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On a generalized class of q-Gegenbauer polynomials and subordination-defined bi-univalent functions

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Abstract: This work introduces a unique family of bi-univalent functions utilising q-Gegenbauer polynomials. The estimates of the initial coefficients $|a_2|$ and $|a_2|$ for functions in this new class, together with the Fekete-Szegö functional, have been obtained. Subsequent to the specialisation of the parameters utilised in our principal findings, many novel outcomes are presented.

Keywords: analytic functions, bi-univalent functions, subordination, q-Gegenbauer polynomials, Fekete-Szegö functional

1. Introduction

Orthogonal polynomials were first identified by Legendre in 1784 [1]. Ordinary differential equations are often resolved via orthogonal polynomials when specific model conditions are satisfied. Furthermore, the orthogonal polynomials [2] play a crucial role in approximation theory.

Two polynomials of order s and k, respectively, are Φ_s and Φ_k, and they are orthogonal if

$$\int_{\mathsf{a}}^{\mathsf{b}} \Phi_s(u)\Phi_k(u)\varpi(u)du = 0, \qquad \text{for} \qquad s \neq k.$$

Since $\varpi(x)$ is an appropriately defined function inside the interval $(\dot{a}, \ddot{b})$; hence, all finite order polynomials $\Phi_n(x)$ possess a well stated integral.

One particular class of orthogonal polynomials is Gegenbauer polynomials. Kiepiela et al. [3] states that the integral representation of typically real functions T_R is conceptually linked to the generating function of Gegenbauer polynomials when standard algebraic formulations are used. This undoubtedly caused certain advantageous inequalities to arise from the Gegenbauer polynomial domain.

Nowadays, the q-orthogonal polynomials are significant in science and mathematics due to the creation of quantum groups. The q-deformed harmonic oscillator has a group-theoretic basis for the q-Laguerre and q-Hermite polynomials. Jackson's q-exponential determines the structure of mathematics needed to describe the properties of these q-polynomials, including generating functions, orthogonality associations, and relations of recurrence. A closed-form studied series of conventional exponentials with known coefficients is what Quesne [4] recently described as Jackson's q-exponential. It is important to consider how this result can affect the theory of q-orthogonal polynomials.

This study made an effort to go in that direction. We utilised the aforementioned finding specifically to derive new nonlinear connection equations for q-Gegenbauer polynomials in relation to their classical counterparts. This study examined many characteristics of the class in question by correlating certain bi-univalent functions with q-Gegenbauer polynomials. This section establishes the groundwork for mathematical notations and definitions.

2. Preliminaries

Let $\mathcal{A}$ represent the set of all analytic functions f determined in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, normalised by the criteria $f'(0) - 1 = f(0) = 0$. Consequently, any function $f \in \mathcal{A}$ possesses a Taylor expansion represented as

$$f(z) = z + \sum_{l=2}^{\infty} a_l z^l, \quad (z \in \mathbb{U}). \tag{1}$$

Assume that $\mathcal{S}$ is the set of all functions $f \in \mathcal{A}$ that are univalent in $\mathbb{U}$.

Let the functions $\hbar(z)$ and $f(z)$ be analytic inside the domain $\mathbb{U}$. The function $f(z)$ is deemed subordinate to $\hbar(z)$, denoted as $f(z) \prec \hbar(z)$, if a Schwarz function κ analytic in $\mathbb{U}$ is found and satisfying

$$|\kappa(z)| < 1 \text{ and } \kappa(0) = 0 \qquad (z \in \mathbb{U}),$$

such that

$$\hbar(\kappa(z)) = f(z).$$

Furthermore, $f(z) \prec \hbar(z)$ iff the following inclusion holds

$$\hbar(0) = f(0) \text{ and } f(\mathbb{U}) \subset \hbar(\mathbb{U}).$$

The function $f \in \mathcal{S}$ has an inverse, which is stated as

$$z = f^{-1}(f(z)) \text{ and } f^{-1}(f(\varpi)) = \varpi \qquad \left(\mathrm{r}_0(f) \geq \frac{1}{4};\ |\varpi| < \mathrm{r}_0(f)\right),$$

where

$$f^{-1}(\varpi) = \varpi_1 + v_2\varpi^2 + v_3\varpi^3 + v_4\varpi^4 + \cdots, \tag{2}$$

with

$$v_2 = -a_2, v_3 = (2a_2^2 - a_3) \text{ and } v_4 = -(5a_2^3 - 5a_2a_3 + a_4).$$

In $\mathbb{U}$, a function is termed bi-univalent if both f and f^{-1} are univalent in the same domain.

As stated by (1), let Σ be the class of bi-univalent functions in $\mathbb{U}$. Functions in the class Σ include, for example,

$$\frac{z}{1-z}, \qquad \log\frac{1}{1-z}.$$

Fekete and Szegö defined an accurate limitation on the functional $a_3 - \sigma a_2^2$ for a univalent function f in 1933 [5]. In this constraint, σ is a real integer that satisfies $(0 \leq \sigma \leq 1)$. Since then, finding the precise limits for this functional of any closed class of functions $f \in \mathcal{A}$ with any complex σ has been considered the well-known Fekete-Szegö issue.

Askey and Ismail (1983) [6] discovered a family of polynomials that may be understood.

$$\mathfrak{G}_q^{(\epsilon)}(\nu, z) = \sum_{n=0}^{\infty} \mathcal{G}_l^{(\epsilon)}(\nu; q) z^l, \tag{3}$$

where $\nu \in [-1, 1]$ and $z \in \mathbb{U}$.

Chakrabarti et al. [7] discovered a category of polynomials in 2006 that, according to the relevant recurrence interaction, may be considered to be q-analogues of the Gegenbauer polynomials:

$$\begin{cases} \mathcal{G}_0^{(\epsilon)}(\nu; q) = 1 \\ \mathcal{G}_1^{(\epsilon)}(\nu; q) = 2[\epsilon]_q \nu \\ \mathcal{G}_2^{(\epsilon)}(\nu; q) = 2\left([\epsilon]_{q^2} + [\epsilon]_q^2\right)\nu^2 - [\epsilon]_{q^2} \end{cases} \tag{4}$$

Alatawi et al. (2023) [8] examined the standard Gegenbauer polynomials $\mathfrak{G}^{(\epsilon)}(\nu, z)$, where $\xi \in \mathbb{U}$ and $\nu \in [-1, 1]$. Since $\mathfrak{G}^{(\epsilon)}$ is analytic in $\mathbb{U}$ for constant ν, it could potentially be extended in a Taylor expansion as

$$\mathfrak{G}_q^{(\epsilon)}(\nu, z) = \sum_{n=0}^{\infty} \mathcal{G}_n^{\alpha}(\nu) z^l,$$

where $\mathcal{G}_n^{\alpha}(\nu)$ is the Gegenbauer polynomial of degree n.

Alatawi et al. [8] have presented subclasses of bi-univalent functions utilising Gegenbauer polynomials. Furthermore, Alsoboh et al. [9] employed the q-Gegenbauer polynomials associated with the generalisation of the q-Poisson distribution. Series to derive another bi-univalent function class. Fekete-Szegö inequalities and bounds for coefficients $|a_2|$ and $|a_3|$ have been determined for functions that fall into these subcategories.

Recently, several writers have begun the investigation of bi-univalent functions associated with orthogonal polynomials, with a limited selection to cite ([10–12]).

This study's main goal is to begin investigating the properties of bi-univalent functions related to q-Gegenbauer polynomials. We attain this by looking at the definitions given below.

3. Definition and examples

The following list includes several new subcategories of bi-univalent functions that rely on the q-Gegenbauer polynomial.

Definition 1. Let $0 \leq \varsigma \leq 1$, $\chi \in \mathbb{C} \setminus \{0\}$. According to (1), a function $f \in \Sigma$ belongs to the class $\mathcal{B}_{\Sigma}(\chi, \varsigma, \mathfrak{G}_q^{(\epsilon)}(\nu, z))$ if the following connections are confirmed:

$$1 + \frac{1}{\chi}\left(\partial_q f(z) + \varsigma z \partial_q^2 f(z) - 1\right) \prec \mathfrak{G}_q^{(\epsilon)}(\nu, z), \tag{5}$$

and

$$1 + \frac{1}{\chi}\left(\partial_q \hbar(\varpi) + \varsigma \varpi \partial_q^2 \hbar(\varpi) - 1\right) \prec \mathfrak{G}_q^{(\epsilon)}(\nu, \varpi), \tag{6}$$

where $\nu \in (\frac{1}{2}, 1]$, the function $\hbar(\varpi) = f^{-1}(\varpi)$ is defined by (2) and $\mathfrak{G}_q^{(\epsilon)}$ is the generating function of the q-Gegenbauer polynomial provided by (3).

Example 1. Let $\varsigma = 1$, $\chi \in \mathbb{C} \setminus \{0\}$. According to (1), a function $f \in \Sigma$ belongs to the class $\mathcal{B}_{\Sigma}(\chi, 1, \mathfrak{G}_q^{(\epsilon)}(\nu, z))$ if the following connections are confirmed:

$$1 + \frac{1}{\chi}\left(\partial_q f(z) + z \partial_q^2 f(z) - 1\right) \prec \mathfrak{G}_q^{(\epsilon)}(\nu, z),$$

and

$$1 + \frac{1}{\chi}\left(\partial_q \hbar(\varpi) + \varpi \partial_q^2 \hbar(\varpi) - 1\right) \prec \mathfrak{G}_q^{(\epsilon)}(\nu, \varpi),$$

where $\nu \in (\frac{1}{2}, 1]$, the function $\hbar(\varpi) = f^{-1}(\varpi)$ is defined by (2) and $\mathfrak{G}_q^{(\epsilon)}$ is the generating function of q-Gegenbauer polynomials given by (3).

Example 2. Let $\varsigma = 0$, $\chi \in \mathbb{C} \setminus \{0\}$. A function $f \in \Sigma$ given by (1) is considered in the class in the class $\mathcal{B}_{\Sigma}(\chi, 0, \mathfrak{G}_q^{(\epsilon)}(\nu, z))$ if the subsequent relationships are held:

$$1 + \frac{1}{\chi}\left(\partial_q f(z) - 1\right) \prec \mathfrak{G}_q,^{(\epsilon)}(\nu, z)$$

and

$$1 + \frac{1}{\chi}\left(\partial_q \hbar(\varpi) - 1\right) \prec \mathfrak{G}_q^{(\epsilon)}(\nu, \varpi),$$

where $\nu \in (\frac{1}{2}, 1]$, the function $\hbar(\varpi) = f^{-1}(\varpi)$ is defined by (2) and $\mathfrak{G}_q^{(\epsilon)}$ is the generating function of q-Gegenbauer polynomials given by (3).

4. Limits values of the class $\mathcal{B}_\Sigma(\chi,\varsigma,\mathfrak{G}_q^{(\epsilon)}(\nu,z))$

For the class $\mathcal{B}_\Sigma(\chi,\varsigma,\mathfrak{G}_q^{(\epsilon)}(\nu,z))$ provided in Definition 1, we first derive the coefficient estimates.

Theorem 1. *Let $f \in \Sigma$ given by (1) belongs to the class $\mathcal{B}_\Sigma(\chi,\varsigma,\mathfrak{G}_q^{(\epsilon)}(\nu,z))$. Then*

$$|a_2| \leq \frac{2|\chi[\epsilon]_q|\nu\sqrt{2[\epsilon]_q\nu}}{\sqrt{\left|\chi\left[\mathcal{G}_1^{(\epsilon)}(\nu;q)\right]^2([3]_q(1+\varsigma[2]_q)) - \mathcal{G}_2^{(\epsilon)}(\nu;q)([2]_q(1+\varsigma))^2\right|}},$$

and

$$|a_3| \leq \frac{4\chi^2[\epsilon]_q^2\nu^2}{([2]_q(1+\varsigma))^2} + \frac{2|\chi[\epsilon]_q|\nu}{[3]_q|1+[2]_q\varsigma|}.$$

Proof. Let $f \in \mathcal{B}_\Sigma(\chi,\varsigma,\mathfrak{G}_q^{(\epsilon)}(\nu,z))$. Then, for some analytic functions ω, v such that $\kappa(0) = v(0) = 0$, using Definition 1 for every $z,\omega \in \mathbb{U}$, and $|\kappa(z)| < 1$, $|v\kappa| < 1$, we consider

$$1 + \frac{1}{\chi}\left(\partial_q f(z) + \varsigma z\partial_q^2 f(z) - 1\right) = \mathfrak{G}_q^{(\epsilon)}(\nu,\kappa(z)), \tag{7}$$

and

$$1 + \frac{1}{\chi}\left(\partial_q \hbar(\varpi) + \varsigma\varpi\partial_q^2 \hbar(\varpi) - 1\right) = \mathfrak{G}_q^{(\epsilon)}(\nu,v(\omega)). \tag{8}$$

The equalities (7) and (8) leads to

$$1 + \frac{1}{\chi}\left(\partial_q f(z) + \varsigma z\partial_q^2 f(z) - 1\right) = 1 + \mathcal{G}_1^{(\epsilon)}(\nu;q)\mathcal{G}_1 z + \left[\mathcal{G}_1^{(\epsilon)}(\nu;q)\mathcal{G}_2 + \mathcal{G}_2^{(\epsilon)}(\nu;q)\mathcal{G}_1^2\right]z^2 + \cdots, \tag{9}$$

and

$$1 + \frac{1}{\chi}\left(\partial_q \hbar(\varpi) + \varsigma\varpi\partial_q^2 \hbar(\varpi) - 1\right) = 1 + \mathcal{G}_1^{(\epsilon)}(\nu;q)v_1\varpi + \left[\mathcal{G}_1^{(\epsilon)}(\nu;q)v_2 + \mathcal{G}_2^{(\epsilon)}(\nu;q)v_1^2\right]\varpi^2 + \cdots. \tag{10}$$

It is equitable familiar that if

$$|\kappa(z)| = \left|\mathcal{G}_1 z + \mathcal{G}_2 z^2 + \mathcal{G}_3 z^3 + \cdots\right| < 1, \quad (z \in \mathbb{U}),$$

and

$$|v(\varpi)| = \left|v_1\varpi + v_2\varpi^2 + v_3\varpi^3 + \cdots\right| < 1, \quad (\varpi \in \mathbb{U}),$$

then

$$|\mathcal{G}_j| \leq 1 \text{ and } |v_j| \leq 1 \text{ for all } j \in \mathbb{N}. \tag{11}$$

From (9), (10), (1), and (2), we derive

$$\begin{aligned}1 &+ \frac{1}{\chi}\Big([2]_q(1+\varsigma)\Big)a_2 z + \frac{1}{\chi}\Big([3]_q(1+[2]_q\varsigma)\Big)a_3 z^2 + \cdots \\ &= 1 + \mathcal{G}_1^{(\epsilon)}(\nu;q)\mathcal{G}_1 z + \left[\mathcal{G}_1^{(\epsilon)}(\nu;q)\mathcal{G}_2 + \mathcal{G}_2^{(\epsilon)}(\nu;q)\mathcal{G}_1^2\right]z^2 + \cdots,\end{aligned}$$

and

$$\begin{aligned}1 &- \frac{1}{\chi}\Big([2]_q\vartheta + 1\Big)a_2\varpi + \frac{1}{\chi}\Big([3]_q(1+[2]_q\varsigma)\Big)(2a_2^2 - a_3)\varpi^2 + \cdots \\ &= 1 + \mathcal{G}_1^{(\epsilon)}(\nu;q)v_1\varpi + \left[\mathcal{G}_1^{(\epsilon)}(\nu;q)v_2 + \mathcal{G}_2^{(\epsilon)}(\nu;q)v_1^2\right]\varpi^2 + \cdots.\end{aligned}$$

Consequently, comparing the upon coefficients in (9) and (10), we get

$$\frac{1}{\chi}\Big([2]_q(1+\varsigma)\Big)a_2 = \mathcal{G}_1^{(\epsilon)}(\nu;q)\mathcal{G}_1, \tag{12}$$

$$\frac{1}{\chi}\Big([3]_q(1+[2]_q\varsigma)\Big)a_3 = \mathcal{G}_1^{(\epsilon)}(\nu;q)\mathcal{G}_2 + \mathcal{G}_2^{(\epsilon)}(\nu;q)\mathcal{G}_1^2, \tag{13}$$

and

$$-\frac{1}{\chi}\Big([2]_q(1+\varsigma)\Big)a_2 = \mathcal{G}_1^{(\epsilon)}(\nu;q)v_1, \tag{14}$$

$$\frac{1}{\chi}\Big([3]_q(1+[2]_q\varsigma)\Big)(2a_2^2-a_3) = \mathcal{G}_1^{(\epsilon)}(\nu;q)v_2 + \mathcal{G}_2^{(\epsilon)}(\nu;q)v_1^2. \tag{15}$$

It follows from (12) and (14) that

$$\mathcal{G}_1 = -v_1, \tag{16}$$

and

$$\begin{aligned}\frac{2}{\chi^2}\Big([2]_q(1+\varsigma)\Big)^2 a_2^2 &= \left[\mathcal{G}_1^{(\epsilon)}(\nu;q)\right]^2\left(\mathcal{G}_1^2+v_1^2\right)\\ a_2^2 &= \frac{\chi^2\left[\mathcal{G}_1^{(\epsilon)}(\nu;q)\right]^2}{2\Big([2]_q(1+\varsigma)\Big)^2}\left(\mathcal{G}_1^2+v_1^2\right).\end{aligned} \tag{17}$$

Adding (13) and (15), we get

$$\frac{2}{\chi}\Big([3]_q(1+\varsigma[2]_q)\Big)a_2^2 = \mathcal{G}_1^{(\epsilon)}(\nu;q)(\mathcal{G}_2+v_2) + \mathcal{G}_2^{(\epsilon)}(\nu;q)(\mathcal{G}_1^2+v_1^2). \tag{18}$$

Substituting the value of $(\mathcal{G}_1^2+v_1^2)$ from (17), we obtain

$$a_2^2 = \frac{\chi^2\left[\mathcal{G}_1^{(\epsilon)}(\nu;q)\right]^3(\mathcal{G}_2+v_2)}{2\left[\chi\left[\mathcal{G}_1^{(\epsilon)}(\nu;q)\right]^2([3]_q(1+[2]_q\varsigma)) - \mathcal{G}_2^{(\epsilon)}(\nu;q)([2]_q(1+\varsigma))^2\right]}.$$

Applying for the coefficients $\mathcal{G}_2$ and v_2 and using (4), we obtain

$$|a_2| \leq \frac{2|\chi[\epsilon]_q|\nu\sqrt{2[\epsilon]_q\nu}}{\sqrt{\left|\chi\left[\mathcal{G}_1^{(\epsilon)}(\nu;q)\right]^2([3]_q(1+[2]_q\varsigma)) - \mathcal{G}_2^{(\epsilon)}(\nu;q)([2]_q(1+\varsigma))^2\right|}}.$$

By subtracting (15) from (13), we get

$$\frac{2}{\chi}\Big([3]_q(1+\varsigma[2]_q)\Big)(a_3-a_2^2) = \mathcal{G}_1^{(\epsilon)}(\nu;q)\left(\mathcal{G}_2-v_2\right) + \mathcal{G}_2^{(\epsilon)}(\nu;q)\left(\mathcal{G}_1^2-v_1^2\right). \tag{19}$$

Then, in view of (16) and (17), Eq. (19) becomes

$$a_3 = \frac{\chi^2\left[\mathcal{G}_1^{(\epsilon)}(\nu;q)\right]^2}{2\Big([2]_q(1+\varsigma)\Big)^2}\left(\mathcal{G}_1^2+v_1^2\right) + \frac{\chi\mathcal{G}_1^{(\epsilon)}(\nu;q)}{2([3]_q(1+[2]_q\varsigma))}\left(\mathcal{G}_2-v_2\right).$$

Thus applying (4), we conclude that

$$|a_3| \leq \frac{4\chi^2[\epsilon]_q^2\nu^2}{([2]_q(1+\varsigma))^2} + \frac{2|\chi[\epsilon]_q|\nu}{[3]_q|1+[2]_q\varsigma|}.$$

The Theorem's proof is now complete. □

Inspired by the investigation of Zaprawa [13], we derive the Fekete–Szegö functional for the class $\mathcal{B}_\Sigma(\chi,\varsigma,\mathfrak{G}_q^{(\epsilon)}(\nu,z))$.

Theorem 2. *Let $f\in\Sigma$ given by (1) belongs to the class $\mathcal{B}_\Sigma(\chi,\varsigma,\mathfrak{G}_q^{(\epsilon)}(\nu,z))$ and $\daleth\in\mathbb{R}$. Then, we have*

$$\left|a_3-\daleth a_2^2\right|\leq\begin{cases}\dfrac{2\left|\chi[\epsilon]_q\right|\nu}{[3]_q\left(1-[2]_q\vartheta\right)}, & |1-\daleth|\leq\left|1-\dfrac{\left([2]_q(1+\varsigma)\right)^2\left(2\left([\epsilon]_{q^2}+[\epsilon]_q^2\right)\nu^2-[\epsilon]_{q^2}\right)}{4[3]_q\chi\,[\epsilon]_q^2\,\nu^2\left(1+[2]_q\varsigma\right)}\right|,\\[2ex] 4\left|[\epsilon]_q\right|\nu\left|\mathcal{H}(\daleth)\right|, & |1-\daleth|\geq\left|1-\dfrac{\left([2]_q(1+\varsigma)\right)^2\left(2\left([\epsilon]_{q^2}+[\epsilon]_q^2\right)\nu^2-[\epsilon]_{q^2}\right)}{4[3]_q\chi\,[\epsilon]_q^2\,\nu^2\left(1+[2]_q\varsigma\right)}\right|,\end{cases}$$

where

$$\mathcal{H}(\daleth)=\frac{(1-\daleth)\chi^2\left[\mathcal{G}_1^{(\epsilon)}(\nu;q)\right]^2}{2\left[\chi\left[\mathcal{G}_1^{(\epsilon)}(\nu;q)\right]^2\left([3]_q(1+[2]_q\varsigma)\right)-\mathcal{G}_2^{(\epsilon)}(\nu;q)\left([2]_q(1+\varsigma)\right)^2\right]}.$$

Proof. If $f\in\mathcal{B}_\Sigma(\chi,\varsigma,\mathfrak{G}_q^{(\epsilon)}(\nu,z))$ is given by (1), from (18) and (19), we have

$$\begin{aligned}a_3-\daleth a_2^2=&\frac{\chi\mathcal{G}_1^{(\epsilon)}(\nu;q)}{2\left([3]_q(1+[2]_q\varsigma)\right)}\left(\mathcal{G}_2-v_2\right)\\&+\frac{(1-\daleth)\chi^2\left[\mathcal{G}_1^{(\epsilon)}(\nu;q)\right]^3\left(\mathcal{G}_2+v_2\right)}{2\left[\chi\left[\mathcal{G}_1^{(\epsilon)}(\nu;q)\right]^2\left([3]_q(1+[2]_q\varsigma)\right)-\mathcal{G}_2^{(\epsilon)}(\nu;q)\left([2]_q(1+\varsigma)\right)^2\right]}\\=&\mathcal{G}_1^{(\epsilon)}(\nu;q)\left[\left(\mathcal{H}(\daleth)+\frac{\chi}{2\left([3]_q(1+[2]_q\varsigma)\right)}\right)\mathcal{G}_2+\left(\mathcal{H}(\daleth)-\frac{\chi}{2\left([3]_q(1+[2]_q\varsigma)\right)}\right)v_2\right],\end{aligned}$$

where

$$\mathcal{H}(\daleth)=\frac{(1-\daleth)\chi^2\left[\mathcal{G}_1^{(\epsilon)}(\nu;q)\right]^2}{2\left[\chi\left[\mathcal{G}_1^{(\epsilon)}(\nu;q)\right]^2\left([3]_q(1+[2]_q\varsigma)\right)-\mathcal{G}_2^{(\epsilon)}(\nu;q)\left([2]_q(1+\varsigma)\right)^2\right]}.$$

Then, we conclude that

$$\left|a_3-\daleth a_2^2\right|\leq\begin{cases}\dfrac{\left|\chi\mathcal{G}_1^{(\epsilon)}(\nu;q)\right|}{[3]_q\left(1-[2]_q\vartheta\right)}, & |\mathcal{H}(\daleth)|\leq\dfrac{\chi}{2\left([3]_q(1+[2]_q\varsigma)\right)},\\[2ex] 2\left|\mathcal{G}_1^{(\epsilon)}(\nu;q)\right|\left|\mathcal{H}(\daleth)\right|, & |\mathcal{H}(\daleth)|\geq\dfrac{\chi}{2\left([3]_q(1+[2]_q\varsigma)\right)}.\end{cases}$$

The Theorem's 2 proof is now complete. □

5. Corollaries

Theorems 1 and 2 yield the subsequent corollaries, which closely correspond to Examples 1 and 2.

Corollary 1. *Let $f\in\Sigma$ given by (1) belongs to the class $\mathcal{B}_\Sigma(\chi,1,\mathfrak{G}_q^{(\epsilon)}(\nu,z))$.*

Then

$$|a_2| \leq \frac{2\left|\chi[\epsilon]_q\right|\nu\sqrt{2[\epsilon]_q\nu}}{\sqrt{\left|\chi\left[\mathcal{G}_1^{(\epsilon)}(\nu;q)\right]^2\left([3]_q(1+[2]_q)\right)-\mathcal{G}_2^{(\epsilon)}(\nu;q)\left(2[2]_q\right)^2\right|}}.$$

$$|a_3| \leq \frac{4\chi^2[\epsilon]_q^2\nu^2}{\left(2[2]_q\right)^2}+\frac{2\left|\chi[\epsilon]_q\right|\nu}{[3]_q\left|1+[2]_q\right|},$$

and

$$\left|a_3-\daleth a_2^2\right| \leq \begin{cases} \frac{2\left|\chi[\epsilon]_q\right|\nu}{[3]_q\left(1-[2]_q\vartheta\right)}, & |1-\daleth| \leq \left|1-\frac{\left(2[2]_q\right)^2\left(2\left([\epsilon]_{q^2}+[\epsilon]_q^2\right)\nu^2-[\epsilon]_{q^2}\right)}{4[3]_q\chi\,[\epsilon]_q^2\,\nu^2\left(1+[2]_q\right)}\right|, \\ 4\left|[\epsilon]_q\right|\nu\left|\mathcal{H}(\daleth)\right|, & |1-\daleth| \geq \left|1-\frac{\left(2[2]_q\right)^2\left(2\left([\epsilon]_{q^2}+[\epsilon]_q^2\right)\nu^2-[\epsilon]_{q^2}\right)}{4[3]_q\chi\,[\epsilon]_q^2\,\nu^2\left(1+[2]_q\right)}\right|, \end{cases}$$

where $\mathcal{H}(\daleth)=\frac{(1-\daleth)\chi^2\left[\mathcal{G}_1^{(\epsilon)}(\nu;q)\right]^2}{2\left[\chi\left[\mathcal{G}_1^{(\epsilon)}(\nu;q)\right]^2\left([3]_q(1+[2]_q)\right)-\mathcal{G}_2^{(\epsilon)}(\nu;q)\left(2[2]_q\right)^2\right]}$.

Corollary 2. *Let $f \in \Sigma$ given by (1) belongs to the class $\mathcal{B}_\Sigma(\chi,0,\mathfrak{G}_q^{(\epsilon)}(\nu,z))$. Then*

$$|a_2| \leq \frac{2\left|\chi[\epsilon]_q\right|\nu\sqrt{2[\epsilon]_q\nu}}{\sqrt{\left|\chi\left[\mathcal{G}_1^{(\epsilon)}(\nu;q)\right]^2\left([3]_q\right)-\mathcal{G}_2^{(\epsilon)}(\nu;q)\left([2]_q\right)^2\right|}},$$

$$|a_3| \leq \frac{4\chi^2[\epsilon]_q^2\nu^2}{\left([2]_q\right)^2}+\frac{2\left|\chi[\epsilon]_q\right|\nu}{[3]_q},$$

and

$$\left|a_3-\daleth a_2^2\right| \leq \begin{cases} \frac{2\left|\chi[\epsilon]_q\right|\nu}{[3]_q\left(1-[2]_q\vartheta\right)}, & |1-\daleth| \leq \left|1-\frac{\left([2]_q\right)^2\left(2\left([\epsilon]_{q^2}+[\epsilon]_q^2\right)\nu^2-[\epsilon]_{q^2}\right)}{4[3]_q\chi\,[\epsilon]_q^2\,\nu^2}\right|, \\ 4\left|[\epsilon]_q\right|\nu\left|\mathcal{H}(\daleth)\right|, & |1-\daleth| \geq \left|1-\frac{\left([2]_q\right)^2\left(2\left([\epsilon]_{q^2}+[\epsilon]_q^2\right)\nu^2-[\epsilon]_{q^2}\right)}{4[3]_q\chi\,[\epsilon]_q^2\,\nu^2}\right|, \end{cases}$$

where $\mathcal{H}(\daleth)=\frac{(1-\daleth)\chi^2\left[\mathcal{G}_1^{(\epsilon)}(\nu;q)\right]^2}{2\left[\chi\left[\mathcal{G}_1^{(\epsilon)}(\nu;q)\right]^2\left([3]_q\right)-\mathcal{G}_2^{(\epsilon)}(\nu;q)\left([2]_q\right)^2\right]}$.

6. Conclusion and concluding remark

The coefficient issues pertaining to each new subcategory of bi-univalent functions inside the open unit disc have been introduced and examined in this study. The formula is $\mathcal{B}_\Sigma(\chi,\varsigma,\mathfrak{G}_q^{(\epsilon)}(\nu,z))$, $\mathcal{B}_\Sigma(\chi,1,\mathfrak{G}_q^{(\epsilon)}(\nu,z))$ and $\mathcal{B}_\Sigma(\chi,0,\mathfrak{G}_q^{(\epsilon)}(\nu,z))$. Bi-univalent function classes are described in Definitions 1. We have computed the estimates of the Fekete-Szegö functional problems and the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for each of these three classes of bi-univalent functions. After fine-tuning the parameters pertinent to our primary findings, we reveal more distinctive results. Determining the bound of $|a_l|$ for $l \geq 4; l \in \mathbb{N}$ for the aforementioned classes remains difficult.

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