \|\psi_2\|_{2,A} = 1$, we obtain

$$\|\mathscr{L}_{\psi_1,\psi_2}(\sigma)\|_{S^1} \leq \int_{\mathbb{R}^2_+} |\sigma(x,\lambda)|\, \mathrm{d}\theta_A(x,\lambda) = \|\sigma\|_{1,\theta_A}.$$

Now, to prove the lower bound in (40), note that

$$\tilde{\sigma}(x,\lambda) = \left\langle \mathscr{L}_{\psi_1,\psi_2}(\sigma)\left(\varphi_\lambda \mathcal{T}_A^x(\psi_1)\right), \varphi_\lambda \mathcal{T}_A^x(\psi_2)\right\rangle_{\mu_A}.$$

By inserting the expansion (41) and using Fubini's theorem:

$$\begin{aligned}\|\tilde{\sigma}\|_{1,\theta_A} &= \int_{\mathbb{R}^2_+} |\tilde{\sigma}(x,\lambda)|\, \mathrm{d}\theta_A(x,\lambda)\\ &\leq \frac{1}{2}\sum_{j=1}^{\infty} s_j \int_{\mathbb{R}^2_+}\left(\left|\left\langle \varphi_\lambda \mathcal{T}_A^x(\psi_1), \phi_j\right\rangle_{\mu_A}\right|^2 + \left|\left\langle h_j, \varphi_\lambda \mathcal{T}_A^x(\psi_2)\right\rangle_{\mu_A}\right|^2\right) \mathrm{d}\theta_A(x,\lambda).\end{aligned}$$

Applying the orthonormality and Parseval's identity again, we get:

$$\|\tilde{\sigma}\|_{1,\theta_A} \leq \frac{1}{2}\sum_{j=1}^{\infty} s_j \left(\|\psi_1\|^2_{2,A} + \|\psi_2\|^2_{2,A}\right) = \sum_{j=1}^{\infty} s_j = \|\mathscr{L}_{\psi_1,\psi_2}(\sigma)\|_{S^1}.$$

Hence,

$$\|\tilde{\sigma}\|_{1,\theta_A} \leq \|\mathscr{L}_{\psi_1,\psi_2}(\sigma)\|_{S^1} \leq \|\sigma\|_{1,\theta_A},$$

and the proof is complete. □

We conclude this section with the following trace formula for the localization operators.

Theorem 6. *Let* $\sigma \in L^1_{\theta_A}(\mathbb{R}^2_+)$*. Then the localization operator*

$$\mathscr{L}_{\psi_1,\psi_2}(\sigma) : L^2_A(\mathbb{R}_+) \longrightarrow L^2_A(\mathbb{R}_+)$$

is trace-class, and its trace is given by

$$\operatorname{Tr}\left(\mathscr{L}_{\psi_1,\psi_2}(\sigma)\right) = \int_{\mathbb{R}^2_+} \sigma(x,\lambda)\, \langle \varphi_\lambda \mathcal{T}^x_A(\psi_1), \varphi_\lambda \mathcal{T}^x_A(\psi_2)\rangle_{\mu_A}\, \mathrm{d}\theta_A(x,\lambda). \tag{42}$$

Proof. Let $\{\phi_j\}_{j=1}^{\infty}$ be an orthonormal basis of $L^2_A(\mathbb{R}_+)$. Since $\mathscr{L}_{\psi_1,\psi_2}(\sigma) \in S_1$ by Theorem 5, we can write its trace as

$$\operatorname{Tr}\left(\mathscr{L}_{\psi_1,\psi_2}(\sigma)\right) = \sum_{j=1}^{\infty} \left\langle \mathscr{L}_{\psi_1,\psi_2}(\sigma)(\phi_j), \phi_j \right\rangle_{\mu_A}.$$

Using the integral representation from Theorem 1 (see relation (25)), Fubini's theorem, and Parseval's identity, we obtain:

$$\begin{aligned}\operatorname{Tr}\left(\mathscr{L}_{\psi_1,\psi_2}(\sigma)\right) &= \int_{\mathbb{R}^2_+} \sigma(x,\lambda) \sum_{j=1}^{\infty} \left\langle \phi_j, \overline{\varphi_\lambda \mathcal{T}^x_A(\psi_1)}\right\rangle_{\mu_A} \left\langle \overline{\varphi_\lambda \mathcal{T}^x_A(\psi_2)}, \phi_j \right\rangle_{\mu_A} \mathrm{d}\theta_A(x,\lambda)\\ &= \int_{\mathbb{R}^2_+} \sigma(x,\lambda)\, \langle \varphi_\lambda \mathcal{T}^x_A(\psi_1), \varphi_\lambda \mathcal{T}^x_A(\psi_2)\rangle_{\mu_A}\, \mathrm{d}\theta_A(x,\lambda),\end{aligned}$$

which proves the result. □

Corollary 1. *If $\psi_1 = \psi_2 = \psi$, and σ is a real-valued, non-negative function in $L^1_{\theta_A}(\mathbb{R}^2_+)$, then the localization operator*

$$\mathscr{L}_\psi(\sigma) := \mathscr{L}_{\psi,\psi}(\sigma) : L^2_A(\mathbb{R}_+) \longrightarrow L^2_A(\mathbb{R}_+)$$

is a positive trace-class operator, and its trace norm satisfies

$$\left\|\mathscr{L}_\psi(\sigma)\right\|_{S_1} = \int_{\mathbb{R}^2_+} \sigma(x,\lambda)\, \|\varphi_\lambda \mathcal{T}^x_A(\psi)\|^2_{2,A}\, \mathrm{d}\theta_A(x,\lambda).$$

Corollary 2 (Main Result)**.** *Let $\sigma \in L^p_{\theta_A}(\mathbb{R}^2_+)$ for $1 \le p \le \infty$. Then the localization operator*

$$\mathscr{L}_{\psi_1,\psi_2}(\sigma) : L^2_A(\mathbb{R}_+) \longrightarrow L^2_A(\mathbb{R}_+)$$

belongs to the Schatten–von Neumann class S^p, and we have the estimate

$$\left\|\mathscr{L}_{\psi_1,\psi_2}(\sigma)\right\|_{S_p} \le \|\sigma\|_{p,\theta_A}.$$

Proof. The result follows from the Hilbert–Schmidt case ($p = 2$) and the trace-class case ($p = 1$) established in Theorems 3 and 5, respectively, combined with interpolation theory (see [19]). □

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