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Some spherical function values for two-row tableaux and Young subgroups with three factors

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Abstract: A Young subgroup of the symmetric group $\mathcal{S}_N$ with three factors, is realized as the stabilizer G_n of a monomial x^λ $\left(= x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_N^{\lambda_N}\right)$ with $\lambda = \left(d_1^{n_1}, d_2^{n_2}, d_3^{n_3}\right)$ (meaning d_j is repeated n_j times, $1 \leq j \leq 3$), thus is isomorphic to the direct product $\mathcal{S}_{n_1} \times \mathcal{S}_{n_2} \times \mathcal{S}_{n_3}$. The orbit of x^λ under the action of $\mathcal{S}_N$ (by permutation of coordinates) spans a module V_λ, the representation induced from the identity representation of G_n. The space V_λ decomposes into a direct sum of irreducible $\mathcal{S}_N$-modules. The spherical function is defined for each of these, it is the character of the module averaged over the group G_n. This paper concerns the value of certain spherical functions evaluated at a cycle which has no more than one entry in each of the three intervals $I_j = \left\{i : \lambda_i = d_j\right\}, 1 \leq j \leq 3$. These values appear in the study of eigenvalues of the Heckman-Polychronakos operators in the paper by V. Gorin and the author (arXiv:2412:01938v1). The present paper determines the spherical function values for $\mathcal{S}_N$-modules V of two-row tableau type, corresponding to Young tableaux of shape $[N-k, k]$. The method is based on analyzing the effect of a cycle on G_n-invariant elements of V. These are constructed in terms of Hahn polynomials in two variables.

Keywords: spherical function, symmetric group, Hahn polynomials

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1. Introduction

Spherical functions arise when an irreducible representation of a group contains the identity representation of a subgroup. This paper concerns the symmetric group and subgroups of Young type. Such groups are defined as stabilizer groups of particular monomials in the context of the symmetric group acting on polynomials by permutation of variables. Specifically we study the Young subgroup of $\mathcal{S}_N$ leaving each of three subintervals $I_1 = [1, n_1]$, $I_2 = [n_1 + 1, n_1 + n_2]$, $I_3 = [n_1 + n_2 + 1, N]$ setwise invariant , where $N = n_1 + n_2 + n_3$ and $\mathcal{S}_N$ is the symmetric group of permutations of $[1, N] = \{1, 2, \ldots, N\}$. The goal is to evaluate the spherical function for the isotype described by two-row tableaux at cycles which have at most one entry in each of the subintervals. This problem comes from a paper by Gorin and the author [1] which analyzed the eigenvalues of certain difference-differential operator.

The basic technique is to specify a submodule of polynomials realizing the isotype $[N-k, k]$ with $2k \leq N$, describe the polynomials invariant under the Young subgroup, act on each of these by the cycle of interest, and then project onto the space of invariants. The spherical function is then computed from this data. In the present situation the invariant polynomials are expressed with the aid of certain Hahn polynomials in two variables.

We begin with a brief sketch of the background from [1]. The commutative family of Heckman-Polychronakos operators is the set $\mathcal{P}_k := \sum_{i=1}^{N} \left(x_i \mathcal{D}_i\right)^k$ $(k \geq 1)$ in terms of Dunkl operators $\mathcal{D}_i f(x) := \frac{\partial}{\partial x_i} f(x) + \kappa \sum_{j=1, j \neq i}^{N} \frac{f(x) - f(x(i,j))}{x_i - x_j}$; $x(i,j)$ denotes x with x_i and x_j interchanged, and κ is a fixed parameter, often satisfying $\kappa > -\frac{1}{N}$ (see Heckman [2], Polychronakos [3]). For $\alpha \in \mathbb{Z}_+^N$, let $x^\alpha := \prod_{i=1}^{N} x_i^{\alpha_i}$. Suppose $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0$ then set $V_\lambda = \operatorname{span}_{\mathbb{F}} \left\{x^\beta : \beta = w\lambda, w \in \mathcal{S}_N\right\}$, that is, β ranges over the permutations of λ, and $\mathbb{F}$ is an extension field of $\mathbb{R}$ containing at least κ. The space V_λ is invariant under the action of $\mathcal{S}_N$. The eigenvalue analysis of $\mathcal{P}_k$ is derived from the restriction of $\mathcal{P}_k V_\lambda$ to V_λ (there is a triangularity based on the dominance order of partitions). Let $\lambda = \left(d_1^{n_1}, d_2^{n_2}, d_3^{n_3}\right)$ (that is, d_j is repeated n_j times, $1 \leq j \leq 3$), with $d_1 > d_2 > d_3 \geq 0$. Let $G_{\mathbf{n}}$ denote the stabilizer group of x^λ, so that $G_{\mathbf{n}} \cong \mathcal{S}_{n_1} \times \mathcal{S}_{n_2} \times \mathcal{S}_{n_3}$. The representation of $\mathcal{S}_N$ realized on V_λ is the induced representation $\operatorname{ind}_{G_{\mathbf{n}}}^{\mathcal{S}_N}$. The space V_λ can be decomposed into a direct sum of $\mathcal{S}_N$-invariant subspaces of various isotypes, which may appear as several copies. The number of copies (the multiplicity) of a particular isotype τ is called a *Kostka* number (see Macdonald [4]). That is, $V_\lambda = \sum_\tau \oplus V_{\lambda;\tau}$. Because $\mathcal{P}_k$ commutes with the group action the restriction of $\mathcal{P}_k V_{\lambda;\tau}$ to V_λ is contained in $V_{\lambda;\tau}$. If the multiplicity of the isotype τ in V_λ is greater than one then the eigenvalues of $\mathcal{P}_k$ realized on $V_{\lambda;\tau}$ are generally not rational in the parameters, but the sum of all the eigenvalues (for any fixed k) can be explicitly found, in terms of the character of τ. In general this may not have a simple explicit form . A closed form was found for hook isotypes, labeled by partitions of the form $\left[N-b, 1^b\right]$ (by the Dunkl [5], for the more general $G_{\mathbf{n}} \cong \mathcal{S}_{n_1} \times \mathcal{S}_{n_{2\ldots}} \times \cdots \times \mathcal{S}_{n_p}$). The formula is based on considering cycles corresponding to subsets $\mathcal{A} = \{a_1, \ldots, a_\ell\}$ of $\{1,2,3\}$, which are of length ℓ with exactly one entry from each interval I_{a_j}. Any such cycle can be used and the order of $a_1, \cdots, a_\ell$ is immaterial. The degrees d_1, d_2, d_3 enter the formula in a shifted way:

$$\widetilde{d}_1 := d_1 + \kappa\left(n_2 + n_3\right), \widetilde{d}_2 := d_2 + \kappa n_3, \widetilde{d}_3 := d_3.$$

Let $h_m^{\mathcal{A}} := h_m\left(\widetilde{d}_{a_1}, \widetilde{d}_{a_2}, \ldots, \widetilde{d}_{a_\ell}\right)$, the complete symmetric polynomial of degree m (the generating function is $\sum_{k\geq 0} h_k\left(c_1, c_2, \ldots, c_q\right) t^k = \prod_{i=1}^q \left(1 - c_i t\right)^{-1}$, see [4]). Denote the character of the representation τ of $\mathcal{S}_N$ by $\chi^\tau(w)$ then the spherical function

$$\Phi^\tau\left(g_{\mathcal{A}}\right) := \frac{1}{\# G_{\mathbf{n}}} \sum_{h \in G_{\mathbf{n}}} \chi^\tau\left(g_{\mathcal{A}}\, h\right),$$

where $g_{\mathcal{A}}$ is an ℓ-cycle labeled by $\mathcal{A}$ as above, and $\# G_{\mathbf{n}} = \prod_{i=1}^3 n_i!$. In [1] the spherical function Φ^τ is denoted by $\chi^\tau\left[\mathcal{A}; \mathbf{n}\right]$, and called an "averaged character."

Now suppose the multiplicity of τ in V_λ is μ then there are $\mu \dim \tau$ eigenfunctions and eigenvalues of $\mathcal{P}_k$, and the sum of all these eigenvalues is ([1, Theorem 5.4])

$$\dim \tau \sum_{\ell=1}^{\min(k+1,3)} (-\kappa)^{\ell-1} \sum_{\mathcal{A} \subset \{1,2,3\}, \# \mathcal{A} = \ell} \Phi^\tau\left(g_{\mathcal{A}}\right) h_{k+1-\ell}^{\mathcal{A}} \prod_{i \in \mathcal{A}} n_i!.$$

Here is an outline of the paper. §2 reviews some general results about spherical functions and the formula proven in [5] which is the basic tool for the computations. The derivation starts with the construction of an irreducible module of polynomials of isotype τ (for practical reasons we choose such a module of minimum polynomial degree). The invariant polynomials in V are described in terms of elementary symmetric polynomials. The dimension of the subspace of invariants is found in terms of the parameters n_1, n_2, n_3 (by Frobenius reciprocity the dimension is the same as μ, the multiplicity of $[N-k, k]$ in V_λ). In §3 the two-variable Hahn polynomials are defined and a basis for the invariants is constructed. §4 determines the spherical functions for the 2-cycles. Lastly §5 produces the spherical function for 3-cycles and also has a discussion (§5.1) about some specific examples, especially those with multiplicity one.

2. Spherical functions and invariants

Definition 1. The action of the symmetric group $\mathcal{S}_N$ on polynomials $P(x)$ is given by $wP(x) = P(xw)$ and $(xw)_i = x_{w(i)}, w \in \mathcal{S}_N, 1 \leq i \leq N$.

Note $(x(vw))_i = (xv)_{w(i)} = x_{v(w(i))} = x_{vw(i)}$, $vwP(x) = (wP)(xv) = P(xvw)$. The projection onto $G_{\mathbf{n}}$-invariant polynomials is given by

$$\rho P(x) := \frac{1}{\# G_{\mathbf{n}}} \sum_{h \in G_{\mathbf{n}}} P(xh).$$

Let $\lambda = \left(d_1^{n_1}, d_2^{n_2}, d_3^{n_3}\right)$ (that is, d_j is repeated n_j times, $1 \le j \le 3$), with $d_1 > d_2 > d_3 \ge 0$. Let $G_{\mathbf{n}}$ denote the stabilizer group of x^λ, so that $G_{\mathbf{n}} \cong \mathcal{S}_{n_1} \times \mathcal{S}_{n_2} \times \mathcal{S}_{n_3}$. Suppose that M_τ is an $\mathcal{S}_N$-module of isotype τ and that $\left\{\psi_j : 1 \le j \le \mu\right\}$ is a basis for the $G_{\mathbf{n}}$-invariants.

Proposition 1. [5, Cor. 2] *Suppose* $g \in \mathcal{S}_N$ *and* $\rho g \xi_i = \sum\limits_{j=1}^{\mu} B_{ji}(g)\, \xi_j$ $(1 \le i, j \le \mu)$ *then* $\Phi^\tau(g) = \operatorname{tr}(B(g))$.

The key fact is that $\rho g \xi_i$ is itself an invariant and thus has a unique expansion in the basis $\left\{\psi_j : 1 \le j \le \mu\right\}$. The approach used in what follows is to determine the action of ρg on each basis element for the cycles described above.

For $1 \le k \le \frac{N}{2}$ let $E \subset \{1, 2, \ldots, N\}$ with $\#E = k$ and let $m_E := \prod\limits_{i \in E} x_i$, and

$$V_k := \left\{ p = \sum_{\#E=k} c_E m_E : \sum_{j=1}^{N} \frac{\partial}{\partial x_i} p = 0 \right\}.$$

Then V_k is of isotype $[N-k, k]$ (an irreducible $\mathcal{S}_N$-module of dimension $\binom{N}{k} - \binom{N}{k-1}$).

To clearly display the action of $G_{\mathbf{n}}$ we introduce a modified coordinate system. Replace

$$(x_1, x_2, \ldots, x_N)\,\tilde{}\, \left(x_1^{(1)}, \ldots, x_{n_1}^{(1)}, x_1^{(2)}, \ldots, x_{n_2}^{(2)}, x_1^{(3)}, \ldots, x_{n_p}^{(3)}\right),$$

that is, $x_i^{(j)}$ stands for x_s with $s = \sum\limits_{i=1}^{j-1} n_i + i$. We use $x_*^{(i)}, x_>^{(i)}$ to denote a generic $x_j^{(i)}$ with $1 \le j \le n_i$, respectively $2 \le j \le n_i$. In the sequel g_ℓ denotes the cycle $\left(x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(\ell)}\right)$ (with $2 \le \ell \le 3$). Let $e_i\left(x_*^{(j)}\right), e_i\left(x_>^{(j)}\right)$ be defined by $\prod\limits_{i=1}^{n_j}\left(1 + t x_i^{(j)}\right) = \sum\limits_{i=0}^{n_i} t^i e_i\left(x_*^{(j)}\right)$, respectively $\prod\limits_{i=2}^{n_j}\left(1 + t x_i^{(j)}\right) = \sum\limits_{i=0}^{n_j-1} t^i e_i\left(x_>^{(j)}\right)$ (elementary symmetric functions).

Lemma 1. $\rho\left(x_1^{(j)} e_{i-1}\left(x_>^{(j)}\right)\right) = \dfrac{i}{n_j} e_i\left(x_*^{(j)}\right)$ *and* $\rho\left(e_i\left(x_>^{(j)}\right)\right) = \dfrac{n_j - i}{n_j} e_i\left(x_*^{(j)}\right)$.

Proof. Let $p = x_{s_1}^{(j)} x_{s_2}^{(j)} \cdots x_{s_i}^{(j)}$ (with $s_1 < \ldots < s_i$) then $\rho p = \binom{n_j}{i}^{-1} e_i\left(x_*^{(j)}\right)$, because $e_i\left(x_*^{(j)}\right)$ is the sum of $\binom{n_j}{i}$ monomials. There are $\binom{n_j-1}{i-1}$ monomials in $x_1^{(j)} e_{i-1}\left(x_>^{(j)}\right)$ and $\binom{n_j-1}{i-1} / \binom{n_j}{i} = \dfrac{i}{n_j}$. There are $\binom{n_j-1}{i}$ monomials in $e_i\left(x_>^{(j)}\right)$ and $\binom{n_j-1}{i} / \binom{n_j}{i} = \dfrac{n_j - i}{n_j}$. $\square$

Proposition 2. [6, Prop. 2.1] *A polynomial* $f(u, v)$ *satisfies*

$$(u - n_1) f(u+1, v) + (v - n_2) f(u, v+1) - (n_3 - k + 1 + u + v) f(u, v) = 0,$$

for $0 \le u \le n_1,\ 0 \le v \le n_2,\ k - n_3 \le u + v \le k$ *, if and only if*

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \sum_{u, v, u+v \le k} f(u, v)\, e_u\left(x_*^{(1)}\right) e_v\left(x_*^{(2)}\right) e_{k-u-v}\left(x_*^{(3)}\right) = 0,$$

that is, the inner sum is an element of V_k.

The formula describes the space of $G_{\mathbf{n}}$-invariant polynomials of isotype $[N-k, k]$.

3. Hahn polynomials in two variables

One convenient orthogonal basis for V_k is defined in terms of Hahn polynomials (see [6]) (the Pochhammer symbol is $(\alpha)_j = \prod_{i=1}^{j}(\alpha+i-1)$)

$$E_m(\alpha,\beta,\gamma,t) := \sum_{i=0}^{m}(-1)^i\binom{m}{i}(\beta-m+1)_i(\alpha-m+1)_{m-i}(-t)_i(t-\gamma)_{m-i},$$

then the basis element ψ_m is given by (in two parts for later convenience)

$$\begin{aligned}\widetilde{\psi}_m^{(1)}(t) &:= E_{k-m}(n_3, n_1+n_2-2m, k-m, k-t)\\ &= \sum_{j=0}^{k-m}\frac{(m-k)_j}{j!}(n_1+n_2-k-m+1)_j(n_3-k+m+1)_{k-m-j}\times(t-k)_j(m-t)_{k-m-j},\end{aligned} \tag{1}$$

$$\begin{cases}\widetilde{\psi}_m^{(2)}(u,v) & := E_m(n_2,n_1,u+v,v) = \sum_{i=0}^{m}\frac{(-m)_i}{i!}(n_1-m+1)_i(n_2-m+1)_{m-i}(-v)_i(-u)_{m-i},\\ \widetilde{\psi}_m(u,v) & := \widetilde{\psi}_m^{(1)}(u+v)\,\widetilde{\psi}_m^{(2)}(u,v),\end{cases} \tag{2}$$

for $0\vee(k-n_3)\leq m\leq n_1\wedge n_2\wedge k\wedge(n_1+n_2-k)$ (the number of these points is the multiplicity of $1_{G_\mathbf{n}}$ in $[N-k,k]$).

Proposition 3. *A basis for the $G_\mathbf{n}$-invariants is given by*

$$\psi_m(x) := \sum_{u,v,u+v\leq k}\widetilde{\psi}_m(u,v)\,e_u\left(x_*^{(1)}\right)e_v\left(x_*^{(2)}\right)e_{k-u-v}\left(x_*^{(3)}\right).$$

This follows from the fact that $\widetilde{\psi}_m(u,v)$ satisfies the difference equation in Proposition 2 (see [6, p.63, (3.11)]).

There are useful special values:

$$\begin{cases}\widetilde{\psi}_m^{(1)}(m) & = (k-m)!\,(n_1+n_2-k-m+1)_{k-m},\\ \widetilde{\psi}_m^{(2)}(u,0) & = (n_2-m+1)_m(-u)_m = (-1)^m(-n_2)_m(-u)_m,\\ \widetilde{\psi}_m(m,0) & = (-1)^{k-m}(m-n_1-n_2)_{k-m}(-n_2)_m\,m!\,(k-m)!.\end{cases} \tag{3}$$

Lemma 2. *If $u+v<m$ then $\widetilde{\psi}_m^{(2)}(u,v)=0$.*

Proof. The term for i in $\widetilde{\psi}_m^{(2)}(u,v)$ is nonzero only if $i\leq v$ and $m-i\leq u$, that is, $m-u\leq i\leq v$. □

An analogous structure is known for isotypes of 3-part partitions, due to Scarabotti [7]. We are not pursuing this situation here because of the complexity due to the added dimension and the numerous conditions on the parameters.

4. Spherical functions at a 2-cycle

For an invariant polynomial $\psi(x) = \sum_{u,v,u+v\leq k} f(u,v)\,e_u\left(x_*^{(1)}\right)e_v\left(x_*^{(2)}\right)e_{k-u-v}\left(x_*^{(3)}\right)$ we will determine $\rho\psi(xg_2)$ where $g_2=\left(x_1^{(1)},x_1^{(2)}\right)$. In this section we will show that $\rho g_2\psi_m = \frac{1}{n_1n_2}\{(m-n_1)(m-n_2)-m\}\psi_m$ for each m. This coefficient is then used in Proposition 1. Compute term-by-term. Let

$$\begin{aligned}p &= e_u\left(x_*^{(1)}\right)e_v\left(x_*^{(2)}\right)e_{k-u-v}\left(x_*^{(3)}\right)\\ &= \left\{x_1^{(1)}e_{u-1}\left(x_>^{(1)}\right)+e_u\left(x_>^{(1)}\right)\right\}\left\{x_1^{(2)}e_{v-1}\left(x_>^{(2)}\right)+e_v\left(x_>^{(2)}\right)\right\}e_{k-u-v}\left(x_*^{(3)}\right)\\ g_2p &= x_1^{(1)}e_{u-1}\left(x_>^{(1)}\right)x_1^{(2)}e_{v-1}\left(x_>^{(2)}\right)e_{k-u-v}\left(x_*^{(3)}\right)+x_1^{(1)}e_u\left(x_>^{(1)}\right)e_{v-1}\left(x_>^{(2)}\right)e_{k-u-v}\left(x_*^{(3)}\right)\end{aligned}$$

$$+e_{u-1}\left(x_{>}^{(1)}\right)x_1^{(2)}e_v\left(x_{>}^{(2)}\right)e_{k-u-v}\left(x_*^{(3)}\right)+e_u\left(x_{>}^{(1)}\right)e_v\left(x_{>}^{(2)}\right)e_{k-u-v}\left(x_*^{(3)}\right).$$

Apply ρ and use Lemma 1

$$\begin{aligned}\rho g_2 p =&\frac{1}{n_1n_2}\left\{\left((n_1-u)(n_2-v)+uv\right)e_u\left(x_*^{(1)}\right)e_v\left(x_*^{(2)}\right)e_{k-u-v}\left(x_*^{(3)}\right)\right.\\&+(u+1)(n_2-v+1)e_{u+1}\left(x_*^{(1)}\right)e_{v-1}\left(x_*^{(2)}\right)e_{k-u-v}\left(x_*^{(3)}\right)\\&\left.+(n_1-u+1)(v+1)e_{u-1}\left(x_*^{(1)}\right)e_{v+1}\left(x_*^{(2)}\right)e_{k-u-v}\left(x_*^{(3)}\right)\right\}.\end{aligned}$$

Thus

$$\begin{aligned}\rho g_2\psi =&\sum_{u,v,u+v\le k}\left\{\left((n_1-u)(n_2-v)+uv\right)f(u,v)+u(n_2-v)f(u-1,v+1)+(n_1-u)vf(u+1,v-1)\right\}\\&\times e_u\left(x_*^{(1)}\right)e_v\left(x_*^{(2)}\right)e_{k-u-v}\left(x_*^{(3)}\right).\end{aligned}$$

Observe that values like $f(-1,v+1)$ or $f(n_1+1,v)$ do not appear. Let $f(u,v)=\widetilde{\psi}_m^{(1)}(u+v)\,\widetilde{\psi}_m^{(2)}(u,v)$ from (1), (2). By Lemma 2 $u+v<m$ implies $\widetilde{\psi}_m^{(2)}(u,v)=0$.

Let $C(u,v,i)$ denote the i-term in the sum (2) for $\widetilde{\psi}_m^{(2)}(u,v)$. We will express $u(n_2-v)C(u-1,v+1)+(n_1-u)vC(u+1,v-1,i)$ in terms of $C(u,v,i-1),C(u,v,i+1)$ and $C(u,v,i)$. It is more readable to display ratios like $C(u+1,v-1,i)/C(u,v,i)$ (resulting from straightforward calculations)

$$\begin{aligned}a_1(i)&:=u(n_2-v)\frac{C(u-1,v+1,i)}{C(u,v,i)}=\frac{(n_2-v)(v+1)(m-i-u)}{i-v-1},\\a_2(i)&:=(n_1-u)v\frac{C(u+1,v-1,i)}{C(u,v,i)}=\frac{(n_1-u)(i-v)(u+1)}{m-u-i-1},\\b_1(i)&:=-(m+1-i)(n_1-m+i)\frac{C(u,v,i-1)}{C(u,v,i)}=\frac{i(m-i-u)(n_2+1-i)}{i-v-1},\\b_2(i)&:=-(i+1)(n_2-i)\frac{C(u,v,i+1)}{C(u,v,i)}=\frac{(-m+i)(i-v)(n_1-m+1+i)}{1+i+u-m}.\end{aligned}$$

Then $a_1(i)-b_1(i)=(i-v+n_2)(m-i-u)$, $a_2(i)-b_2(i)=(i-m+n_1-u)(v-i)$. Thus

$$\begin{aligned}&u(n_2-v)C(u-1,v+1,i)+(n_1-u)vC(u+1,v-1,i)\\&\qquad=-(m+1-i)(n_1-m+i)C(u,v,i-1)-(i+1)(n_2-i)C(u,v,i+1)\\&\qquad\quad+\left(a_1(i)-b_1(i)+a_2(i)-b_2(i)\right)C(u,v,i).\end{aligned}$$

Apply $\sum_{i=0}^{m}$ to each line: the first line gives $u(n_2-v)\widetilde{\psi}_m^{(2)}(u-1,v+1)+(n_1-u)v\widetilde{\psi}_m^{(2)}(u+1,v-1)$, the second and third yield

$$\begin{aligned}&-\sum_{i=1}^{m}(m+1-i)(n_1-m+i)C(u,v,i-1)-\sum_{i=0}^{m-1}(i+1)(n_2-i)C(u,v,i+1)\\&+\sum_{i=0}^{m}\left(a_1(i)-b_1(i)+a_2(i)-b_2(i)\right)C(u,v,i)\\&=\sum_{i=0}^{m}\left\{(i-m)(n_1-m+1-i)-i(n_2-i)+a_1(i)-b_1(i)+a_2(i)-b_2(i)\right\}C(u,v,i)\\&=\sum_{i=0}^{m}\left\{m(m-n_1-n_2-1)+n_2u+n_1v-2uv\right\}C(u,v,i),\end{aligned}$$

a multiple of $\widetilde{\psi}_m^{(2)}(u,v)$. The changes of summation variable are valid since $b_1(0)=0=b_2(m)$. Now add $\left((n_1-u)(n_2-v)+uv\right)\widetilde{\psi}_m^{(2)}(u,v)$ to both sides and obtain $\rho g_2\psi_m=\frac{1}{n_1n_2}\left\{(m-n_1)(m-n_2)-m\right\}\psi_m$.

Let $m_L := 0 \vee (k - n_3)$ and $m_U := n_1 \wedge n_2 \wedge k \wedge (n_1 + n_2 - k)$. Thus the spherical function

$$\Phi^{[N-k,k]}(g_2) = \frac{1}{n_1 n_2} \sum_{m=m_L}^{m_U} \{(m - n_1)(m - n_2) - m\},$$

if $k \le n_1 \wedge n_2 \wedge n_3$ then $m_L = 0, m_U = k$ and

$$\Phi^{[N-k,k]}(g_2) = \frac{k+1}{n_1 n_2} \left(n_1 n_2 - \frac{1}{2} k (n_1 + n_2) + \frac{1}{3} k (k-1) \right).$$

More generally let $\mu = m_U - m_L$ then

$$\Phi^{[N-k,k]}(g_2) = \frac{\mu+1}{n_1 n_2} \left(m_L^2 + n_1 n_2 - \left(m_L + \frac{\mu}{2} \right)(n_1 + n_2) + \left(m_L + \frac{\mu}{3} \right)(\mu - 1) \right). \tag{4}$$

The corresponding situations for the 2-cycles $\left(x_1^{(2)}, x_1^{(3)}\right)$ and $\left(x_1^{(1)}, x_1^{(3)}\right)$ are obtained by suitably permuting the parameters n_1, n_2, n_3 in the formula. The multiplicity of $1_{G_{\mathbf{n}}}$ in V_k is symmetric in $\{n_i\}$, namely $\min\{k, n_1, \ldots, n_1 + n_2 - k, \ldots\} + 1$.

4.1. Case $n_1 + n_2 = N$

The same scheme can be used for $\mathbf{n} = (n_1, n_2)$ (with $N = n_1 + n_2$): the multiplicity of $[N-k, k]$ is one for $0 \le k \le n_1 \wedge n_2$, the unique invariant polynomial (with $\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \psi(x) = 0$) is

$$\begin{aligned}
\psi(x) &:= \sum_{u=0}^{k} f(u)\, e_u\left(x_*^{(1)}\right) e_{k-u}\left(x_*^{(2)}\right),\\
f(u) &:= (-1)^u (n_2 - k + 1)_u (n_1 - k + 1)_{k-u},
\end{aligned}$$

and with a similar calculation to the previous one

$$\rho g_2 \psi = \sum_{u=0}^{k} \{((n_1 - u)(n_2 - v) + uv) f(u) + u(n_2 - v) f(u-1) + v(n_1 - u) f(u+1)\}\, e_u\left(x_*^{(1)}\right) e_{k-u}\left(x_*^{(2)}\right),$$

with $v = k - u$. We find

$$u(n_2 - v)\frac{f(u-1)}{f(u)} + v(n_1 - u)\frac{f(u+1)}{f(u)} = -u(n_1 + 1 - u) - (k-u)(n_2 - k + 1 + u),$$

and adding $(n_1 - u)(n_2 - k - u) + u(k - u)$ to both sides we obtain

$$\begin{aligned}
\rho g_2 \psi &= \left(n_1 n_2 - (n_1 + n_2) k + k^2 - k \right) \sum_{u=0}^{k} f(u)\, e_u\left(x_*^{(1)}\right) e_{k-u}\left(x_*^{(2)}\right)\\
&= \left(n_1 n_2 - (n_1 + n_2) k + k^2 - k \right) \psi.
\end{aligned}$$

Thus the spherical function

$$\Phi^{[N-k,k]}(g_2) = \frac{1}{n_1 n_2} \left(n_1 n_2 - (n_1 + n_2) k + k^2 - k \right).$$

5. Spherical functions at a 3-cycle

We use the 3 -cycle $g_3 = \left(x_1^{(1)}, x_1^{(2)}, x_1^{(3)}\right)$. We will determine $\rho\psi(xg_3)$ for an invariant polynomial

$$\psi(x) = \sum_{u,v,u+v \le k} f(u,v) \left\{ x_1^{(1)} e_{u-1}\left(x_>^{(1)}\right) + e_u\left(x_>^{(1)}\right) \right\} \left\{ x_1^{(2)} e_{v-1}\left(x_>^{(2)}\right) + e_v\left(x_>^{(2)}\right) \right\}$$

$$\times\left\{x_1^{(3)}e_{k-u-v-1}\left(x_>^{(3)}\right)+e_{k-u-v}\left(x_>^{(3)}\right)\right\}.$$

The computation is quite a bit more involved than the 2-cycle case. Apply g_3 to the (u,v)-term; there are 8 terms in the expansion (and abbreviate $k-u-v=w$)

$$\begin{aligned}
&x_1^{(2)}e_{u-1}\left(x_>^{(1)}\right)x_1^{(3)}e_{v-1}\left(x_>^{(2)}\right)x_1^{(1)}e_{w-1}\left(x_>^{(3)}\right)+x_1^{(2)}e_{u-1}\left(x_>^{(1)}\right)x_1^{(3)}e_{v-1}\left(x_>^{(2)}\right)e_w\left(x_>^{(3)}\right)\\
&+x_1^{(2)}e_{u-1}\left(x_>^{(1)}\right)e_v\left(x_>^{(2)}\right)x_1^{(1)}e_{w-1}\left(x_>^{(3)}\right)+x_1^{(2)}e_{u-1}\left(x_>^{(1)}\right)e_v\left(x_>^{(2)}\right)e_w\left(x_>^{(3)}\right)\\
&+e_u\left(x_>^{(1)}\right)x_1^{(3)}e_{v-1}\left(x_>^{(2)}\right)x_1^{(1)}e_{w-1}\left(x_>^{(3)}\right)+e_u\left(x_>^{(1)}\right)x_1^{(3)}e_{v-1}\left(x_>^{(2)}\right)e_w\left(x_>^{(3)}\right)\\
&+e_u\left(x_>^{(1)}\right)e_v\left(x_>^{(2)}\right)x_1^{(1)}e_{w-1}\left(x_>^{(3)}\right)+e_u\left(x_>^{(1)}\right)e_v\left(x_>^{(2)}\right)e_w\left(x_>^{(3)}\right).
\end{aligned}$$

Symmetrize each term using Lemma 1, and denote

$$P(u,v,w):=\frac{1}{n_1n_2n_3}e_u\left(x_*^{(1)}\right)e_v\left(x_*^{(2)}\right)e_w\left(x_*^{(3)}\right)$$

:

$$\begin{aligned}
&uvwP(u,v,w)+(n_1-u+1)\,v\,(w+1)\,P(u-1,v,w+1)\\
&+u\,(v+1)\,(n_3-w+1)\,P(u,v+1,w-1)+(n_1-u+1)\,(v+1)\,(n_3-w)\,P(u-1,v+1,w)\\
&+(u+1)\,(n_2-v+1)\,wP(u+1,v-1,w)+(n_1-u)\,(n_2-v+1)\,(w+1)\,P(u,v-1,w+1)\\
&+(u+1)\,(n_2-v)\,(n_3-w+1)\,P(u+1,v,w-1)+(n_1-u)\,(n_2-v)\,(n_3-w)\,P(u,v,w)
\end{aligned}$$

respectively. Changing indices as appropriate we obtain

$$\begin{aligned}
\rho\psi(xg_3)=\sum_{u,v,u+v\le k}&P(u,v,w)\left\{uvwf(u,v)+(n_1-u)\,vwf(u+1,v)\right.\\
&+uv\,(n_3-w)\,f(u,v-1)+(n_1-u)\,v\,(n_3-w)\,f(u+1,v-1)\\
&+u\,(n_2-v)\,wf(u-1,v+1)+(n_1-u)\,(n_2-v)\,wf(u,v+1)\\
&\left.+u\,(n_2-v)\,(n_3-w)\,f(u-1,v)+(n_1-u)\,(n_2-v)\,(n_3-w)\,f(u,v)\right\}.
\end{aligned}$$

Mow set $f(u,v)=\widetilde{\psi}_m(u,v)$ and determine the coefficient c_m in $\rho\psi_m(xg_3)=\sum_n c_n\psi_n(x)$ (equivalently the expression in $\{\cdot\}$ denoted $S_m(u,v)$ equals $\sum_n c_n\widetilde{\psi}_n(u,v)$).

In $\psi_m^{(1)}(m)$ only the $j=k-m$ term is nonzero, and in $\widetilde{\psi}_m^{(2)}(u,0)$ only the $i=0$ term is nonzero. Furthermore $u<m$ implies $\widetilde{\psi}_m(u,0)=0$, (because of the factor $(-u)_m$ in $\widetilde{\psi}_m^{(2)}(u,v)$). By Lemma 2 $u+v<n$ implies $\widetilde{\psi}_n(u,v)=0$. Thus $S_m(u,0)=0$ for $u<m-1$ (note the terms $vf(*,v-1)=0$). The following is used to determine the coefficients needed for Proposition 1.

Proposition 4. *Suppose* $f=\sum_{n=0}^{k}c_n\widetilde{\psi}_n$ *and* $f(u,0)=0$ *for* $u<m-1$ *then*

$$c_m=\frac{1}{\widetilde{\psi}_m(m,0)}\left\{f(m.0)-\frac{m(k-m-n_3)}{n_1+n_2-2m+2}f(m-1,0)\right\}. \tag{5}$$

Proof. The coefficients $c_j=0$ for $j<m-1$. Indeed let $i:=\min\{j:c_j\neq 0\}$ then $f(i,0)=c_i\psi_i(i,0)\neq 0$, and thus $i\ge m-1$. It remains to show that $\widetilde{\psi}_{m-1}(m,0)-\frac{m(k-m-n_3)}{n_1+n_2-2m+2}\widetilde{\psi}_{m-1}(m-1,0)=0$. From $\widetilde{\psi}_{m-1}^{(2)}(u,0)$ $=(n_2-m+2)_{m-1}(-u)_{m-1}$ we find $\widetilde{\psi}_{m-1}^{(2)}(m,0)/\widetilde{\psi}_{m-1}^{(2)}(m-1,0)=m$. In the sum for $\widetilde{\psi}_{m-1}^{(1)}(u)$ at $u=m-1$ only the $j=k-m+1$ term appears and at $u=m$ only the $j=k-m$ term appears. Using this fact we find

$$\frac{\widetilde{\psi}_{m-1}^{(1)}(m)}{\widetilde{\psi}_{m-1}^{(1)}(m-1)}=\frac{k-m-n_3}{n_1+n_2-2m+2},$$

$$\frac{\widetilde{\psi}_{m-1}(m,0)}{\widetilde{\psi}_{m-1}(m-1,0)}=\frac{m(k-m-n_3)}{n_1+n_2-2m+2},$$

and this concludes the proof. $\square$

We need to evaluate $\widetilde{\psi}_m(u-1,1), \widetilde{\psi}_m(u,1), \widetilde{\psi}_m(u-1,0), \widetilde{\psi}_m(u,0)$ at $u=m-1,m$. The first and third of these vanish at $u=m-1$, by Lemma 2. Besides the values in formulas (3) the following are needed:

$$\begin{aligned}
\widetilde{\psi}_m^{(1)}(m+1) &= -(k-m)!\,(n_1+n_2-k-m+1)_{k-m-1}(n_3-k+m+1)\\
\widetilde{\psi}_m^{(2)}(m,0) &= m!\,(-n_2)_m\\
\widetilde{\psi}_m^{(2)}(m-1,1) &= m!\,(n_1-m+1)(1-n_2)_{m-1}\\
\widetilde{\psi}_m^{(2)}(m,1) &= (-1)^m m!\,(n_2-m+1)_{m-1}(n_2-m(n_1-m+1)).
\end{aligned}$$

To organize the calculations let

$$\begin{aligned}
A &:= \widetilde{\psi}_m^{(1)}(m+1)/\widetilde{\psi}_m^{(1)}(m) = -\frac{n_3-k+m+1}{n_1+n_2-2m}\\
B_1 &:= \widetilde{\psi}_m^{(2)}(m-1,1)/\widetilde{\psi}_m^{(2)}(m,0) = -\frac{n_1-m+1}{n_2}\\
B_2 &:= \widetilde{\psi}_m^{(2)}(m,1)/\widetilde{\psi}_m^{(2)}(m,0) = \frac{n_2-m(n_1-m+1)}{n_2}\\
\mathcal{T}f(u,v) &:= \frac{1}{\widetilde{\psi}_m^{(1)}(m)\,\widetilde{\psi}_m^{(2)}(m,0)}\{f(u,0)-C_m f(u-1,0)\}\\
C_m &:= \frac{m(k-m-n_3)}{n_1+n_2-2m+2}.
\end{aligned}$$

Three of the $\mathcal{T}$- evaluations are nonzero:

$$\begin{aligned}
\mathcal{T}\{u(n_2-v)wf(u-1,v+1)\} &= mn_2(k-m)B_1,\\
\mathcal{T}\{(n_1-u)(n_2-v)wf(u,v+1)\} &= (n_1-m)n_2(k-m)AB_2-(n_1-m+1)n_2(k-m+1)CB_1,\\
\mathcal{T}\{(n_1-u)(n_2-v)(n_3-w)f(u,v)\} &= (n_1-m)n_2(n_3-k+m).
\end{aligned}$$

The omitted cases are due to $\widetilde{\psi}_m^{(2)}(m-1,0)=0=\widetilde{\psi}_m^{(2)}(m-2,0)$. Let

$$\xi(k,m) := (m+1)(n_3-k+m+1)(k-m)\frac{n_1n_2-m^2}{n_1+n_2-2m},$$

then

$$\begin{aligned}
(n_1-m)n_2(k-m)AB_2 &= (n_1+1)m(k-m)(n_3-k+m+1)-\xi(k,m)\\
-(n_1-m+1)n_2(k-m+1)CB_1 &= -n_1m(k-m+1)(n_3-k-m)+\xi(k,m-1).
\end{aligned}$$

We must deal with the exceptional case $2m=n_1+n_2$: if $k<(n_1\wedge n_2)$ then $m\le k$ and $2m<n_1+n_2$, so suppose $n_1\wedge n_2\le k$ and $m=n_1\wedge n_2$, and $2m=n_1+n_2$ implies $m=n_1=n_2$ so that the term AB_2 does not occur. In fact there is more detail: if $k>n_1=n_2$ then $n_1+n_2-k<n_1$ giving the bound $m\le n_1+n_2-k$ and $n_1+n_2-2m\ge k-m>0$, else if $k=m$ then $\widetilde{\psi}_1=1$ and $A=1$.

Adding the six terms we have shown that the coefficient c_m in the expansion $\rho g_3\psi_m=\sum_n c_n\psi_n$ is

$$\begin{aligned}
c_m &= \frac{1}{n_1n_2n_3}\{\zeta(k,m)-\xi(k,m)+\xi(k,m-1)\}\\
\zeta(k,m) &:= m^2(3k-2m)+m\left(n_3^2-k^2\right)-(n_3-k+m)(m(n_1+n_2+n_3)-n_1n_2).
\end{aligned}$$

Thus the spherical function $\Phi^{[N-k,k]}(g_3) = \frac{1}{n_1n_2n_3}\sum_{m=m_L}^{m_U}\zeta(k,m) - \xi(k,m_U) + \xi(k,m_L-1)$, by telescoping. Recall $m_L := 0 \vee (k-n_3)$ and $m_U := n_1 \wedge n_2 \wedge k \wedge (n_1+n_2-k)$. By definition $\xi(k,-1) = 0 = \xi(k,k-n_3-1)$ so that $\xi(k,m_L-1) = 0$, also $\xi(k,k) = 0$. The nonzero values of $\xi(k,m_U)$ are

$$\xi(k,m_U) = \begin{cases} (m_U+1)(m_U-(k-n_3)+1)\,m_U(k-m_U)\,, & m_U = n_1 \wedge n_2 \\ (m_U+1)(m_U-(k-n_3)+1)\left(n_1n_2-m_U^2\right), & m_U = n_1+n_2-k. \end{cases}$$

One of the first two factors is $(m_U - m_L + 1)$ since $m_L = 0 \vee k - n_3$..We point out that the value of a spherical function times $n_1n_2n_3$ is an integer, because the character table of $\mathcal{S}_N$ has all integer entries, and necessarily the denominator in $\xi(k,\mu)$ cancels out. If $k \le n_3$ ($m_L = 0$) let $\mu = m_U$ and then

$$\begin{aligned}\Phi^{[N-k,k]}(g_3) =& \frac{\mu+1}{n_1n_2n_3}\left\{n_1n_2n_3 - \frac{1}{2}\mu(n_1n_2+n_1n_3+n_1n_3) + \frac{1}{6}N\mu(\mu-1)\right.\\ &\left.+\frac{1}{2}(k-\mu)(\mu(N-k+\mu+1)-2n_1n_2) - \frac{1}{\mu+1}\xi(k,\mu)\right\}.\end{aligned}$$

The simplest case is $\mu = k$, $\xi(k,k) = 0$. The sum can be explicitly found in general but tends to be complicated. Here is one way to display the sum (with $\mu = m_U - m_L$, $\nu = m_L$, $\delta = k - m_U$), omitting the factor $\frac{\mu+1}{n_1n_2n_3}$

$$\begin{aligned}&\frac{1}{6}N\mu(\mu-1) + \frac{1}{2}N(\mu\nu+\mu\delta+2\nu\delta) - \frac{1}{2}(\mu+2\nu)(n_1n_2+n_1n_3+n_1n_3) + n_1n_2n_3\\ &+\frac{1}{2}\mu\nu(\nu-1) + (\nu-\delta)(n_1n_2+\nu\delta) - \frac{1}{2}\mu\delta(\delta-1) - \frac{1}{\mu+1}\xi(k,m_U).\end{aligned} \tag{6}$$

5.1. Special situations

Multiplicity equal to one arises when $m_L = m_U$ at (1) $k = n_1+n_3$, (2) $k = n_2+n_3$, (3) $k = n_1+n_2$, (4) $k = \frac{N}{2}$. For case (1) $m_L = k - n_3 = m_U = n_1$, then $\mu = 0$, $\nu = n_1$, $\delta = n_3$ and by formula (6) $\Phi^{[N-k,k]}(g_3) = -\frac{1}{n_2}$. A similar calculation shows case (2) yields $-\frac{1}{n_1}$. For case (3) $m_L = 0 = n_1+n_2-k = m_U$, then $\mu = 0$, $\nu = 0$, $\delta = n_1+n_2$ and by the same formula $\Phi^{[N-k,k]}(g_3) = -\frac{1}{n_3}$. In these cases there are implicit bounds such as $n_2 \ge \frac{N}{2}, n_1+n_3 \le \frac{N}{2}$, following from $2k \le N$. Applying these parameters for the 2-cycle case when $g_2 = \left(x_1^{(2)}, x_1^{(1)}\right)$ and using formula (4) for $\Phi^{[N-k,k]}(g_2)$ one obtains (1) $-\frac{1}{n_2}$, (2) $-\frac{1}{n_1}$, (3) 1.

For case (4) (when N is even) $m_L = \frac{N}{2} - n_3 = m_U = n_1+n_2-\frac{N}{2}$, $\delta = n_3$. The resulting values can be written as

$$\begin{aligned}\Phi^{[N/2,N/2]}(g_3) &= \frac{1}{n_1n_2n_3}\left\{-\prod_{i=1}^{3}\left(\frac{N}{2}-n_i\right) - \sum_{1\le i<j\le 3}\left(\frac{N}{2}-n_i\right)\left(\frac{N}{2}-n_j\right)\right\},\\ \Phi^{[N/2,N/2]}(g_2) &= \frac{1}{n_1n_2}\left(\left(\frac{N}{2}-n_1\right)\left(\frac{N}{2}-n_2\right) - \frac{N}{2} + n_3\right).\end{aligned}$$

Another example is $n_1 = n_2 = n_3 = n$ (and $N = 3n$). If $k \le n$ then $\nu = m_L = 0, m_U = k, \mu = k, \delta = 0$ and

$$\Phi^{[N-k,k]}(g_3) = \frac{k+1}{n^2}\left(n^2 - \frac{3}{2}nk + \frac{1}{2}k(k-1)\right).$$

If $n \le k \le \frac{3}{2}n$ then $\nu = m_L = k-n, m_U = 2n-k$ (since $n \ge 2n-k$), $\mu = 3n-2k, \delta = 2k-2n$ and

$$\Phi^{[N-k,k]}(g_3) = \frac{3n-2k+1}{n^2}\left(n^2 - \frac{3}{2}nk + \frac{1}{2}k(k-1)\right).$$

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