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New approximation inversion formulas using Weinstein-Gabor transform

Fethi Soltani[1,2,*]

[1] Faculté des Sciences de Tunis, Laboratoire d'Analyse Mathématique et Applications LR11ES11, Université de Tunis El Manar, Tunis 2092, Tunisia

[2] Ecole Nationale d'Ingénieurs de Carthage, Université de Carthage, Tunis 2035, Tunisia

* Correspondence: fethi.soltani@fst.utm.tn

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Abstract: In this paper, we extend the one-dimensional Gabor transform discussed to the Weinstein harmonic analysis setting. We obtain the expected properties of extended Gabor transform such as inversion formula and Calderón's reproducing formula.

Keywords: Weinstein-Gabor transform, inversion formula, Calderón's reproducing formula

MSC: 42B10, 44A05, 44A20

1. Introduction

The Fourier transform stands out as a significant discovery in mathematical sciences, that plays a crucial role in modern scientific and technological advancements. In signal processing, extensive research has utilized the Fourier transform to analyze stationary signals or processes with statistically invariant properties over time. Although, Fourier transforms have many successful applications that fascinated the mathematical, physical and engineering communities over decades, they still have numerous shortcomings.

One of the significant disadvantages of the Fourier transforms is that they do not give any information about the occurrence of the frequency component at a particular time. They only enable us to analyse the signals either in time domain or frequency domain, but not simultaneously in both domains [1,2].

A suitable redress of these limitations was given by Gabor [3] in the form of windowed Fourier transform using a Gaussian distribution function as a window function in order to construct efficient time-frequency localized expansions of finite energy signals $f \in L^2(\mathbb{R})$ as

$$V_g(f)(\xi,b) := \int_{\mathbb{R}} f(x)\overline{g(x-b)}e^{-i\xi x}\mathrm{d}x, \quad \xi, b \in \mathbb{R}.$$

The spectral contents of non-transient signals in localized neighbourhoods of time can be analyzed. This astonishing feature of the Gabor transform provides the local characteristics of the Fourier transform with a time resolution equal to the size of the window. The Gabor transform, also known as the short-time Fourier transform (STFT), marked a breakthrough in time-frequency analysis. This method involves decomposing non-transient signals using time and frequency-shifted basis functions, termed Gabor window functions. The STFT, with its clear resemblance to the classical Fourier transform, has gained considerable attention in the past few decades. Soon after its inception in quantum mechanics, the Gabor transform profound influenced diverse branches of science and engineering including harmonic analysis, signal and image processing, pseudo-differential operators, sampling theory, wave propagation, quantum optics, geophysics, astrophysics, medicine [3–5], and others. Besides its applications, the theoretical skeleton of Gabor transform has likewise been extensively studied and investigated in other groups including the locally compact Abelian and non-Abelian groups [6–8], hypergroups [9], Gelfand pairs [10] and so on. For more about Gabor transforms and their applications, we allude to [11–14].

As the harmonic analysis associated with Weinstein operator has known remarkable development, it is natural that there is an equivalent of the Gabor transform in the Weinstein harmonic analysis setting [15]. This article aims to develop two applications of the Weinstein-Gabor transform (WGT) as an inversion formula and a Calderón's reproducing formula by means of the theory of Weinstein transform. Precisely, let $\alpha > -1/2$ and $\mathbb{K} := \mathbb{R}^{d-1} \times \mathbb{R}_+$. We denote by $L^p(\mathbb{K}, \nu_\alpha)$, $p \in [1, \infty]$, the space of measurable functions f on $\mathbb{K}$, such that

$$\|f\|_{L^p(\mathbb{K},\nu_\alpha)} := \left[\int_{\mathbb{K}} |f(x)|^p \mathrm{d}\nu_\alpha(x)\right]^{1/p} < \infty, \quad p \in [1,\infty),$$
$$\|f\|_{L^\infty(\mathbb{K},\nu_\alpha)} := \operatorname*{ess\,sup}_{x\in\mathbb{K}} |f(x)| < \infty,$$

where

$$\mathrm{d}\nu_\alpha(x) := \mathrm{d}\nu_\alpha(x', x_d) = \frac{x_d^{2\alpha+1}}{\pi^{(d-1)/2} 2^{\alpha+(d-1)/2} \Gamma(\alpha+1)} \mathrm{d}x' \mathrm{d}x_d,$$

and $\mathrm{d}x' = \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_{d-1}$.

For $f \in L^1(\mathbb{K}, \nu_\alpha)$, the Weinstein transform $\mathscr{F}_W$ (see [16,17]) of f is defined by

$$\mathscr{F}_W(f)(\xi) := \int_{\mathbb{K}} f(x) \Psi_\xi^\alpha(x) \mathrm{d}\nu_\alpha(x), \quad \xi = (\xi', \xi_d) \in \mathbb{K},$$

where $\Psi_\xi^\alpha(x)$ is the Weinstein kernel given by

$$\Psi_\xi^\alpha(x) = e^{-i\langle x', \xi'\rangle} j_\alpha(x_d \xi_d), \quad x = (x', x_d) \in \mathbb{K}.$$

Here j_α is the spherical Bessel function, whose definition will be recalled from [18] in §2 below. This transform extends uniquely to an isometric isomorphism on $L^2(\mathbb{K}, \nu_\alpha)$, that is $\|f\|_{L^2(\mathbb{K},\nu_\alpha)} = \|\mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_\alpha)}$. And its inverse is denoted by $\mathscr{F}_W^{-1}$.

For $f, g \in L^2(\mathbb{K}, \nu_\alpha)$. The Weinstein convolution product (see [17]) of f and g is defined by

$$f * g(y) := \int_{\mathbb{K}} f(x) \tau_y g(-x', x_d) \mathrm{d}\nu_\alpha(x), \quad y \in \mathbb{K},$$

where τ_y, $y \in \mathbb{K}$ are the Weinstein translation operators (see [19]) defined in Section 2 below.

Let $g \in L^2(\mathbb{K}, \nu_\alpha)$. The Weinstein-Gabor transform $\mathscr{S}_g$ is the mapping defined for $f \in L^2(\mathbb{K}, \nu_\alpha)$ by

$$\mathscr{S}_g(f)(x, y) := f * g_y(x), \quad x, y \in \mathbb{K},$$

where g_y is the modulation of g by y defined by

$$g_y := \mathscr{F}_W^{-1}\left(\sqrt{\tau_y |\mathscr{F}_W(g)|^2}\right).$$

The Weinstein-Gabor transform $\mathscr{S}_g$ is studied in [15]; and in this work we will establish the following inversion formula.

Theorem 1 (Inversion formula)**.** *Let $g \in L^2(\mathbb{K}, \nu_\alpha)$ be a non-zero function. For all $f \in L^1 \cap L^2(\mathbb{K}, \nu_\alpha)$ such that $\mathscr{F}_W(f) \in L^1(\mathbb{K}, \nu_\alpha)$, we have*

$$f(z) = \frac{1}{\|g\|^2_{L^2(\mathbb{K},\nu_\alpha)}} \int_{\mathbb{K}} \mathscr{S}_g(f)(., y) * g_y(z) d\nu_\alpha(y), \quad z \in \mathbb{K}.$$

Let $g \in L^2(\mathbb{K}, \nu_\alpha)$ be a non-zero function, such that $\mathscr{F}_W(g) \in L^\infty(\mathbb{K}, \nu_\alpha)$. For $f \in L^2(\mathbb{K}, \nu_\alpha)$, we define the reconstruction function f_Δ associated with $\mathscr{S}_g$, by

$$f_\Delta(z) := \frac{1}{\|g\|^2_{L^2(\mathbb{K},\nu_\alpha)}} \int_{\Delta} \mathscr{S}_g(f)(., y) * g_y(z) \mathrm{d}\nu_\alpha(y), \quad z \in \mathbb{K},$$

where Δ is the pavement of $\mathbb{K}$ defined by $\Delta := \prod_{j=1}^{d}[a_j, b_j]$, with

$$-\infty < a_j < b_j < \infty, \quad j = 1, \ldots, d-1, \quad 0 < a_d < b_d < \infty.$$

We prove the following Calderón's reproducing formula.

Theorem 2 (Calderón's reproducing formula). *Let $g \in L^2(\mathbb{K}, \nu_\alpha)$ be a non-zero function, such that $\mathscr{F}_W(g) \in L^\infty(\mathbb{K}, \nu_\alpha)$. Then, for $f \in L^2(\mathbb{K}, \nu_\alpha)$, the function f_Δ belongs to $L^2(\mathbb{K}, \nu_\alpha)$ and satisfies*

$$\lim_{\Delta \to \mathbb{K}} \|f_\Delta - f\|_{L^2(\mathbb{K}, \nu_\alpha)} = 0.$$

The paper is organized as follows. In §2, we recall some results about the harmonic analysis associated to Weinstein operator on $\mathbb{K}$ (Weinstein transform $\mathscr{F}_W$, Weinstein translation operators τ_y, $y \in \mathbb{K}$, Weinstein convolution product $*$,...). In §3 we recall some results about the Weinstein-Gabor transform $\mathscr{S}_g$, and we establish an inversion formula. Finally, in §4, we prove Calderón's reproducing formula for the Weinstein-Gabor transform $\mathscr{S}_g$.

2. The Weinstein-harmonic analysis

In this section we recall some basic results related the Weinstein harmonic analysis [17,19–22]. We consider the Weinstein operator Δ_W (also called Laplace-Bessel operator), defined on $\mathbb{R}^{d-1} \times \mathbb{R}_+^*$ by

$$\Delta_W := \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha+1}{x_d}\frac{\partial}{\partial x_d} = \Delta_{d-1} + B_\alpha, \quad d \geq 2, \ \alpha > -1/2,$$

where Δ_{d-1} is the Laplacian operator in $\mathbb{R}^{d-1}$ and B_α is the Bessel operator with respect to the variable x_d defined on $\mathbb{R}_+^*$ by

$$B_\alpha := \frac{\partial^2}{\partial x_d^2} + \frac{2\alpha+1}{x_d}\frac{\partial}{\partial x_d}.$$

The Weinstein operator has several applications in pure and applied mathematics especially in fluid mechanics [23,24].

For all $\xi \in \mathbb{K}$, the system (see [21])

$$B_\alpha u(x) = -\xi_d^2 u(x), \quad \frac{\partial^2 u}{\partial x_j^2}(x) = -\xi_j^2 u(x), \quad j = 1, \ldots, d-1,$$

$$u(0) = 1, \quad \frac{\partial u}{\partial x_d}(0) = 0, \quad \frac{\partial u}{\partial x_j}(0) = -i\xi_j, \quad j = 1, \ldots, d-1,$$

admits a unique solution $\Psi_\xi^\alpha(x)$, given by

$$\Psi_\xi^\alpha(x) = e^{-i\langle x', \xi' \rangle} j_\alpha(x_d \xi_d), \quad x \in \mathbb{K},$$

where j_α is the spherical Bessel function [18] given by

$$j_\alpha(x) := \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} \left(\frac{x}{2}\right)^{2n}.$$

For all $x, \xi \in \mathbb{K}$, the Weinstein kernel $\Psi_\xi^\alpha(x)$ satisfies

$$|\Psi_\xi^\alpha(x)| \leq 1.$$

The Weinstein kernel $\Psi^{\alpha}_{\xi}(x)$ gives rise to an integral transform, which is called Weinstein transform on $\mathbb{R}^d_+$, where many basic properties had been established [17,19,20]. The Weinstein transform (or Laplace-Bessel transform) $\mathscr{F}_W$ is defined for $f \in L^1(\mathbb{K}, \nu_\alpha)$ by

$$\mathscr{F}_W(f)(\xi) := \int_{\mathbb{K}} f(x)\Psi^{\alpha}_{\xi}(x)\mathrm{d}\nu_\alpha(x), \quad \xi \in \mathbb{K}.$$

Moreover if $f \in L^1(\mathbb{K}, \nu_\alpha)$, then

$$\|\mathscr{F}_W(f)\|_{L^\infty(\mathbb{K},\nu_\alpha)} \leq \|f\|_{L^1(\mathbb{K},\nu_\alpha)}.$$

Theorem 3 (See [17]). *(i) (Plancherel formula). The Weinstein transform $\mathscr{F}_W$ extends uniquely to an isometric isomorphism on $L^2(\mathbb{K}, \nu_\alpha)$, that is,*

$$\|\mathscr{F}_W(f)\|_{L^2(\mathbb{K},\nu_\alpha)} = \|f\|_{L^2(\mathbb{K},\nu_\alpha)}, \quad f \in L^2(\mathbb{K}, \nu_\alpha).$$

(ii) (Inversion formula). If f and $\mathscr{F}_W(f)$ are both in $L^1(\mathbb{K}, \nu_\alpha)$, the inverse Weinstein transform is defined by

$$f(x) = \int_{\mathbb{K}} \mathscr{F}_W(f)(\xi)\Psi^{\alpha}_{-\xi}(x)d\nu_\alpha(\xi), \quad a.e \quad x \in \mathbb{K}.$$

The Weinstein kernel $\Psi^{\alpha}_{\xi}(x)$ satisfies also the following product formula.

Theorem 4 (See [18]). *For $\xi \in \mathbb{K}$ and $x, y \in \mathbb{K}$, the product $\Psi^{\alpha}_{\xi}(x)\Psi^{\alpha}_{\xi}(y)$ admits the following integral representation*

$$\Psi^{\alpha}_{\xi}(x)\Psi^{\alpha}_{\xi}(y) = \int_0^\infty \Psi^{\alpha}_{\xi}(x' + y', \rho)q_\alpha(x_d, y_d, \rho)\rho^{2\alpha+1}d\rho,$$

where

$$q_\alpha(x_d, y_d, \rho) = a_\alpha \frac{\left[(x_d + y_d)^2 - \rho^2\right]^{\alpha-\frac{1}{2}} \left[\rho^2 - (x_d - y_d)^2\right]^{\alpha-\frac{1}{2}}}{(x_d y_d \rho)^{2\alpha}} \mathbf{1}_A,$$

where $a_\alpha = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}2^{2\alpha-1}\Gamma(\alpha+\frac{1}{2})}$ and $\mathbf{1}_A$ is the characteristic function of the interval

$$A = (|x_d - y_d|, x_d + y_d).$$

We denote by $C(\mathbb{K})$, the space of continuous functions f on $\mathbb{K}$. For $f \in C(\mathbb{K})$, the linear operator

$$\tau_y f(x) := \int_0^\infty f(x' + y', \rho)q_\alpha(x_d, y_d, \rho)\rho^{2\alpha+1}\mathrm{d}\rho, \quad x, y \in \mathbb{K},$$

will be called the Weinstein translation operator (see [19]).

As a first remark, we note that

$$\int_{\mathbb{K}} \tau_y f(x)\mathrm{d}\nu_\alpha(x) = \int_{\mathbb{K}} f(x)\mathrm{d}\nu_\alpha(x), \quad f \in L^1(\mathbb{K}, \nu_\alpha), \tag{1}$$

and

$$\|\tau_y f\|_{L^1(\mathbb{K},\nu_\alpha)} \leq \|f\|_{L^1(\mathbb{K},\nu_\alpha)}, \quad f \in L^1(\mathbb{K}, \nu_\alpha).$$

Let $f, g \in L^1(\mathbb{K}, \nu_\alpha)$. The Weinstein convolution product (see [17]) of f and g is defined by

$$f * g(y) := \int_{\mathbb{K}} f(x)\tau_y g(-x', x_d)\mathrm{d}\nu_\alpha(x), \quad y \in \mathbb{K}.$$

The Weinstein translation operator is connected with the Weinstein transform $\mathscr{F}_W$ via the following formula.

Theorem 5 (See [17], page 6). *For $f \in L^2(\mathbb{K}, \nu_\alpha)$ and $y \in \mathbb{K}$, we have*

$$\mathscr{F}_W(\tau_y f)(\xi) = \Psi^\alpha_\xi(-y)\mathscr{F}_W(f)(\xi), \quad \xi \in \mathbb{K}.$$

Remark 1. From Theorem 3 (i) and Theorem 5 we have

$$\|\tau_y f\|_{L^2(\mathbb{K},\nu_\alpha)} \leq \|f\|_{L^2(\mathbb{K},\nu_\alpha)}, \quad y \in \mathbb{K},\ f \in L^2(\mathbb{K}, \nu_\alpha). \tag{2}$$

Theorem 6 (See [21,22]). (i) *For $f \in L^1(\mathbb{K}, \nu_\alpha)$ and $g \in L^2(\mathbb{K}, \nu_\alpha)$, the function $f * g$ belongs to $L^2(\mathbb{K}, \nu_\alpha)$, and*

$$\mathscr{F}_W(f * g)(\xi) = \mathscr{F}_W(f)(\xi)\mathscr{F}_W(g)(\xi), \quad \xi \in \mathbb{K}.$$

(ii) *Let $f, g \in L^2(\mathbb{K}, \nu_\alpha)$. Then*

$$f * g(x) = \mathscr{F}_W^{-1}(\mathscr{F}_W(f)\mathscr{F}_W(g))(x), \quad x \in \mathbb{K}.$$

(iii) *Let $f, g \in L^2(\mathbb{K}, \nu_\alpha)$. Then $f * g$ belongs to $L^2(\mathbb{K}, \nu_\alpha)$ if and only if $\mathscr{F}_W(f)\mathscr{F}_W(g)$ belongs to $L^2(\mathbb{K}, \nu_\alpha)$, and*

$$\mathscr{F}_W(f * g) = \mathscr{F}_W(f)\mathscr{F}_W(g), \quad \text{in the } L^2(\mathbb{K}, \nu_\alpha) - \text{case}.$$

(iv) *Let $f, g \in L^2(\mathbb{K}, \nu_\alpha)$. Then*

$$\int_{\mathbb{K}} |f * g(x)|^2 d\nu_\alpha(x) = \int_{\mathbb{K}} |\mathscr{F}_W(f)(\xi)|^2 |\mathscr{F}_W(g)(\xi)|^2 d\nu_\alpha(\xi),$$

where both sides are finite or infinite.

3. The Weinstein-Gabor transform

In the following we establish a reproducing inversion formula for the Weinstein-Gabor transform $\mathscr{S}_g$. Let $g \in L^2(\mathbb{K}, \nu_\alpha)$ and $y \in \mathbb{K}$. The modulation of g by y is the function g_y defined by

$$g_y := \mathscr{F}_W^{-1}\left(\sqrt{\tau_y |\mathscr{F}_W(g)|^2}\right).$$

From Theorem 3 (i) and (1) we have

$$\|g_y\|_{L^2(\mathbb{K},\nu_\alpha)} = \|g\|_{L^2(\mathbb{K},\nu_\alpha)}. \tag{3}$$

Let $g \in L^2(\mathbb{K}, \nu_\alpha)$. The Weinstein-Gabor transform is the mapping $\mathscr{S}_g$ defined for $f \in L^2(\mathbb{K}, \nu_\alpha)$ by

$$\mathscr{S}_g(f)(x, y) := f * g_y(x) = \int_{\mathbb{K}} f(t)\tau_x g_y(-t', t_d)\mathrm{d}\nu_\alpha(t), \quad x, y \in \mathbb{K}. \tag{4}$$

From Remark 1 and (3) we have

$$\|\mathscr{S}_g(f)\|_{L^\infty(\mathbb{K}\times\mathbb{K},\nu_\alpha\otimes\nu_\alpha)} \leq \|f\|_{L^2(\mathbb{K},\nu_\alpha)} \|g\|_{L^2(\mathbb{K},\nu_\alpha)}.$$

The Weinstein-Gabor transform $\mathscr{S}_g$ possesses the following property.

Theorem 7. *Let $f, g \in L^2(\mathbb{K}, \nu_\alpha)$. Then*

$$\mathscr{S}_g(f)(x, y) = \int_{\mathbb{K}} \Psi^\alpha_{-\xi}(x)\mathscr{F}_W(f)(\xi)\sqrt{\tau_y |\mathscr{F}_W(g)|^2(\xi)}d\nu_\alpha(\xi), \quad x, y \in \mathbb{K}.$$

Proof. From Theorem 3 (ii) and Theorem 6 (ii) we have

$$\mathscr{S}_g(f)(x, y) = \int_{\mathbb{K}} \Psi^\alpha_{-\xi}(x)\mathscr{F}_W(f)(\xi)\mathscr{F}_W(g_y)(\xi)\mathrm{d}\nu_\alpha(\xi).$$

We obtain the result from the fact that

$$\mathscr{F}_W(g_y)(\xi) = \sqrt{\tau_y|\mathscr{F}_W(g)|^2(\xi)}. \tag{5}$$

The theorem is proved. □

Theorem 8 (See [15]). *Let $g \in L^2(\mathbb{K}, \nu_\alpha)$.*
(i) *(Plancherel formula). For all $f \in L^2(\mathbb{K}, \nu_\alpha)$, we have*

$$\|\mathscr{S}_g(f)\|^2_{L^2(\mathbb{K}\times\mathbb{K},\nu_\alpha\otimes\nu_\alpha)} = \|g\|^2_{L^2(\mathbb{K},\nu_\alpha)}\|f\|^2_{L^2(\mathbb{K},\nu_\alpha)}.$$

(ii) *(Parseval formula). For all $f, h \in L^2(\mathbb{K}, \nu_\alpha)$, we have*

$$\langle \mathscr{S}_g(f), \mathscr{S}_g(h)\rangle_{L^2(\mathbb{K}\times\mathbb{K},\nu_\alpha\otimes\nu_\alpha)} = \|g\|^2_{L^2(\mathbb{K},\nu_\alpha)}\langle f, h\rangle_{L^2(\mathbb{K},\nu_\alpha)}.$$

Proof. From (4) and Theorem 6 (iv), we obtain

$$\begin{aligned}\int_{\mathbb{K}}\int_{\mathbb{K}} |\mathscr{S}_g(f)(x,y)|^2 \mathrm{d}\nu_\alpha(x)\mathrm{d}\nu_\alpha(y) &= \int_{\mathbb{K}}\int_{\mathbb{K}} |f * g_y(x)|^2 \mathrm{d}\nu_\alpha(x)\mathrm{d}\nu_\alpha(y)\\ &= \int_{\mathbb{K}}\int_{\mathbb{K}} |\mathscr{F}_W(f)(\xi)|^2|\mathscr{F}_W(g_y)(\xi)|^2 \mathrm{d}\nu_\alpha(\xi)\mathrm{d}\nu_\alpha(y).\end{aligned}$$

Using Theorem 3 (i), (1), (5) and Fubini-Tonelli theorem, we deduce

$$\begin{aligned}\int_{\mathbb{K}}\int_{\mathbb{K}} |\mathscr{S}_g(f)(x,y)|^2 \mathrm{d}\nu_\alpha(x)\mathrm{d}\nu_\alpha(y) &= \int_{\mathbb{K}}\int_{\mathbb{K}} |\mathscr{F}_W(f)(\xi)|^2 \tau_y|\mathscr{F}_W(g)|^2(\xi) \mathrm{d}\nu_\alpha(\xi)\mathrm{d}\nu_\alpha(y)\\ &= \|g\|^2_{L^2(\mathbb{K},\nu_\alpha)}\|f\|^2_{L^2(\mathbb{K},\nu_\alpha)}.\end{aligned}$$

The (i) is proved and as in the same way we prove (ii). □

Theorem 9 (Inversion formula). *Let $g \in L^2(\mathbb{K}, \nu_\alpha)$ be a non-zero function. For all $f \in L^1 \cap L^2(\mathbb{K}, \nu_\alpha)$ such that $\mathscr{F}_W(f) \in L^1(\mathbb{K}, \nu_\alpha)$, we have*

$$f(z) = \frac{1}{\|g\|^2_{L^2(\mathbb{K},\nu_\alpha)}} \int_{\mathbb{K}} \mathscr{S}_g(f)(.,y) * g_y(z) d\nu_\alpha(y), \quad z \in \mathbb{K}.$$

Proof. By Theorem 6 (i), we have $\mathscr{S}_g(f)(.,y) \in L^2(\mathbb{K}, \nu_\alpha)$. Then, by Theorem 6 (ii), we obtain

$$\mathscr{S}_g(f)(.,y) * g_y(z) = \int_{\mathbb{K}} \Psi^\alpha_{-\xi}(z)\mathscr{F}_W(\mathscr{S}_g(f)(.,y))(\xi)\mathscr{F}_W(g_y)(\xi)\mathrm{d}\nu_\alpha(\xi).$$

But by Theorem 6 (i) and (5), we have

$$\mathscr{F}_W(\mathscr{S}_g(f)(.,y))(\xi) = \mathscr{F}_W(f)(\xi)\mathscr{F}_W(g_y)(\xi) = \mathscr{F}_W(f)(\xi)\sqrt{\tau_y|\mathscr{F}_W(g)|^2(\xi)}.$$

Thus,

$$\mathscr{S}_g(f)(.,y) * g_y(z) = \int_{\mathbb{K}} \Psi^\alpha_{-\xi}(z)\mathscr{F}_W(f)(\xi)\tau_y|\mathscr{F}_W(g)|^2(\xi)\mathrm{d}\nu_\alpha(\xi).$$

Therefore, by Fubini's theorem, Theorem 3 (ii) and (1), we deduce that

$$\begin{aligned}\int_{\mathbb{K}} \mathscr{S}_g(f)(.,y) * g_y(z)\mathrm{d}\nu_\alpha(y) &= \|g\|^2_{L^2(\mathbb{K},\alpha)} \int_{\mathbb{K}} \Psi_{-\xi}(z)\mathscr{F}_W(f)(\xi)\mathrm{d}\nu_\alpha(\xi)\\ &= \|g\|^2_{L^2(\mathbb{K},\nu_\alpha)} f(z).\end{aligned}$$

This completes the proof of the theorem. □

4. Calderón's reproducing formula for $\mathscr{S}_g$

In the following we establish a Calderón's reproducing inversion formula for the Weinstein-Gabor transform $\mathscr{S}_g$.

Let Δ be the pavement of $\mathbb{K}$ defined by $\Delta := \prod_{j=1}^{d}[a_j, b_j]$, where

$$-\infty < a_j < b_j < \infty, \quad j = 1, \dots, d-1, \quad 0 < a_d < b_d < \infty.$$

We use the notation $\Delta \to \mathbb{K}$ if and only if

$$a_j \to -\infty, \quad b_j \to \infty, \quad j = 1, \dots, d-1, \quad a_d \to 0, \quad b_d \to \infty.$$

Theorem 10 (Calderón's reproducing formula). *Let $g \in L^2(\mathbb{K}, \nu_\alpha)$ be a non-zero function, such that $\mathscr{F}_W(g) \in L^\infty(\mathbb{K}, \nu_\alpha)$. Then, for $f \in L^2(\mathbb{K}, \nu_\alpha)$, the function f_Δ given by*

$$f_\Delta(z) := \frac{1}{\|g\|^2_{L^2(\mathbb{K},\nu_\alpha)}} \int_\Delta \mathscr{S}_g(f)(.,y) * g_y(z) d\nu_\alpha(y), \quad z \in \mathbb{K},$$

belongs to $L^2(\mathbb{K}, \nu_\alpha)$ and satisfies

$$\lim_{\Delta \to \mathbb{K}} \|f_\Delta - f\|_{L^2(\mathbb{K},\nu_\alpha)} = 0. \tag{6}$$

Proof. By Theorem 6 (iii), we have $\mathscr{S}_g(f)(.,y) \in L^2(\mathbb{K}, \nu_\alpha)$, then by Theorem 6 (ii), we obtain

$$\mathscr{S}_g(f)(.,y) * g_y(z) = \int_{\mathbb{K}} \Psi^\alpha_{-\xi}(z) \mathscr{F}_W(\mathscr{S}_g(f)(.,y))(\xi) \mathscr{F}_W(g_y)(\xi) \mathrm{d}\nu_\alpha(\xi).$$

But by Theorem 6 (iii) and (5), we deduce that

$$\mathscr{F}_W(\mathscr{S}_g(f)(.,y))(\xi) = \mathscr{F}_W(f)(\xi)\mathscr{F}_W(g_y)(\xi) = \mathscr{F}_W(f)(\xi)\sqrt{\tau_y|\mathscr{F}_W(g)|^2(\xi)}.$$

Thus,

$$\mathscr{S}_g(f)(.,y) * g_y(z) = \int_{\mathbb{K}} \Psi^\alpha_{-\xi}(z) \mathscr{F}_W(f)(\xi) \tau_y |\mathscr{F}_W(g)|^2(\xi) \mathrm{d}\nu_\alpha(\xi),$$

and

$$f_\Delta(z) = \frac{1}{\|g\|^2_{L^2(\mathbb{K},\nu_\alpha)}} \int_\Delta \int_{\mathbb{K}} \Psi^\alpha_{-\xi}(z) \mathscr{F}_W(f)(\xi) \tau_y |\mathscr{F}_W(g)|^2(\xi) \mathrm{d}\nu_\alpha(\xi) \mathrm{d}\nu_\alpha(y).$$

Then, by Fubini's theorem we get

$$f_\Delta(z) = \int_{\mathbb{K}} \Psi^\alpha_{-\xi}(z) \mathscr{F}_W(f)(\xi) K_\Delta(\xi) \mathrm{d}\nu_\alpha(\xi), \tag{7}$$

where

$$K_\Delta(\xi) = \frac{1}{\|g\|^2_{L^2(\mathbb{K},\nu_\alpha)}} \int_\Delta \tau_y |\mathscr{F}_W(g)|^2(\xi) \mathrm{d}\nu_\alpha(y).$$

From (1), it is easily to see that

$$\|K_\Delta\|_{L^\infty(\mathbb{K},\nu_\alpha)} \leq 1.$$

On the other hand, by Hölder's inequality, we deduce that

$$|K_\Delta(\xi)|^2 \leq \frac{\nu_\alpha(\Delta)}{\|g\|^4_{L^2(\mathbb{K},\nu_\alpha)}} \int_\Delta |\tau_y |\mathscr{F}_W(g)|^2(\xi)|^2 \mathrm{d}\nu_\alpha(y).$$

Hence, by (2) we find

$$\|K_\Delta\|^2_{L^2(\mathbb{K},\nu_\alpha)} \leq \frac{(\nu_\alpha(\Delta))^2}{\|g\|^4_{L^2(\mathbb{K},\nu_\alpha)}} \int_{\mathbb{K}} |\mathscr{F}_W(g)(\xi)|^4 \mathrm{d}\nu_\alpha(\xi) \leq \frac{(\nu_\alpha(\Delta))^2 \|\mathscr{F}_W(g)\|^2_{L^\infty(\mathbb{K},\nu_\alpha)}}{\|g\|^2_{L^2(\mathbb{K},\nu_\alpha)}}.$$

Thus $K_\Delta \in L^\infty \cap L^2(\mathbb{K}, \nu_\alpha)$. Therefore and by (7), we have $f_\Delta = \mathscr{F}_W^{-1}(K_\Delta \mathscr{F}_W(f))$ and by Theorem 3 (i), $f_\Delta \in L^2(\mathbb{K}, \nu_\alpha)$ and

$$\mathscr{F}_W(f_\Delta) = K_\Delta \mathscr{F}_W(f).$$

From this relation it follows that

$$\|f_\Delta - f\|^2_{L^2(\mathbb{K},\nu_\Delta)} = \int_{\mathbb{K}} |\mathscr{F}_W(f)(\xi)|^2 (1 - K_\Delta(\xi))^2 d\nu_\alpha(\xi).$$

But by (1) we have

$$\lim_{\Delta \to \mathbb{K}} K_\Delta(\xi) = 1, \quad \text{for all } \xi \in \mathbb{K},$$

and

$$|\mathscr{F}_W(f)(\xi)|^2 (1 - K_\Delta(\xi))^2 \leq |\mathscr{F}_W(f)(\xi)|^2, \quad \text{for all } \xi \in \mathbb{K}.$$

So, the relation (6) follows from the dominated convergence theorem. □

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