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Article

Study of Hardy-Hilbert-type integral inequalities with non-homogeneous power-max kernel functions

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Abstract: Hardy-Hilbert-type integral inequalities are among the classics of mathematical analysis. In particular, this includes well-known variants involving homogeneous power-max kernel functions. In this article, we extend the theory by studying the non-homogeneous case using specially designed power-max kernel functions. Additionally, we explore different integration domains to increase the flexibility of our results in a variety of mathematical contexts. We also establish several equivalences, modifications and generalizations of our main integral inequalities. The proofs are detailed and self-contained. To support the theory, we provide numerical examples together with the corresponding implementation codes for transparency and reproducibility.

Keywords: Hardy-Hilbert-type integral inequalities, non-homogeneous kernel functions, double integrals, Hölder integral inequality, R codes

MSC: 26D15, 33E20.

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1. Introduction

1.1. Context

G.H. Hardy and D. Hilbert made significant contributions to the study of integral inequalities. In particular, D. Hilbert established the classical Hilbert integral inequality in 1908, which G.H. Hardy later extended and generalized in 1925, leading to the well-known Hardy-Hilbert (HH) integral inequality. This inequality plays an important role in mathematical analysis, especially in the study of function spaces, operator theory, and Fourier analysis. The essence of these classical results can be found in [1,2].

To lay the foundation for this article, a formal statement of the HH integral inequality is given below, using conventional notation. Let $p > 1$, $q = p/(p-1)$ be the Hölder conjugate of p, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. Then we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy \leq \frac{\pi}{\sin(\pi/p)} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

under the condition that the integrals defining the upper bound converge, i.e., $\int_0^{+\infty} f^p(x)dx < +\infty$ and $\int_0^{+\infty} g^q(y)dy < +\infty$. The left-hand side term is thus a double integral of the form

$$\int_0^{+\infty} \int_0^{+\infty} k(x,y)f(x)g(y)dxdy,$$

where k is the kernel function defined by $k(x,y) = 1/(x+y)$. A key property of this function is its homogeneity, meaning that there exists $\epsilon \in \mathbb{R}$ such that, for any $x, y > 0$ and $\lambda > 0$, the following scaling property holds:

$$k(\lambda x, \lambda y) = \lambda^{\epsilon} k(x,y).$$

Here, we have $\epsilon = -1$. Notably, if we take $p = 2$, the HH integral inequality reduces to the classical Hilbert integral inequality, with π as the constant factor. Beyond these fundamental results, various modifications and generalizations have been studied. Some of these are characterized by the inclusion of different kernel functions. Despite the extensive literature on the subject, this remains an active area of research with ongoing developments. For further references and recent advancements, see [2–40]. Important results from some of these references are discussed below.

1.2. Power-max kernel functions

In this article, we focus on HH-type integral inequalities for a certain class of kernel functions that depend on the power and the maximum of the two variables x and y, i.e., $\max(x, y)$. These are called power-max kernel functions (including the case where the power exponent is reduced to 1 for simplicity). A fundamental result in this area was established in [2], which states that, within the framework of the HH integral inequality,

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\max(x,y)}f(x)g(y)dxdy \leq pq\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q},$$

(again under the condition that the integrals defining the upper bound converge, as for all upper bounds presented in this section). The considered power-max kernel function is simply given by $k(x, y) = 1/\max(x, y)$. This result was revisited in [29] by introducing a parameter $\gamma > 0$, which leads to

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\max(x^\gamma,y^\gamma)}f(x)g(y)dxdy \leq \frac{pq}{\gamma}\left[\int_0^{+\infty}x^{p-1-\gamma}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q-1-\gamma}g^q(y)dy\right]^{1/q}.$$

The corresponding kernel function is $k(x, y) = 1/\max(x^\gamma, y^\gamma)$. Note that, if we take $\gamma = 1$, we do not exactly recover the previous result, dealing with weighted L_p and L_q integral norms of f and g, respectively. A more original HH-type integral inequality was introduced in [18] for the case $p = 2$, given by

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y+\max(x,y)}f(x)g(y)dxdy \leq \sqrt{2}\left[\pi-2\arctan(\sqrt{2})\right]\left[\int_0^{+\infty}f^2(x)dx\right]^{1/2}\left[\int_0^{+\infty}g^2(y)dy\right]^{1/2}.$$

Here, the kernel function considered is $k(x, y) = 1/[x + y + \max(x, y)]$. This result was further generalized in [23] by including an arbitrary norm parameter $p > 1$, yielding

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y+\max(x,y)}f(x)g(y)dxdy\\&\leq \sqrt{2}\left[\pi-2\arctan(\sqrt{2})\right]\left[\int_0^{+\infty}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{q/2-1}g^q(y)dy\right]^{1/q}.\end{aligned}$$

Still in [23], a one-power-parameter extension of this result was investigated. It ensures that, for any $\mu > 1$,

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[x+y+\max(x,y)]^\mu}f(x)g(y)dxdy\\&\leq \frac{1}{2(\mu-1)}\left(\frac{1}{2^{\mu-2}}-\frac{1}{3^{\mu-1}}\right)\left[\int_0^{+\infty}x^{1-\mu}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{1-\mu}g^q(y)dy\right]^{1/q}.\end{aligned}$$

Here, the kernel function is

$$k(x,y) = \frac{1}{[x+y+\max(x,y)]^\mu}. \tag{1}$$

The more technical case $\mu \in (0, 1)$ was also examined in [23], along with functional generalizations. We complete this brief state of the art by mentioning that general homogeneous power-max kernel functions were introduced in [2,5,18,19,25,26], with the use of intermediate functions and adjustable parameters. Based on these, HH-type integral inequalities were established with detailed developments and original techniques.

1.3. Contributions

A key observation in all the HH-type integral inequalities discussed above is that the considered kernel functions are homogeneous. This raises a natural question:

Question 1. *Can we establish relevant HH-type integral inequalities with non-homogeneous power-max functions?*

To answer this question, we first propose to study a new class of kernel functions of the form

$$k(x,y)=\frac{1}{[xy+\max(x,y)]^{\alpha}},$$

where $\alpha>1$. Compared to the kernel function in Eq. (1), the sum term $x+y$ is thus replaced by xy. Although this change may seem minor at first glance, it has a significant impact on the behavior of the kernel function. Indeed, the presence of this product term disrupts its homogeneity (as detailed later), and leads to an innovative mathematical framework. In addition, we extend the analysis by introducing a varying integration domain, assuming that $(x,y)\in(\beta,+\infty)^2$ with $\beta\geq 0$. The upper bounds obtained are of the form

$$\kappa\left[\int_{\beta}^{+\infty}w(x)f^p(x)dx\right]^{1/p}\left[\int_{\beta}^{+\infty}w(y)g^q(y)dy\right]^{1/q},$$

where $\kappa>0$ is a certain constant and w is a carefully chosen weight function. An integral equivalence result is also established, as well as more tractable integral inequalities. To improve applicability, we also give a functional generalization using two intermediate functions.

Next, we introduce a class of related kernel functions where the maximum term is modulated by a parameter δ. More specifically, we consider

$$k(x,y)=\frac{1}{[xy+\delta\max(x,y)]^{2}},$$

with $\delta>0$. Three integration domains are considered, assuming that $(x,y)\in(0,1)^2$, $(x,y)\in(1,+\infty)^2$ and $(x,y)\in(0,+\infty)^2$. An integral equivalence of our main result is given. Then some variants are studied, one of which is

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)g(y)dxdy\leq\frac{2}{\delta^2}\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}.$$

Beyond these cases, we establish additional HH-type integral inequalities with other non-homogeneous kernel functions, as well as functional generalizations that extend their applicability.

To support the theory, we provide numerical examples wherever possible. For transparency and reproducibility, the implementation codes used in the computations are also included.

1.4. Article organization

§2 and §3 present the first and second HH-type integral inequalities, respectively. §4 provides the conclusion. The appendix includes the codes used in the numerical examples.

2. First HH-type integral inequalities

2.1. Main result

The proposition below is an integral result for an integrand involving a (ratio) power-max function. Note also an integration domain that can vary depending on an intermediate parameter.

Proposition 1. *Let $\alpha>1$ and $\beta\geq 0$. Then, for any $x\geq\beta$, we have*

$$\int_{\beta}^{+\infty}\frac{1}{[xy+\max(x,y)]^{\alpha}}dy=\frac{1}{(\alpha-1)x^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(x+1)^{\alpha}}\right].$$

Proof of Proposition 1. By means of the Chasles integral theorem, combined with the definition of $\max(x,y)$, standard power primitive rules and $\alpha > 1$, for any $x \geq \beta$, we get

$$\begin{aligned}
\int_{\beta}^{+\infty} \frac{1}{[xy+\max(x,y)]^{\alpha}} dy &= \int_{\beta}^{x} \frac{1}{[xy+\max(x,y)]^{\alpha}} dy + \int_{x}^{+\infty} \frac{1}{[xy+\max(x,y)]^{\alpha}} dy \\
&= \int_{\beta}^{x} \frac{1}{(xy+x)^{\alpha}} dy + \int_{x}^{+\infty} \frac{1}{(xy+y)^{\alpha}} dy \\
&= \frac{1}{x^{\alpha}} \int_{\beta}^{x} \frac{1}{(y+1)^{\alpha}} dy + \frac{1}{(x+1)^{\alpha}} \int_{x}^{+\infty} \frac{1}{y^{\alpha}} dy \\
&= \frac{1}{x^{\alpha}} \left[\frac{1}{1-\alpha} \times \frac{1}{(y+1)^{\alpha-1}} \right]_{y=\beta}^{y=x} + \frac{1}{(x+1)^{\alpha}} \left[\frac{1}{1-\alpha} \times \frac{1}{y^{\alpha-1}} \right]_{y=x}^{y\to+\infty} \\
&= \frac{1}{\alpha-1} \times \frac{1}{x^{\alpha}} \left[\frac{1}{(\beta+1)^{\alpha-1}} - \frac{1}{(x+1)^{\alpha-1}} \right] + \frac{1}{\alpha-1} \times \frac{1}{x^{\alpha-1}} \times \frac{1}{(x+1)^{\alpha}} \\
&= \frac{1}{(\alpha-1)x^{\alpha}} \left[\frac{1}{(\beta+1)^{\alpha-1}} - \frac{1}{(x+1)^{\alpha-1}} + \frac{x}{(x+1)^{\alpha}} \right] \\
&= \frac{1}{(\alpha-1)x^{\alpha}} \left[\frac{1}{(\beta+1)^{\alpha-1}} - \frac{1}{(x+1)^{\alpha}} \right].
\end{aligned}$$

The proof of the proposition ends. □

The expression of the considered integral is therefore relatively manageable. Based on this, we concentrate on a double integral of the form

$$\int_{\beta}^{+\infty} \int_{\beta}^{+\infty} k(x,y) f(x) g(y) dx dy,$$

where $\alpha > 1$, $\beta \geq 0$ and k is the power-max kernel function given by

$$k(x,y) = \frac{1}{[xy+\max(x,y)]^{\alpha}}.$$

This kernel function is non-homogeneous, meaning there is no $\epsilon \in \mathbb{R}$ such that, for any $x, y > 0$ and $\lambda > 0$, $k(\lambda x, \lambda y) = \lambda^{\epsilon} k(x,y)$. To substantiate this claim, notice that

$$k(\lambda x, \lambda y) = \frac{1}{[(\lambda x)(\lambda y)+\max(\lambda x, \lambda y)]^{\alpha}} = \frac{1}{[\lambda^2 xy + \lambda \max(x,y)]^{\alpha}} = \frac{1}{\lambda^{\alpha}[\lambda xy+\max(x,y)]^{\alpha}}.$$

This suggests that the only possible candidate for ϵ is $\epsilon = -\alpha$. However, due to the additional factor λ in the xy term of the denominator, the equality $k(\lambda x, \lambda y) = \lambda^{\epsilon} k(x,y)$ does not hold. The kernel function thus deviates from the standard homogeneous form, introducing a distinct analytical framework compared to those explored in [2,18,23,25,26,29]. In addition, we emphasize the adaptable integration domain, i.e., $(\beta, +\infty)^2$, with $\beta \geq 0$, which includes the commonly studied case $(0, +\infty)^2$ when $\beta = 0$.

In this context, the theorem below presents the first HH-type integral inequalities. Proposition 1 is central to the proof.

Theorem 1. *Let $\alpha > 1$, $\beta \geq 0$, $p > 1$, $q = p/(p-1)$ and $f, g : (\beta, +\infty) \mapsto (0, +\infty)$ be two functions. Then we have*

$$\begin{aligned}
\int_{\beta}^{+\infty} \int_{\beta}^{+\infty} \frac{1}{[xy+\max(x,y)]^{\alpha}} f(x) g(y) dx dy \leq{}& \frac{1}{\alpha-1} \left\{ \int_{\beta}^{+\infty} \frac{1}{x^{\alpha}} \left[\frac{1}{(\beta+1)^{\alpha-1}} - \frac{1}{(x+1)^{\alpha}} \right] f^{p}(x) dx \right\}^{1/p} \\
&\times \left\{ \int_{\beta}^{+\infty} \frac{1}{y^{\alpha}} \left[\frac{1}{(\beta+1)^{\alpha-1}} - \frac{1}{(y+1)^{\alpha}} \right] g^{q}(y) dy \right\}^{1/q},
\end{aligned}$$

under the condition that the integrals defining the upper bound converge.

Proof of Theorem 1. A suitable product decomposition of the integrand using the equality $1/p + 1/q = 1$, followed by an application of the Hölder integral inequality with the parameters p and q, gives

$$\begin{aligned}
&\int_\beta^{+\infty}\int_\beta^{+\infty}\frac{1}{[xy+\max(x,y)]^\alpha}f(x)g(y)dxdy\\
&=\int_\beta^{+\infty}\int_\beta^{+\infty}\frac{1}{[xy+\max(x,y)]^{\alpha/p}}f(x)\times\frac{1}{[xy+\max(x,y)]^{\alpha/q}}g(y)dxdy\\
&\le\left\{\int_\beta^{+\infty}\int_\beta^{+\infty}\frac{1}{[xy+\max(x,y)]^\alpha}f^p(x)dxdy\right\}^{1/p}\times\left\{\int_\beta^{+\infty}\int_\beta^{+\infty}\frac{1}{[xy+\max(x,y)]^\alpha}g^q(y)dxdy\right\}^{1/q}. \qquad (2)
\end{aligned}$$

By means of the Fubini-Tonelli integral theorem (which allows the exchange of the order of integration, the integrand being of constant sign, i.e., non-negative here) and Proposition 1, we have

$$\begin{aligned}
\int_\beta^{+\infty}\int_\beta^{+\infty}\frac{1}{[xy+\max(x,y)]^\alpha}f^p(x)dxdy&=\int_\beta^{+\infty}f^p(x)\left\{\int_\beta^{+\infty}\frac{1}{[xy+\max(x,y)]^\alpha}dy\right\}dx\\
&=\frac{1}{\alpha-1}\int_\beta^{+\infty}\frac{1}{x^\alpha}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(x+1)^\alpha}\right]f^p(x)dx. \qquad (3)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\int_\beta^{+\infty}\int_\beta^{+\infty}\frac{1}{[xy+\max(x,y)]^\alpha}g^q(y)dxdy&=\int_\beta^{+\infty}g^q(y)\left\{\int_\beta^{+\infty}\frac{1}{[xy+\max(x,y)]^\alpha}dx\right\}dy\\
&=\frac{1}{\alpha-1}\int_\beta^{+\infty}\frac{1}{y^\alpha}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^\alpha}\right]g^q(y)dy. \qquad (4)
\end{aligned}$$

Combining Eqs. (2), (3) and (4) and using the equality $1/p + 1/q = 1$, we get

$$\begin{aligned}
\int_\beta^{+\infty}\int_\beta^{+\infty}\frac{1}{[xy+\max(x,y)]^\alpha}f(x)g(y)dxdy&\le\left\{\frac{1}{\alpha-1}\int_\beta^{+\infty}\frac{1}{x^\alpha}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(x+1)^\alpha}\right]f^p(x)dx\right\}^{1/p}\\
&\quad\times\left\{\frac{1}{\alpha-1}\int_\beta^{+\infty}\frac{1}{y^\alpha}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^\alpha}\right]g^q(y)dy\right\}^{1/q}\\
&=\frac{1}{\alpha-1}\left\{\int_\beta^{+\infty}\frac{1}{x^\alpha}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(x+1)^\alpha}\right]f^p(x)dx\right\}^{1/p}\\
&\quad\times\left\{\int_\beta^{+\infty}\frac{1}{y^\alpha}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^\alpha}\right]g^q(y)dy\right\}^{1/q}.
\end{aligned}$$

This ends the proof of the theorem. □

To the best of our knowledge, this theorem provides a new integral inequality result in the literature. It is one of the few that mixes a non-homogeneous power-max kernel function and an adaptive integration domain. An equivalent formulation of this theorem with only one function will be described later, in Proposition 2.

Remark 1. Some basic or technical remarks on Theorem 1 are formulated below.

- Noticing that $xy = \max(x,y)\min(x,y)$, the main double integral can be rewritten with $\min(x,y)$ and $\max(x,y)$, as follows:

$$\int_\beta^{+\infty}\int_\beta^{+\infty}\frac{1}{[xy+\max(x,y)]^\alpha}f(x)g(y)dxdy=\int_\beta^{+\infty}\int_\beta^{+\infty}\frac{1}{\max^\alpha(x,y)[\min(x,y)+1]^\alpha}f(x)g(y)dxdy.$$

  From this point of view, we deal with a non-homogeneous min-max-power kernel function, which is $k(x,y) = 1/\{\max^\alpha(x,y)[\min(x,y)+1]^\alpha\}$.

- Using the expression $\max(x,y) = (1/2)(x+y+|x-y|)$, the main double integral can be rewritten as follows:

$$\int_{\beta}^{+\infty}\int_{\beta}^{+\infty}\frac{1}{[xy+\max(x,y)]^{\alpha}}f(x)g(y)dxdy = 2^{\alpha}\int_{\beta}^{+\infty}\int_{\beta}^{+\infty}\frac{1}{[2xy+x+y+|x-y|]^{\alpha}}f(x)g(y)dxdy.$$

So we have

$$\begin{aligned}&\int_{\beta}^{+\infty}\int_{\beta}^{+\infty}\frac{1}{[2xy+x+y+|x-y|]^{\alpha}}f(x)g(y)dxdy\\ &\leq \frac{1}{2^{\alpha}(\alpha-1)}\left\{\int_{\beta}^{+\infty}\frac{1}{x^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(x+1)^{\alpha}}\right]f^{p}(x)dx\right\}^{1/p}\\ &\quad\times\left\{\int_{\beta}^{+\infty}\frac{1}{y^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^{\alpha}}\right]g^{q}(y)dy\right\}^{1/q}.\end{aligned}$$

To the best of our knowledge, this is a new HH-type integral inequality. It is defined with a power-absolute value kernel function.

- For any $x>0$, we have $-1/(x+1)^{\alpha}\leq 0$, and, for any $y>0$, we have $-1/(y+1)^{\alpha}\leq 0$. Using this and the equality $1/p+1/q=1$, we get

$$\begin{aligned}&\int_{\beta}^{+\infty}\int_{\beta}^{+\infty}\frac{1}{[xy+\max(x,y)]^{\alpha}}f(x)g(y)dxdy\\ &\leq \frac{1}{\alpha-1}\left[\frac{1}{(\beta+1)^{\alpha-1}}\int_{\beta}^{+\infty}\frac{1}{x^{\alpha}}f^{p}(x)dx\right]^{1/p}\left[\frac{1}{(\beta+1)^{\alpha-1}}\int_{\beta}^{+\infty}\frac{1}{y^{\alpha}}g^{q}(y)dy\right]^{1/q}\\ &= \frac{1}{(\alpha-1)(\beta+1)^{\alpha-1}}\left[\int_{\beta}^{+\infty}\frac{1}{x^{\alpha}}f^{p}(x)dx\right]^{1/p}\left[\int_{\beta}^{+\infty}\frac{1}{y^{\alpha}}g^{q}(y)dy\right]^{1/q}.\end{aligned}$$

The weight functions in the integrals of the upper bounds are thus reduced to power functions, with an original constant factor that can be small, depending on the values of α and β.

2.2. Numerical work

We now illustrate Theorem 1 with some numerical examples. The values of α and β are arbitrarily chosen from those to which the integrals of the upper bound converge. For the sake of reproducibility, the free software R is used with the package named `pracma`. See [41]. The corresponding codes are given in the appendix.

Example 1. If we take $\alpha = 2.5$, $\beta = 1$, $p = 2$ (so $q = 2$), $f(x) = 1$ and $g(y) = 1$, we have

$$\int_{\beta}^{+\infty}\int_{\beta}^{+\infty}\frac{1}{[xy+\max(x,y)]^{\alpha}}f(x)g(y)dxdy = \int_{1}^{+\infty}\int_{1}^{+\infty}\frac{1}{[xy+\max(x,y)]^{2.5}}dxdy \approx 0.09268775,$$

and

$$\begin{aligned}&\frac{1}{\alpha-1}\left\{\int_{\beta}^{+\infty}\frac{1}{x^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(x+1)^{\alpha}}\right]f^{p}(x)dx\right\}^{1/p}\times\left\{\int_{\beta}^{+\infty}\frac{1}{y^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^{\alpha}}\right]g^{q}(y)dy\right\}^{1/q}\\ &\quad=\frac{1}{2.5-1}\left\{\int_{1}^{+\infty}\frac{1}{x^{2.5}}\left[\frac{1}{(1+1)^{2.5-1}}-\frac{1}{(x+1)^{2.5}}\right]dx\right\}\approx 0.1170915.\end{aligned}$$

Since $0.09268775 < 0.1170915$, this is consistent with Theorem 1.

Example 2. If we take $\alpha = 1.1$, $\beta = 2$, $p = 3$ (so $q = 3/2$), $f(x) = 1$ and $g(y) = e^{-y}$, we have

$$\int_{\beta}^{+\infty}\int_{\beta}^{+\infty}\frac{1}{[xy+\max(x,y)]^{\alpha}}f(x)g(y)dxdy = \int_{2}^{+\infty}\int_{2}^{+\infty}\frac{1}{[xy+\max(x,y)]^{1.1}}e^{-y}dxdy \approx 0.2857713,$$

and

$$\frac{1}{\alpha-1}\left\{\int_{\beta}^{+\infty}\frac{1}{x^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(x+1)^{\alpha}}\right]f^{p}(x)dx\right\}^{1/p}\times\left\{\int_{\beta}^{+\infty}\frac{1}{y^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^{\alpha}}\right]g^{q}(y)dy\right\}^{1/q}$$
$$=\frac{1}{1.1-1}\left\{\int_{2}^{+\infty}\frac{1}{x^{1.1}}\left[\frac{1}{(2+1)^{1.1-1}}-\frac{1}{(x+1)^{1.1}}\right]dx\right\}^{1/3}$$
$$\times\left\{\int_{2}^{+\infty}\frac{1}{y^{1.1}}\left[\frac{1}{(2+1)^{1.1-1}}-\frac{1}{(y+1)^{1.1}}\right]e^{-3y/2}dy\right\}^{2/3}\approx 0.7654846.$$

Since $0.2857713 < 0.7654846$, this is in accordance with Theorem 1.

Example 3. If we take $\alpha = 2$, $\beta = 0$, $p = 4/3$ (so $q = 4$), $f(x) = e^{-1/x^2}$ and $g(y) = e^{-1/y}$, we have

$$\int_{\beta}^{+\infty}\int_{\beta}^{+\infty}\frac{1}{[xy+\max(x,y)]^{\alpha}}f(x)g(y)dxdy=\int_{0}^{+\infty}\int_{0}^{+\infty}\frac{1}{[xy+\max(x,y)]^{2}}e^{-1/x^2}e^{-1/y}dxdy\approx 0.220281,$$

and

$$\frac{1}{\alpha-1}\left\{\int_{\beta}^{+\infty}\frac{1}{x^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(x+1)^{\alpha}}\right]f^{p}(x)dx\right\}^{1/p}\times\left\{\int_{\beta}^{+\infty}\frac{1}{y^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^{\alpha}}\right]g^{q}(y)dy\right\}^{1/q}$$
$$=\frac{1}{2-1}\left\{\int_{0}^{+\infty}\frac{1}{x^{2}}\left[\frac{1}{(0+1)^{2-1}}-\frac{1}{(x+1)^{2}}\right]e^{-4/(3x^2)}dx\right\}^{3/4}$$
$$\times\left\{\int_{0}^{+\infty}\frac{1}{y^{2}}\left[\frac{1}{(0+1)^{2-1}}-\frac{1}{(y+1)^{2}}\right]e^{-4/y}dy\right\}^{1/4}\approx 0.5251761.$$

Since $0.220281 < 0.5251761$, this is consistent with Theorem 1.

2.3. Additional results

The proposition below is complementary to Theorem 1, dealing with only one function.

Proposition 2. *Let $\alpha > 1$, $\beta \geq 0$, $p > 1$, $q = p/(p-1)$ and $f : (0,+\infty) \mapsto (0,+\infty)$ be a function. Then Theorem 1 is rigorously equivalent to*

$$\int_{\beta}^{+\infty}\left\{\frac{1}{y^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^{\alpha}}\right]\right\}^{1-p}\left\{\int_{\beta}^{+\infty}\frac{1}{[xy+\max(x,y)]^{\alpha}}f(x)dx\right\}^{p}dy$$
$$\leq\frac{1}{(\alpha-1)^{p}}\int_{\beta}^{+\infty}\frac{1}{x^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(x+1)^{\alpha}}\right]f^{p}(x)dx,$$

under the condition that the integral defining the upper bound converges.

Proof of Proposition 2. Let us first prove that Theorem 1 implies the stated inequality. To do this, let us set

$$\mathcal{I}=\int_{\beta}^{+\infty}\left\{\frac{1}{y^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^{\alpha}}\right]\right\}^{1-p}\left\{\int_{\beta}^{+\infty}\frac{1}{[xy+\max(x,y)]^{\alpha}}f(x)dx\right\}^{p}dy.$$

A suitable product decomposition of the integrand gives

$$\mathcal{I}=\int_{\beta}^{+\infty}\left\{\frac{1}{y^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^{\alpha}}\right]\right\}^{1-p}\left\{\int_{\beta}^{+\infty}\frac{1}{[xy+\max(x,y)]^{\alpha}}f(x)dx\right\}$$
$$\times\left\{\int_{\beta}^{+\infty}\frac{1}{[xy+\max(x,y)]^{\alpha}}f(x)dx\right\}^{p-1}dy$$
$$=\int_{\beta}^{+\infty}\int_{\beta}^{+\infty}\frac{1}{[xy+\max(x,y)]^{\alpha}}f(x)g_{\ddagger}(y)dxdy, \quad (5)$$

where
$$g_{\ddagger}(y)=\left\{\frac{1}{y^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^{\alpha}}\right]\right\}^{1-p}\left\{\int_{\beta}^{+\infty}\frac{1}{[xy+\max(x,y)]^{\alpha}}f(x)dx\right\}^{p-1}.$$

It follows from Theorem 1 applied to the functions f and $g_{\ddagger}$ that

$$\int_{\beta}^{+\infty}\int_{\beta}^{+\infty}\frac{1}{[xy+\max(x,y)]^{\alpha}}f(x)g_{\ddagger}(y)dxdy\leq\frac{1}{\alpha-1}\left\{\int_{\beta}^{+\infty}\frac{1}{x^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(x+1)^{\alpha}}\right]f^{p}(x)dx\right\}^{1/p}$$
$$\times\left\{\int_{\beta}^{+\infty}\frac{1}{y^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^{\alpha}}\right]g_{\ddagger}^{q}(y)dy\right\}^{1/q}. \tag{6}$$

Using $q=p/(p-1)$ and the definition of $g_{\ddagger}$, we have

$$\begin{aligned}
&\int_{\beta}^{+\infty}\frac{1}{y^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^{\alpha}}\right]g_{\ddagger}^{q}(y)dy\\
&=\int_{\beta}^{+\infty}\frac{1}{y^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^{\alpha}}\right]\\
&\quad\times\left\{\frac{1}{y^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^{\alpha}}\right]\right\}^{q(1-p)}\left\{\int_{\beta}^{+\infty}\frac{1}{[xy+\max(x,y)]^{\alpha}}f(x)dx\right\}^{q(p-1)}dy\\
&=\int_{\beta}^{+\infty}\frac{1}{y^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^{\alpha}}\right]\\
&\quad\times\left\{\frac{1}{y^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^{\alpha}}\right]\right\}^{-p}\left\{\int_{\beta}^{+\infty}\frac{1}{[xy+\max(x,y)]^{\alpha}}f(x)dx\right\}^{p}dy\\
&=\int_{\beta}^{+\infty}\left\{\frac{1}{y^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^{\alpha}}\right]\right\}^{1-p}\left\{\int_{\beta}^{+\infty}\frac{1}{[xy+\max(x,y)]^{\alpha}}f(x)dx\right\}^{p}dy\\
&=\mathcal{I}.
\end{aligned}\tag{7}$$

Combining Eqs. (5), (6) and (7), we obtain

$$\mathcal{I}\leq\frac{1}{\alpha-1}\left\{\int_{\beta}^{+\infty}\frac{1}{x^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(x+1)^{\alpha}}\right]f^{p}(x)dx\right\}^{1/p}\mathcal{I}^{1/q}.$$

Using the equality $1/p+1/q=1$, we get

$$\mathcal{I}^{1/p}=\mathcal{I}^{1-1/q}\leq\frac{1}{\alpha-1}\left\{\int_{\beta}^{+\infty}\frac{1}{x^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(x+1)^{\alpha}}\right]f^{p}(x)dx\right\}^{1/p},$$

so that, by the definition of $\mathcal{I}$,

$$\begin{aligned}
&\int_{\beta}^{+\infty}\left\{\frac{1}{y^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^{\alpha}}\right]\right\}^{1-p}\left\{\int_{\beta}^{+\infty}\frac{1}{[xy+\max(x,y)]^{\alpha}}f(x)dx\right\}^{p}dy\\
&\leq\frac{1}{(\alpha-1)^{p}}\int_{\beta}^{+\infty}\frac{1}{x^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(x+1)^{\alpha}}\right]f^{p}(x)dx.
\end{aligned}$$

This is the desired inequality.

Let us now prove that this inequality implies Theorem 1. A suitable product decomposition of the integrand, the Hölder integral inequality with respect to y and with the parameters p and q, and the assumed inequality, give

$$\begin{aligned}
&\int_{\beta}^{+\infty}\int_{\beta}^{+\infty}\frac{1}{[xy+\max(x,y)]^{\alpha}}f(x)g(y)dxdy\\
&=\int_{\beta}^{+\infty}\left[\left\{\frac{1}{y^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^{\alpha}}\right]\right\}^{-1/q}\int_{\beta}^{+\infty}\frac{1}{[xy+\max(x,y)]^{\alpha}}f(x)dx\right]
\end{aligned}$$

$$
\begin{aligned}
&\times\left\{\frac{1}{y^\alpha}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^\alpha}\right]\right\}^{1/q} g(y)dy \\
\leq &\left[\int_\beta^{+\infty}\left\{\frac{1}{y^\alpha}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^\alpha}\right]\right\}^{-p/q}\left\{\int_\beta^{+\infty}\frac{1}{[xy+\max(x,y)]^\alpha}f(x)dx\right\}^p dy\right]^{1/p} \\
&\times\left\{\int_\beta^{+\infty}\frac{1}{y^\alpha}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^\alpha}\right]g^q(y)dy\right\}^{1/q} \\
= &\left[\int_\beta^{+\infty}\left\{\frac{1}{y^\alpha}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^\alpha}\right]\right\}^{1-p}\left\{\int_\beta^{+\infty}\frac{1}{[xy+\max(x,y)]^\alpha}f(x)dx\right\}^p dy\right]^{1/p} \\
&\times\left\{\int_\beta^{+\infty}\frac{1}{y^\alpha}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^\alpha}\right]g^q(y)dy\right\}^{1/q} \\
\leq &\left\{\frac{1}{(\alpha-1)^p}\int_\beta^{+\infty}\frac{1}{x^\alpha}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(x+1)^\alpha}\right]f^p(x)dx\right\}^{1/p} \\
&\times\left\{\int_\beta^{+\infty}\frac{1}{y^\alpha}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^\alpha}\right]g^q(y)dy\right\}^{1/q} \\
= &\frac{1}{\alpha-1}\left\{\int_\beta^{+\infty}\frac{1}{x^\alpha}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(x+1)^\alpha}\right]f^p(x)dx\right\}^{1/p} \\
&\times\left\{\int_\beta^{+\infty}\frac{1}{y^\alpha}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^\alpha}\right]g^q(y)dy\right\}^{1/q}.
\end{aligned}
$$

The inequality in Theorem 1 is established, proving the equivalence. This concludes the proof of the proposition. □

The proposition below provides a simplified upper bound for the HH-type integral inequality established in Theorem 1.

Proposition 3. *Let $\alpha > 1$, $p > 1$, $q = p/(p-1)$ and $f, g : (0,+\infty) \mapsto (0,+\infty)$ be two functions. Then we have*

$$
\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\max(x,y)]^\alpha}f(x)g(y)dxdy \\
&\leq \frac{1}{4(\alpha-1)}\left[\int_0^{+\infty}\left(\frac{x+1}{x}\right)^\alpha f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\left(\frac{y+1}{y}\right)^\alpha g^q(y)dy\right]^{1/q},
\end{aligned}
$$

under the condition that the integrals defining the upper bound converge.

Proof of Proposition 3. Applying Theorem 1 to $\beta = 0$, we have

$$
\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\max(x,y)]^\alpha}f(x)g(y)dxdy \\
&\leq \frac{1}{\alpha-1}\left\{\int_0^{+\infty}\frac{1}{x^\alpha}\left[1-\frac{1}{(x+1)^\alpha}\right]f^p(x)dx\right\}^{1/p}\left\{\int_0^{+\infty}\frac{1}{y^\alpha}\left[1-\frac{1}{(y+1)^\alpha}\right]g^q(y)dy\right\}^{1/q}.
\end{aligned}
\tag{8}
$$

It is well known that, for any $z \in [0,1]$, we have $z(1-z) \in [0,1/4]$. For any $x > 0$, applying this to $z = 1/(x+1)^\alpha \in [0,1]$, we obtain

$$
\frac{1}{x^\alpha}\left[1-\frac{1}{(x+1)^\alpha}\right] = \left(\frac{x+1}{x}\right)^\alpha\frac{1}{(x+1)^\alpha}\left[1-\frac{1}{(x+1)^\alpha}\right] \leq \frac{1}{4}\left(\frac{x+1}{x}\right)^\alpha.
$$

Similarly, for any $y > 0$, we have

$$
\frac{1}{y^\alpha}\left[1-\frac{1}{(y+1)^\alpha}\right] = \left(\frac{y+1}{y}\right)^\alpha\frac{1}{(y+1)^\alpha}\left[1-\frac{1}{(y+1)^\alpha}\right] \leq \frac{1}{4}\left(\frac{y+1}{y}\right)^\alpha.
$$

Using these inequalities and the equality $1/p+1/q=1$, we get

$$\frac{1}{\alpha-1}\left\{\int_0^{+\infty}\frac{1}{x^\alpha}\left[1-\frac{1}{(x+1)^\alpha}\right]f^p(x)dx\right\}^{1/p}\left\{\int_0^{+\infty}\frac{1}{y^\alpha}\left[1-\frac{1}{(y+1)^\alpha}\right]g^q(y)dy\right\}^{1/q}$$
$$\leq\frac{1}{\alpha-1}\left\{\frac{1}{4}\int_0^{+\infty}\left(\frac{x+1}{x}\right)^\alpha f^p(x)dx\right\}^{1/p}\left\{\frac{1}{4}\int_0^{+\infty}\left(\frac{y+1}{y}\right)^\alpha g^q(y)dy\right\}^{1/q}$$
$$=\frac{1}{4(\alpha-1)}\left[\int_0^{+\infty}\left(\frac{x+1}{x}\right)^\alpha f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\left(\frac{y+1}{y}\right)^\alpha g^q(y)dy\right]^{1/q}. \tag{9}$$

Combining Eqs. (8) and (9), we obtain

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\max(x,y)]^\alpha}f(x)g(y)dxdy$$
$$\leq\frac{1}{4(\alpha-1)}\left[\int_0^{+\infty}\left(\frac{x+1}{x}\right)^\alpha f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\left(\frac{y+1}{y}\right)^\alpha g^q(y)dy\right]^{1/q}.$$

This concludes the proof of the proposition. □

The result below completes Proposition 3 by providing an alternative upper bound for a varying integration domain.

Proposition 4. *Let $\alpha>1$, $\beta\geq 0$, $p>1$, $q=p/(p-1)$ and $f,g:(\beta,+\infty)\mapsto(0,+\infty)$ be two functions. Then we have*

$$\int_\beta^{+\infty}\int_\beta^{+\infty}\frac{1}{[xy+\max(x,y)]^\alpha}f(x)g(y)dxdy$$
$$\leq\frac{1}{\alpha-1}\left\{\int_\beta^{+\infty}\frac{1}{x^\alpha}\left[\frac{1}{(\beta+1)^{\alpha-1}}+\alpha x-1\right]f^p(x)dx\right\}^{1/p}\times\left\{\int_\beta^{+\infty}\frac{1}{y^\alpha}\left[\frac{1}{(\beta+1)^{\alpha-1}}+\alpha y-1\right]g^q(y)dy\right\}^{1/q},$$

under the condition that the integrals defining the upper bound converge.

Proof of Proposition 4. It follows from Theorem 1 that

$$\int_\beta^{+\infty}\int_\beta^{+\infty}\frac{1}{[xy+\max(x,y)]^\alpha}f(x)g(y)dxdy$$
$$\leq\frac{1}{\alpha-1}\left\{\int_\beta^{+\infty}\frac{1}{x^\alpha}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(x+1)^\alpha}\right]f^p(x)dx\right\}^{1/p}\times\left\{\int_\beta^{+\infty}\frac{1}{y^\alpha}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^\alpha}\right]g^q(y)dy\right\}^{1/q}. \tag{10}$$

The well-known Bernoulli inequality gives, for any $z\geq -1$ and $r\in\mathbb{R}\backslash(0,1)$,

$$(1+z)^r\geq 1+rz,$$

(and the reversed inequality holds for $r\in[0,1]$). For any $x\geq\beta\geq 0\geq -1$, applying this to $z=x$ and $r=-\alpha\in\mathbb{R}\backslash(0,1)$, we get $(1+x)^{-\alpha}\geq 1-\alpha x$, so that

$$-\frac{1}{(x+1)^\alpha}\leq\alpha x-1.$$

Similarly, for any $y\geq\beta$, we have

$$-\frac{1}{(y+1)^\alpha}\leq\alpha y-1.$$

On the other hand, applying the Bernoulli inequality to $z = \beta$ and $r = 1 - \alpha < 0$, we get

$$\frac{1}{(\beta+1)^{\alpha-1}} + \alpha\beta - 1 = (1+\beta)^{1-\alpha} + \alpha\beta - 1 \geq 1 + (1-\alpha)\beta + \alpha\beta - 1 = \beta \geq 0.$$

It follows from these inequalities (the last one being necessary to check that the terms in square brackets are positive), we obtain

$$\begin{aligned}
&\frac{1}{\alpha-1}\left\{\int_{\beta}^{+\infty}\frac{1}{x^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(x+1)^{\alpha}}\right]f^{p}(x)dx\right\}^{1/p}\times\left\{\int_{\beta}^{+\infty}\frac{1}{y^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y+1)^{\alpha}}\right]g^{q}(y)dy\right\}^{1/q}\\
&\leq\frac{1}{\alpha-1}\left\{\int_{\beta}^{+\infty}\frac{1}{x^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}+\alpha x-1\right]f^{p}(x)dx\right\}^{1/p}\times\left\{\int_{\beta}^{+\infty}\frac{1}{y^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}+\alpha y-1\right]g^{q}(y)dy\right\}^{1/q}.
\end{aligned} \tag{11}$$

Combining Eqs. (10) and (11), we find that

$$\begin{aligned}
&\int_{\beta}^{+\infty}\int_{\beta}^{+\infty}\frac{1}{[xy+\max(x,y)]^{\alpha}}f(x)g(y)dxdy\\
&\leq\frac{1}{\alpha-1}\left\{\int_{\beta}^{+\infty}\frac{1}{x^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}+\alpha x-1\right]f^{p}(x)dx\right\}^{1/p}\times\left\{\int_{\beta}^{+\infty}\frac{1}{y^{\alpha}}\left[\frac{1}{(\beta+1)^{\alpha-1}}+\alpha y-1\right]g^{q}(y)dy\right\}^{1/q}.
\end{aligned}$$

This concludes the proof of the proposition. □

In particular, if we take $\beta = 0$, the following integral inequality is established:

$$\int_{0}^{+\infty}\int_{0}^{+\infty}\frac{1}{[xy+\max(x,y)]^{\alpha}}f(x)g(y)dxdy \leq \frac{\alpha}{\alpha-1}\left[\int_{0}^{+\infty}x^{1-\alpha}f^{p}(x)dx\right]^{1/p}\left[\int_{0}^{+\infty}y^{1-\alpha}g^{q}(y)dy\right]^{1/q}.$$

The form of the upper bounds is similar to some obtained for HH-type integral inequalities defined with homogeneous power-max kernel functions. See, for example, [5,19,23,29].

The theorem below can be seen as a functional generalization of Theorem 1. It follows the methodological spirit of [23], but with a different approach based on changes of variables.

Theorem 2. *Let $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$ with $a < b$, $c, d \in \mathbb{R} \cup \{-\infty, +\infty\}$ with $c < d$, $\alpha > 1$, $\beta \geq 0$, $p > 1$, $q = p/(p-1)$, $f : (a, b) \mapsto (0, +\infty)$ and $g : (c, d) \mapsto (0, +\infty)$ be two functions, and $\phi : (a, b) \mapsto (0, +\infty)$ and $\psi : (c, d) \mapsto (0, +\infty)$ be two differentiable non-decreasing functions such that $\lim_{z\to a}\phi(z) = \beta$, $\lim_{z\to b}\phi(z) = +\infty$, $\lim_{z\to c}\psi(z) = \beta$ and $\lim_{z\to d}\psi(z) = +\infty$. Then we have*

$$\begin{aligned}
&\int_{c}^{d}\int_{a}^{b}\frac{1}{\{\phi(x)\psi(y)+\max[\phi(x),\psi(y)]\}^{\alpha}}f(x)g(y)dxdy\\
&\leq\frac{1}{\alpha-1}\left[\int_{a}^{b}\frac{1}{\phi^{\alpha}(x)}\left\{\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{[\phi(x)+1]^{\alpha}}\right\}f^{p}(x)\frac{1}{[\phi'(x)]^{p-1}}dx\right]^{1/p}\\
&\quad\times\left[\int_{c}^{d}\frac{1}{\psi^{\alpha}(y)}\left\{\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{[\psi(y)+1]^{\alpha}}\right\}g^{q}(y)\frac{1}{[\psi'(y)]^{q-1}}dy\right]^{1/q},
\end{aligned}$$

under the condition that the integrals defining the upper bound converge.

Proof of Theorem 2. Applying the changes of variables $u = \phi(x)$ and $v = \psi(y)$, so that $x = \phi^{-1}(u)$ and $y = \psi^{-1}(v)$, with $dx = \left[1/\phi'(\phi^{-1}(u))\right] du$ and $dy = \left[1/\psi'(\psi^{-1}(v))\right] dv$, and paying attention to the integration domain, we get

$$\begin{aligned}
&\int_{c}^{d}\int_{a}^{b}\frac{1}{\{\phi(x)\psi(y)+\max[\phi(x),\psi(y)]\}^{\alpha}}f(x)g(y)dxdy\\
&=\int_{\beta}^{+\infty}\int_{\beta}^{+\infty}\frac{1}{[uv+\max(u,v)]^{\alpha}}f(\phi^{-1}(u))g(\psi^{-1}(v))\frac{1}{\phi'(\phi^{-1}(u))\psi'(\psi^{-1}(v))}dudv
\end{aligned}$$

$$= \int_{\beta}^{+\infty}\int_{\beta}^{+\infty} \frac{1}{[uv+\max(u,v)]^{\alpha}} f_{\dagger}(u) g_{\dagger}(v) du dv, \tag{12}$$

where

$$f_{\dagger}(u) = f(\phi^{-1}(u)) \frac{1}{\phi'(\phi^{-1}(u))}, \quad g_{\dagger}(v) = g(\psi^{-1}(v)) \frac{1}{\psi'(\psi^{-1}(v))}.$$

Applying Theorem 1 to the functions $f_{\dagger}$ and $g_{\dagger}$, we obtain

$$\begin{aligned} &\int_{\beta}^{+\infty}\int_{\beta}^{+\infty} \frac{1}{[uv+\max(u,v)]^{\alpha}} f_{\dagger}(u) g_{\dagger}(v) du dv \leq \frac{1}{\alpha-1} \left\{ \int_{\beta}^{+\infty} \frac{1}{u^{\alpha}} \left[\frac{1}{(\beta+1)^{\alpha-1}} - \frac{1}{(u+1)^{\alpha}} \right] f_{\dagger}^{p}(u) du \right\}^{1/p} \\ &\times \left\{ \int_{\beta}^{+\infty} \frac{1}{v^{\alpha}} \left[\frac{1}{(\beta+1)^{\alpha-1}} - \frac{1}{(v+1)^{\alpha}} \right] g_{\dagger}^{q}(v) dv \right\}^{1/q} \\ &= \frac{1}{\alpha-1} \left\{ \int_{\beta}^{+\infty} \frac{1}{u^{\alpha}} \left[\frac{1}{(\beta+1)^{\alpha-1}} - \frac{1}{(u+1)^{\alpha}} \right] f^{p}(\phi^{-1}(u)) \frac{1}{[\phi'(\phi^{-1}(u))]^{p}} du \right\}^{1/p} \\ &\times \left\{ \int_{\beta}^{+\infty} \frac{1}{v^{\alpha}} \left[\frac{1}{(\beta+1)^{\alpha-1}} - \frac{1}{(v+1)^{\alpha}} \right] g^{q}(\psi^{-1}(v)) \frac{1}{[\psi'(\psi^{-1}(v))]^{q}} dv \right\}^{1/q}. \end{aligned} \tag{13}$$

Now, reversing the changes of variables process in the upper bound by setting $u = \phi(x)$ and $v = \psi(y)$, we get

$$\begin{aligned} &\frac{1}{\alpha-1} \left\{ \int_{\beta}^{+\infty} \frac{1}{u^{\alpha}} \left[\frac{1}{(\beta+1)^{\alpha-1}} - \frac{1}{(u+1)^{\alpha}} \right] f^{p}(\phi^{-1}(u)) \frac{1}{[\phi'(\phi^{-1}(u))]^{p}} du \right\}^{1/p} \\ &\times \left\{ \int_{\beta}^{+\infty} \frac{1}{v^{\alpha}} \left[\frac{1}{(\beta+1)^{\alpha-1}} - \frac{1}{(v+1)^{\alpha}} \right] g^{q}(\psi^{-1}(v)) \frac{1}{[\psi'(\psi^{-1}(v))]^{q}} dv \right\}^{1/q} \\ &= \frac{1}{\alpha-1} \left[\int_{a}^{b} \frac{1}{\phi^{\alpha}(x)} \left\{ \frac{1}{(\beta+1)^{\alpha-1}} - \frac{1}{[\phi(x)+1]^{\alpha}} \right\} f^{p}(x) \frac{1}{[\phi'(x)]^{p}} \phi'(x) dx \right]^{1/p} \\ &\times \left[\int_{c}^{d} \frac{1}{\psi^{\alpha}(y)} \left\{ \frac{1}{(\beta+1)^{\alpha-1}} - \frac{1}{[\psi(y)+1]^{\alpha}} \right\} g^{q}(y) \frac{1}{[\psi'(y)]^{q}} \psi'(y) dy \right]^{1/q} \\ &= \frac{1}{\alpha-1} \left[\int_{a}^{b} \frac{1}{\phi^{\alpha}(x)} \left\{ \frac{1}{(\beta+1)^{\alpha-1}} - \frac{1}{[\phi(x)+1]^{\alpha}} \right\} f^{p}(x) \frac{1}{[\phi'(x)]^{p-1}} dx \right]^{1/p} \\ &\times \left[\int_{c}^{d} \frac{1}{\psi^{\alpha}(y)} \left\{ \frac{1}{(\beta+1)^{\alpha-1}} - \frac{1}{[\psi(y)+1]^{\alpha}} \right\} g^{q}(y) \frac{1}{[\psi'(y)]^{q-1}} dy \right]^{1/q}. \end{aligned} \tag{14}$$

Combining Eqs. (12), (13) and (14), we get

$$\begin{aligned} &\int_{c}^{d}\int_{a}^{b} \frac{1}{\{\phi(x)\psi(y)+\max[\phi(x),\psi(y)]\}^{\alpha}} f(x) g(y) dx dy \\ &\leq \frac{1}{\alpha-1} \left[\int_{a}^{b} \frac{1}{\phi^{\alpha}(x)} \left\{ \frac{1}{(\beta+1)^{\alpha-1}} - \frac{1}{[\phi(x)+1]^{\alpha}} \right\} f^{p}(x) \frac{1}{[\phi'(x)]^{p-1}} dx \right]^{1/p} \\ &\times \left[\int_{c}^{d} \frac{1}{\psi^{\alpha}(y)} \left\{ \frac{1}{(\beta+1)^{\alpha-1}} - \frac{1}{[\psi(y)+1]^{\alpha}} \right\} g^{q}(y) \frac{1}{[\psi'(y)]^{q-1}} dy \right]^{1/q}. \end{aligned}$$

This concludes the proof of the theorem. □

In particular, if we take $a = 0$, "$b = +\infty$", $c = 0$, "$d = +\infty$", $\beta = 0$, $\phi(x) = x^{\tau}$ with $\tau > 0$ and $\psi(y) = y^{\iota}$ with $\iota > 0$, since $\phi'(x) = \tau x^{\tau-1}$ and $\psi'(y) = \iota y^{\iota-1}$, we have

$$\begin{aligned} &\int_{0}^{+\infty}\int_{0}^{+\infty} \frac{1}{[x^{\tau}y^{\iota}+\max(x^{\tau},y^{\iota})]^{\alpha}} f(x) g(y) dx dy \\ &\leq \frac{1}{\alpha-1} \left[\int_{0}^{+\infty} \frac{1}{x^{\alpha\tau}} \left\{ \frac{1}{(\beta+1)^{\alpha-1}} - \frac{1}{(x^{\tau}+1)^{\alpha}} \right\} f^{p}(x) \frac{1}{(\tau x^{\tau-1})^{p-1}} dx \right]^{1/p} \end{aligned}$$

$$\times\left[\int_0^{+\infty}\frac{1}{y^{\alpha\iota}}\left\{\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y^\iota+1)^\alpha}\right\}g^q(y)\frac{1}{(\iota y^{\iota-1})^{q-1}}dy\right]^{1/q}$$
$$=\frac{1}{(\alpha-1)\tau^{1-1/p}\iota^{1-1/q}}\left[\int_0^{+\infty}\frac{1}{x^{\tau(\alpha+p-1)-p+1}}\left\{\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(x^\tau+1)^\alpha}\right\}f^p(x)dx\right]^{1/p}$$
$$\times\left[\int_0^{+\infty}\frac{1}{y^{\iota(\alpha+q-1)-q+1}}\left\{\frac{1}{(\beta+1)^{\alpha-1}}-\frac{1}{(y^\iota+1)^\alpha}\right\}g^q(y)dy\right]^{1/q}.$$

If we take $\tau=1$ and $\iota=1$, we obtain the inequality in Theorem 1. Much more functional examples of this kind can be presented, extending the scope of this theorem.

The rest of the article is devoted to the second HH-type integral inequalities, defined with other non-homogeneous kernel functions.

3. Second HH-type integral inequalities

3.1. Main result

The proposition below is a variant of Proposition 1. It provides an integral result for an integrand involving a (ratio) power-max function under three integration domains. The main novelty lies in the presence of the parameter δ, which modulates the term $\max(x,y)$ in the denominator.

Proposition 5. *Let $\delta>0$.*

1. *For any $x>0$, we have*

$$\int_0^1\frac{1}{[xy+\delta\max(x,y)]^2}dy=\begin{cases}\dfrac{(2-x)\delta+x}{\delta x(x+\delta)^2} & \text{if } x\in(0,1],\\ \dfrac{1}{\delta(1+\delta)x^2} & \text{if } x>1.\end{cases}$$

2. *For any $x>0$, we have*

$$\int_1^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}dy=\begin{cases}\dfrac{1}{(x+\delta)^2} & \text{if } x\in(0,1],\\ \dfrac{(2x-1)\delta+x^2}{(\delta+1)x^2(x+\delta)^2} & \text{if } x>1.\end{cases}$$

3. *For any $x>0$, we have*

$$\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}dy=\frac{2\delta+x}{\delta x(x+\delta)^2}.$$

Proof of Proposition 5. 1. For $x\in(0,1]$, using the Chasles integral theorem, combined with the definition of $\max(x,y)$ and standard power primitive rules, we get

$$\begin{aligned}\int_0^1\frac{1}{[xy+\delta\max(x,y)]^2}dy&=\int_0^x\frac{1}{[xy+\delta\max(x,y)]^2}dy+\int_x^1\frac{1}{[xy+\delta\max(x,y)]^2}dy\\&=\int_0^x\frac{1}{(xy+\delta x)^2}dy+\int_x^1\frac{1}{(xy+\delta y)^2}dy\\&=\frac{1}{x^2}\int_0^x\frac{1}{(y+\delta)^2}dy+\frac{1}{(x+\delta)^2}\int_x^1\frac{1}{y^2}dy\\&=\frac{1}{x^2}\left[-\frac{1}{y+\delta}\right]_{y=0}^{y=x}+\frac{1}{(x+\delta)^2}\left[-\frac{1}{y}\right]_{y=x}^{y=1}\\&=\frac{1}{x^2}\left(\frac{1}{\delta}-\frac{1}{x+\delta}\right)+\frac{1}{(x+\delta)^2}\left(\frac{1}{x}-1\right)\\&=\frac{(2-x)\delta+x}{\delta x(x+\delta)^2}.\end{aligned}$$

For $x > 1$, since $y \in (0,1)$, we have $\max(x,y) = x$. Using this and a standard power primitive rule, we have

$$\begin{aligned}\int_0^1 \frac{1}{[xy+\delta\max(x,y)]^2}dy &= \int_0^1 \frac{1}{(xy+\delta x)^2}dy \\ &= \frac{1}{x^2}\int_0^1 \frac{1}{(y+\delta)^2}dy \\ &= \frac{1}{x^2}\left[-\frac{1}{y+\delta}\right]_{y=0}^{y=1} \\ &= \frac{1}{x^2}\left(\frac{1}{\delta}-\frac{1}{1+\delta}\right) \\ &= \frac{1}{\delta(1+\delta)x^2}.\end{aligned}$$

The desired results are established.

2. For $x \in (0,1]$, since $y > 1$, we have $\max(x,y) = y$. Using this and a standard power primitive rule, we get

$$\begin{aligned}\int_1^{+\infty} \frac{1}{[xy+\delta\max(x,y)]^2}dy = \int_1^{+\infty} \frac{1}{(xy+\delta y)^2}dy = \frac{1}{(x+\delta)^2}\int_1^{+\infty}\frac{1}{y^2}dy = \frac{1}{(x+\delta)^2}\left[-\frac{1}{y}\right]_{y=1}^{y\to+\infty} \\ = \frac{1}{(x+\delta)^2}.\end{aligned}$$

For $x > 1$, using the Chasles integral theorem, combined with the definition of $\max(x,y)$, and standard power primitive rules, we obtain

$$\begin{aligned}\int_1^{+\infty} \frac{1}{[xy+\delta\max(x,y)]^2}dy &= \int_1^{x} \frac{1}{[xy+\delta\max(x,y)]^2}dy + \int_x^{+\infty} \frac{1}{[xy+\delta\max(x,y)]^2}dy \\ &= \int_1^{x} \frac{1}{(xy+\delta x)^2}dy + \int_x^{+\infty} \frac{1}{(xy+\delta y)^2}dy \\ &= \frac{1}{x^2}\int_1^{x} \frac{1}{(y+\delta)^2}dy + \frac{1}{(x+\delta)^2}\int_x^{+\infty}\frac{1}{y^2}dy \\ &= \frac{1}{x^2}\left[-\frac{1}{y+\delta}\right]_{y=1}^{y=x} + \frac{1}{(x+\delta)^2}\left[-\frac{1}{y}\right]_{y=x}^{y\to+\infty} \\ &= \frac{1}{x^2}\left(\frac{1}{1+\delta}-\frac{1}{x+\delta}\right) + \frac{1}{(x+\delta)^2}\times\frac{1}{x} \\ &= \frac{(2x-1)\delta+x^2}{(\delta+1)x^2(x+\delta)^2}.\end{aligned}$$

The desired results are obtained.

3. Combining the results of the two points above and using the Chasles integral theorem at the intersection value "$x = 1$", for any $x > 0$, we get

$$\begin{aligned}\int_0^{+\infty} \frac{1}{[xy+\delta\max(x,y)]^2}dy &= \int_0^{1} \frac{1}{[xy+\delta\max(x,y)]^2}dy + \int_1^{+\infty} \frac{1}{[xy+\delta\max(x,y)]^2}dy \\ &= \begin{cases} \dfrac{(2-x)\delta+x}{\delta x(x+\delta)^2} + \dfrac{1}{(x+\delta)^2} & \text{if } x \in (0,1], \\ \dfrac{1}{\delta(1+\delta)x^2} + \dfrac{(2x-1)\delta+x^2}{(\delta+1)x^2(x+\delta)^2} & \text{if } x > 1. \end{cases}\end{aligned}$$

After some developments, we find that

$$\frac{(2-x)\delta+x}{\delta x(x+\delta)^2} + \frac{1}{(x+\delta)^2} = \frac{2\delta+x}{\delta x(x+\delta)^2},$$

and

$$\frac{1}{\delta(1+\delta)x^2}+\frac{(2x-1)\delta+x^2}{(\delta+1)x^2(x+\delta)^2}=\frac{2\delta+x}{\delta x(x+\delta)^2}.$$

For any $x>0$, we therefore have

$$\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}dy=\frac{2\delta+x}{\delta x(x+\delta)^2}.$$

The desired formula is obtained.

This concludes the proof of the proposition. □

The expressions for the integrals are therefore tractable. Based on this, we concentrate on a double integral of the form

$$\int\int_D k(x,y)f(x)g(y)dxdy,$$

where $D\in\{(0,1)^2,(1,+\infty)^2,(0,+\infty)^2\}$, $\alpha>1$, $\beta\geq 0$ and k is the power-max kernel function given by

$$k(x,y)=\frac{1}{[xy+\delta\max(x,y)]^2},$$

with $\delta>0$. Compared to the first HH-type integral inequality, a new parameter δ is introduced, which modulates the maximum term, and the exponent α is set to 2. The presence of δ clearly adds a degree of flexibility. The choice of 2 for the power parameter is for reasons of integrability and simplicity. This new kernel function is also non-homogeneous, so there is no $\epsilon\in\mathbb{R}$ such that, for any $x,y\in D$ and $\lambda>0$, $k(\lambda x,\lambda y)=\lambda^{\epsilon}k(x,y)$. To illustrate this, we develop

$$k(\lambda x,\lambda y)=\frac{1}{[(\lambda x)(\lambda y)+\delta\max(\lambda x,\lambda y)]^2}=\frac{1}{[\lambda^2xy+\delta\lambda\max(x,y)]^2}=\frac{1}{\lambda^2[\lambda xy+\delta\max(x,y)]^2}.$$

This suggests that the only possible candidate for ϵ is $\epsilon=-2$. However, due to the additional factor λ in the xy term of the denominator, the equality $k(\lambda x,\lambda y)=\lambda^{\epsilon}k(x,y)$ does not hold. In addition, we emphasize the adaptable integration domain D which can be either $(0,1)^2$, $(1,+\infty)^2$ or $(0,+\infty)^2$.

In this context, the theorem below presents a new HH-type integral inequality. The proof is mainly based on Proposition 5.

Theorem 3. *Let $\delta>0$, $p>1$ and $q=p/(p-1)$.*

1. *Let $f,g:(0,1)\mapsto(0,+\infty)$ be two functions. Then we have*

$$\int_0^1\int_0^1\frac{1}{[xy+\delta\max(x,y)]^2}f(x)g(y)dxdy\leq\frac{1}{\delta}\left[\int_0^1\frac{(2-x)\delta+x}{x(x+\delta)^2}f^p(x)dx\right]^{1/p}\left[\int_0^1\frac{(2-y)\delta+y}{y(y+\delta)^2}g^q(y)dy\right]^{1/q},$$

under the condition that the integrals defining the upper bound converge.

2. *Let $f,g:(1,+\infty)\mapsto(0,+\infty)$ be two functions. Then we have*

$$\int_1^{+\infty}\int_1^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)g(y)dxdy$$
$$\leq\frac{1}{\delta+1}\left[\int_1^{+\infty}\frac{(2x-1)\delta+x^2}{x^2(x+\delta)^2}f^p(x)dx\right]^{1/p}\left[\int_1^{+\infty}\frac{(2y-1)\delta+y^2}{y^2(y+\delta)^2}g^q(y)dy\right]^{1/q},$$

under the condition that the integrals defining the upper bound converge.

3. *Let $f,g:(0,+\infty)\mapsto(0,+\infty)$ be two functions. Then we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)g(y)dxdy$$

$$\leq \frac{1}{\delta}\left[\int_0^{+\infty} \frac{2\delta+x}{x(x+\delta)^2} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} \frac{2\delta+y}{y(y+\delta)^2} g^q(y)dy\right]^{1/q},$$

under the condition that the integrals defining the upper bound converge.

Proof of Theorem 3. Let us prove the items 1, 2 and 3 in turn.

1. A suitable product decomposition of the integrand using the equality $1/p+1/q=1$, followed by an application of the Hölder integral inequality with the parameters p and q, gives

$$\begin{aligned}
&\int_0^1\int_0^1 \frac{1}{[xy+\delta\max(x,y)]^2} f(x)g(y)dxdy \\
&= \int_0^1\int_0^1 \frac{1}{[xy+\delta\max(x,y)]^{2/p}} f(x) \times \frac{1}{[xy+\delta\max(x,y)]^{2/q}} g(y)dxdy \\
&\leq \left\{\int_0^1\int_0^1 \frac{1}{[xy+\delta\max(x,y)]^2} f^p(x)dxdy\right\}^{1/p} \times \left\{\int_0^1\int_0^1 \frac{1}{[xy+\delta\max(x,y)]^2} g^q(y)dxdy\right\}^{1/q}.
\end{aligned} \tag{15}$$

Using the Fubini-Tonelli integral theorem and the item 1 of Proposition 5 for the case $x \in (0,1]$, we have

$$\begin{aligned}
\int_0^1\int_0^1 \frac{1}{[xy+\delta\max(x,y)]^2} f^p(x)dxdy &= \int_0^1 f^p(x)\left\{\int_0^1 \frac{1}{[xy+\delta\max(x,y)]^2}dy\right\}dx \\
&= \frac{1}{\delta}\int_0^1 \frac{(2-x)\delta+x}{x(x+\delta)^2} f^p(x)dx.
\end{aligned} \tag{16}$$

Similarly, we have

$$\begin{aligned}
\int_0^1\int_0^1 \frac{1}{[xy+\delta\max(x,y)]^2} g^q(y)dxdy &= \int_0^1 g^q(y)\left\{\int_0^1 \frac{1}{[xy+\delta\max(x,y)]^2}dx\right\}dy \\
&= \frac{1}{\delta}\int_0^1 \frac{(2-y)\delta+y}{y(y+\delta)^2} g^q(y)dy.
\end{aligned} \tag{17}$$

Combining Eqs. (15), (16) and (17) and using the equality $1/p+1/q=1$, we get

$$\begin{aligned}
&\int_0^1\int_0^1 \frac{1}{[xy+\delta\max(x,y)]^2} f(x)g(y)dxdy \\
&\leq \left[\frac{1}{\delta}\int_0^1 \frac{(2-x)\delta+x}{x(x+\delta)^2} f^p(x)dx\right]^{1/p}\left[\frac{1}{\delta}\int_0^1 \frac{(2-y)\delta+y}{y(y+\delta)^2} g^q(y)dy\right]^{1/q} \\
&= \frac{1}{\delta}\left[\int_0^1 \frac{(2-x)\delta+x}{x(x+\delta)^2} f^p(x)dx\right]^{1/p}\left[\int_0^1 \frac{(2-y)\delta+y}{y(y+\delta)^2} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

The desired inequality is established.

2. A suitable product decomposition of the integrand using the equality $1/p+1/q=1$, followed by an application of the Hölder integral inequality with the parameters p and q, gives

$$\begin{aligned}
&\int_1^{+\infty}\int_1^{+\infty} \frac{1}{[xy+\delta\max(x,y)]^2} f(x)g(y)dxdy \\
&= \int_1^{+\infty}\int_1^{+\infty} \frac{1}{[xy+\delta\max(x,y)]^{2/p}} f(x) \times \frac{1}{[xy+\delta\max(x,y)]^{2/q}} g(y)dxdy \\
&\leq \left\{\int_1^{+\infty}\int_1^{+\infty} \frac{1}{[xy+\delta\max(x,y)]^2} f^p(x)dxdy\right\}^{1/p} \times \left\{\int_1^{+\infty}\int_1^{+\infty} \frac{1}{[xy+\delta\max(x,y)]^2} g^q(y)dxdy\right\}^{1/q}.
\end{aligned} \tag{18}$$

By means of the Fubini-Tonelli integral theorem and the item 2 of Proposition 5 for the case $x > 1$, we have

$$\begin{aligned}\int_1^{+\infty}\int_1^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f^p(x)dxdy &= \int_1^{+\infty}f^p(x)\left\{\int_1^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}dy\right\}dx\\ &= \frac{1}{\delta+1}\int_1^{+\infty}\frac{(2x-1)\delta+x^2}{x^2(x+\delta)^2}f^p(x)dx. \end{aligned}\tag{19}$$

Similarly, we obtain

$$\begin{aligned}\int_1^{+\infty}\int_1^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}g^q(y)dxdy &= \int_1^{+\infty}g^q(y)\left\{\int_1^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}dx\right\}dy\\ &= \frac{1}{\delta+1}\int_1^{+\infty}\frac{(2y-1)\delta+y^2}{y^2(y+\delta)^2}g^q(y)dy. \end{aligned}\tag{20}$$

Combining Eqs. (18), (19) and (20) and using the equality $1/p+1/q=1$, we get

$$\begin{aligned}&\int_1^{+\infty}\int_1^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)g(y)dxdy\\ &\leq \left[\frac{1}{\delta+1}\int_1^{+\infty}\frac{(2x-1)\delta+x^2}{x^2(x+\delta)^2}f^p(x)dx\right]^{1/p}\left[\frac{1}{\delta+1}\int_1^{+\infty}\frac{(2y-1)\delta+y^2}{y^2(y+\delta)^2}g^q(y)dy\right]^{1/q}\\ &= \frac{1}{\delta+1}\left[\int_1^{+\infty}\frac{(2x-1)\delta+x^2}{x^2(x+\delta)^2}f^p(x)dx\right]^{1/p}\left[\int_1^{+\infty}\frac{(2y-1)\delta+y^2}{y^2(y+\delta)^2}g^q(y)dy\right]^{1/q}.\end{aligned}$$

The desired inequality is established.

3. A suitable product decomposition of the integrand using the equality $1/p+1/q=1$, followed by an application of the Hölder integral inequality with the parameters p with q, gives

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)g(y)dxdy\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^{2/p}}f(x)\times\frac{1}{[xy+\delta\max(x,y)]^{2/q}}g(y)dxdy\\ &\leq \left\{\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f^p(x)dxdy\right\}^{1/p}\times\left\{\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}g^q(y)dxdy\right\}^{1/q}.\end{aligned}\tag{21}$$

Using the Fubini-Tonelli integral theorem and the item 3 of Proposition 5, we have

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f^p(x)dxdy &= \int_0^{+\infty}f^p(x)\left\{\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}dy\right\}dx\\ &= \frac{1}{\delta}\int_0^{+\infty}\frac{2\delta+x}{x(x+\delta)^2}f^p(x)dx. \end{aligned}\tag{22}$$

Similarly, we have

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}g^q(y)dxdy &= \int_0^{+\infty}g^q(y)\left\{\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}dx\right\}dy\\ &= \frac{1}{\delta}\int_0^{+\infty}\frac{2\delta+y}{y(y+\delta)^2}g^q(y)dy. \end{aligned}\tag{23}$$

Combining Eqs. (21), (22) and (23) and using the equality $1/p+1/q=1$, we get

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)g(y)dxdy$$

$$\leq \left[\frac{1}{\delta}\int_0^{+\infty}\frac{2\delta+x}{x(x+\delta)^2}f^p(x)dx\right]^{1/p}\left[\frac{1}{\delta}\int_0^{+\infty}\frac{2\delta+y}{y(y+\delta)^2}g^q(y)dy\right]^{1/q}$$
$$=\frac{1}{\delta}\left[\int_0^{+\infty}\frac{2\delta+x}{x(x+\delta)^2}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{2\delta+y}{y(y+\delta)^2}g^q(y)dy\right]^{1/q}.$$

The desired inequality is established.

The proof of the theorem ends. □

If we take $\delta = 1$, the item 2 of Theorem 3 recovers Theorem 1 with $\alpha = 2$ and $\beta = 1$, and the item 3 of Theorem 3 recovers Theorem 1 with $\alpha = 2$ and $\beta = 0$. These cases emphasize their complementary nature. Like Theorem 1, Theorem 3 stands out as one of the few results that explore HH-type integral inequalities with a non-homogeneous power-max kernel function. Proposition 6 will describe an equivalent formulation of this theorem with only one function.

Remark 2. Let us now formulate some remarks on the item 3 of Theorem 3, considering the standard integration domain $(0, +\infty)^2$.

- Using the fact that $xy = \max(x, y)\min(x, y)$, the main double integral can be expressed in terms of $\min(x, y)$ and $\max(x, y)$, as follows:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)g(y)dxdy=\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\max^2(x,y)[\min(x,y)+\delta]^2}f(x)g(y)dxdy.$$

  From this point of view, we deal with a one-parameter non-homogeneous min-max kernel function, which is $k(x, y) = 1/\{\max^2(x, y)[\min(x, y) + \delta]^2\}$.
- Noticing that $\max(x, y) = (1/2)(x + y + |x - y|)$, the main double integral can be rewritten as follows:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)g(y)dxdy=4\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[2xy+\delta(x+y+|x-y|)]^2}f(x)g(y)dxdy.$$

  So we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[2xy+\delta(x+y+|x-y|)]^2}f(x)g(y)dxdy$$
$$\leq\frac{1}{4\delta}\left[\int_0^{+\infty}\frac{2\delta+x}{x(x+\delta)^2}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{2\delta+y}{y(y+\delta)^2}g^q(y)dy\right]^{1/q}.$$

  This is a new HH-type integral inequality, also defined with non-homogeneous kernel function.

3.2. Numerical work

We now illustrate Theorem 3 with some numerical examples. The values of δ are taken arbitrarily among the values for which the integrals of the corresponding upper bounds converge.

Example 4. We consider the item 2 of the theorem. If we take $\delta = 2$, $p = 2$ (so $q = 2$), $f(x) = 1$ and $g(y) = 1$, we have

$$\int_1^{+\infty}\int_1^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)g(y)dxdy=\int_1^{+\infty}\int_1^{+\infty}\frac{1}{[xy+2\max(x,y)]^2}dxdy\approx 0.1765864,$$

and

$$\frac{1}{\delta+1}\left[\int_1^{+\infty}\frac{(2x-1)\delta+x^2}{x^2(x+\delta)^2}f^p(x)dx\right]^{1/p}\left[\int_1^{+\infty}\frac{(2y-1)\delta+y^2}{y^2(y+\delta)^2}g^q(y)dy\right]^{1/q}$$
$$=\frac{1}{2+1}\int_1^{+\infty}\frac{(2x-1)\times 2+x^2}{x^2(x+2)^2}dx\approx 0.2159728.$$

Since $0.1765864 < 0.2159728$, this is in accordance with the item 2.

Example 5. We consider again the item 2 of the theorem. If we take $\delta = 1.5$, $p = 3$ (so $q = 3/2$), $f(x) = 1$ and $g(y) = e^{-y}$, we have

$$\int_1^{+\infty}\int_1^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)g(y)dxdy = \int_1^{+\infty}\int_1^{+\infty}\frac{1}{[xy+1.5\max(x,y)]^2}e^{-y}dxdy \approx 0.02270384,$$

and

$$\begin{aligned}&\frac{1}{\delta+1}\left[\int_1^{+\infty}\frac{(2x-1)\delta+x^2}{x^2(x+\delta)^2}f^p(x)dx\right]^{1/p}\left[\int_1^{+\infty}\frac{(2y-1)\delta+y^2}{y^2(y+\delta)^2}g^q(y)dy\right]^{1/q}\\&=\frac{1}{1.5+1}\left[\int_1^{+\infty}\frac{(2x-1)\times 1.5+x^2}{x^2(x+1.5)^2}dx\right]^{1/3}\left[\int_1^{+\infty}\frac{(2y-1)\times 1.5+y^2}{y^2(y+1.5)^2}e^{-3y/2}dy\right]^{2/3}\\&\approx 0.04039752.\end{aligned}$$

Since $0.02270384 < 0.04039752$, this is consistent with the item 2.

Example 6. We consider the item 3 of the theorem. If we take $\delta = 3$, $p = 4/3$ (so $q = 4$), $f(x) = e^{-1/x^2}$ and $g(y) = e^{-1/y}$, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)g(y)dxdy = \int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+3\max(x,y)]^2}e^{-1/x^2}e^{-1/y}dxdy \approx 0.07572791$$

and

$$\begin{aligned}&\frac{1}{\delta}\left[\int_0^{+\infty}\frac{2\delta+x}{x(x+\delta)^2}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{2\delta+y}{y(y+\delta)^2}g^q(y)dy\right]^{1/q}\\&=\frac{1}{3}\left[\int_0^{+\infty}\frac{2\times 3+x}{x(x+3)^2}e^{-4/(3x^2)}dx\right\}^{3/4}\left[\int_0^{+\infty}\frac{2\times 3+y}{y(y+3)^2}e^{-4/y}dy\right\}^{1/4}\approx 0.1407864.\end{aligned}$$

Since $0.07572791 < 0.1407864$, this is in accordance with the item 3.

3.3. Additional results

The proposition below complements the item 3 of Theorem 3, dealing with only one function.

Proposition 6. *Let $\delta > 0$, $p > 1$, $q = p/(p-1)$ and $f : (0,+\infty) \mapsto (0,+\infty)$ be a function. Then the item 3 of Theorem 3 is rigorously equivalent to*

$$\int_0^{+\infty}\left[\frac{2\delta+y}{y(y+\delta)^2}\right]^{1-p}\left\{\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)dx\right\}^p dy \le \frac{1}{\delta^p}\int_0^{+\infty}\frac{2\delta+x}{x(x+\delta)^2}f^p(x)dx,$$

under the condition that the integral defining the upper bound converges.

Proof of Proposition 6. Let us first prove that the item 3 of Theorem 1 implies the stated inequality. To simplify the notation, let us set

$$\mathcal{J} = \int_0^{+\infty}\left[\frac{2\delta+y}{y(y+\delta)^2}\right]^{1-p}\left\{\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)dx\right\}^p dy.$$

A suitable product decomposition of the integrand gives

$$\begin{aligned}\mathcal{J} &= \int_0^{+\infty}\left[\frac{2\delta+y}{y(y+\delta)^2}\right]^{1-p}\left\{\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)dx\right\}\times\left\{\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)dx\right\}^{p-1}dy\\&= \int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)g_\curlywedge(y)dxdy,\end{aligned} \tag{24}$$

where
$$g_{\curlywedge}(y)=\left[\frac{2\delta+y}{y(y+\delta)^2}\right]^{1-p}\left\{\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)dx\right\}^{p-1}.$$

The item 3 of Theorem 3 applied to the functions f and $g_{\curlywedge}$ ensures that

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)g_{\curlywedge}(y)dxdy\\&\leq\frac{1}{\delta}\left[\int_0^{+\infty}\frac{2\delta+x}{x(x+\delta)^2}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{2\delta+y}{y(y+\delta)^2}g_{\curlywedge}^q(y)dy\right]^{1/q}.\end{aligned}\tag{25}$$

Using $q=p/(p-1)$ and the definition of $g_{\curlywedge}$, we have

$$\begin{aligned}\int_0^{+\infty}\frac{2\delta+y}{y(y+\delta)^2}g_{\curlywedge}^q(y)dy&=\int_0^{+\infty}\frac{2\delta+y}{y(y+\delta)^2}\left[\frac{2\delta+y}{y(y+\delta)^2}\right]^{q(1-p)}\left\{\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)dx\right\}^{q(p-1)}dy\\&=\int_0^{+\infty}\frac{2\delta+y}{y(y+\delta)^2}\left[\frac{2\delta+y}{y(y+\delta)^2}\right]^{-p}\left\{\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)dx\right\}^{p}dy\\&=\int_0^{+\infty}\left[\frac{2\delta+y}{y(y+\delta)^2}\right]^{1-p}\left\{\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)dx\right\}^{p}dy\\&=\mathcal{J}.\end{aligned}\tag{26}$$

Combining Eqs. (24), (25) and (26), we get

$$\mathcal{J}\leq\frac{1}{\delta}\left[\int_0^{+\infty}\frac{2\delta+x}{x(x+\delta)^2}f^p(x)dx\right]^{1/p}\mathcal{J}^{1/q}.$$

Using the equality $1/p+1/q=1$, we obtain

$$\mathcal{J}^{1/p}=\mathcal{J}^{1-1/q}\leq\frac{1}{\delta}\left[\int_0^{+\infty}\frac{2\delta+x}{x(x+\delta)^2}f^p(x)dx\right]^{1/p},$$

so that, by the definition of $\mathcal{J}$,

$$\int_0^{+\infty}\left[\frac{2\delta+y}{y(y+\delta)^2}\right]^{1-p}\left\{\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)dx\right\}^{p}dy\leq\frac{1}{\delta^p}\int_0^{+\infty}\frac{2\delta+x}{x(x+\delta)^2}f^p(x)dx.$$

This is the desired inequality.

Let us now prove that this inequality implies the item 3 of Theorem 3. A suitable product decomposition of the integrand, the Hölder integral inequality with respect to y and with the parameters p and q, and the assumed inequality, give

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)g(y)dxdy\\&=\int_0^{+\infty}\left\{\left[\frac{2\delta+y}{y(y+\delta)^2}\right]^{-1/q}\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)dx\right\}\left[\frac{2\delta+y}{y(y+\delta)^2}\right]^{1/q}g(y)dy\\&\leq\left\{\int_0^{+\infty}\left[\frac{2\delta+y}{y(y+\delta)^2}\right]^{-p/q}\left\{\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)dx\right\}^{p}dy\right\}^{1/p}\times\left[\int_0^{+\infty}\frac{2\delta+y}{y(y+\delta)^2}g^q(y)dy\right]^{1/q}\\&=\left\{\int_0^{+\infty}\left[\frac{2\delta+y}{y(y+\delta)^2}\right]^{1-p}\left\{\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)dx\right\}^{p}dy\right\}^{1/p}\times\left[\int_0^{+\infty}\frac{2\delta+y}{y(y+\delta)^2}g^q(y)dy\right]^{1/q}\\&\leq\left[\frac{1}{\delta^p}\int_0^{+\infty}\frac{2\delta+x}{x(x+\delta)^2}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{2\delta+y}{y(y+\delta)^2}g^q(y)dy\right]^{1/q}\\&=\frac{1}{\delta}\left[\int_0^{+\infty}\frac{2\delta+x}{x(x+\delta)^2}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{2\delta+y}{y(y+\delta)^2}g^q(y)dy\right]^{1/q}.\end{aligned}$$

The inequality in the item 3 of Theorem 1 is demonstrated, establishing the equivalence. This ends the proof of the proposition. □

Similar results can be obtained with the items 1 and 2 of Theorem 3. The details are omitted for the sake of redundancy.

An important proposition is given below, which suggests a simplified and elegant integral inequality derived from the item 3 of Theorem 3.

Proposition 7. *Let $\delta > 0$ and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. Then we have*

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{[xy + \delta \max(x, y)]^2} f(x) g(y) dx dy \le \frac{2}{\delta^2} \left[\int_0^{+\infty} \frac{1}{x} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} \frac{1}{y} g^q(y) dy \right]^{1/q},$$

under the condition that the integrals defining the upper bound converge.

Proof of Proposition 7. It follows from the item 3 of Theorem 3 that

$$\begin{aligned}
\int_0^{+\infty} \int_0^{+\infty} \frac{1}{[xy + \delta \max(x, y)]^2} f(x) g(y) dx dy &\le \frac{1}{\delta} \left[\int_0^{+\infty} \frac{2\delta + x}{x(x + \delta)^2} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} \frac{2\delta + y}{y(y + \delta)^2} g^q(y) dy \right]^{1/q} \\
&= \frac{1}{\delta} \left[\int_0^{+\infty} \frac{1}{x} \ell(x) f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} \frac{1}{y} \ell(y) g^q(y) dy \right]^{1/q}, \qquad (27)
\end{aligned}$$

where

$$\ell(x) = \frac{2\delta + x}{(x + \delta)^2},$$

with $\delta > 0$. Let us study this function. For any $x > 0$, standard differentiation rules give

$$\ell'(x) = -\frac{3\delta + x}{(x + \delta)^3} \le 0.$$

So $\ell(x)$ is non-increasing, and, for any $x > 0$, $\ell(x) \le \ell(0) = 2/\delta$. This, followed by the equality $1/p + 1/q = 1$, gives

$$\begin{aligned}
\frac{1}{\delta} \left[\int_0^{+\infty} \frac{1}{x} \ell(x) f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} \frac{1}{y} \ell(y) g^q(y) dy \right]^{1/q} &\le \frac{1}{\delta} \left[\int_0^{+\infty} \frac{1}{x} \times \frac{2}{\delta} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} \frac{1}{y} \times \frac{2}{\delta} g^q(y) dy \right]^{1/q} \\
&= \frac{2}{\delta^2} \left[\int_0^{+\infty} \frac{1}{x} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} \frac{1}{y} g^q(y) dy \right]^{1/q}. \qquad (28)
\end{aligned}$$

Combining Eqs. (27) and (28), we find that

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{[xy + \delta \max(x, y)]^2} f(x) g(y) dx dy \le \frac{2}{\delta^2} \left[\int_0^{+\infty} \frac{1}{x} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} \frac{1}{y} g^q(y) dy \right]^{1/q}.$$

This concludes the proof of the proposition. □

The fact that only the constant factor depends on δ in the upper bound is an advantage. We can manipulate it in various ways to derive new integral inequalities, starting with the proposition below.

Proposition 8. *Let $\theta > 1$ and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. Then we have*

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{[xy + \max(x, y)][xy + \theta \max(x, y)]} f(x) g(y) dx dy \le \frac{2}{\theta} \left[\int_0^{+\infty} \frac{1}{x} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} \frac{1}{y} g^q(y) dy \right]^{1/q},$$

under the condition that the integrals defining the upper bound converge.

Proof of Proposition 8. It follows from Proposition 7 that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)g(y)dxdy\leq\frac{2}{\delta^2}\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q},$$

with $\delta>0$. Let us now consider δ as a variable. Integrating both sides with respect to δ with $\delta\in(1,\theta)$ and expressing only the upper bound, we get

$$\begin{aligned}\int_1^{\theta}\left\{\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)g(y)dxdy\right\}d\delta&\leq\int_1^{\theta}\left\{\frac{2}{\delta^2}\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}\right\}d\delta\\&=\left(\int_1^{\theta}\frac{2}{\delta^2}d\delta\right)\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}\\&=\left[-\frac{2}{\delta}\right]_{\delta=1}^{\delta=\theta}\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}\\&=\frac{2(\theta-1)}{\theta}\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}.\end{aligned}\tag{29}$$

For the left-hand side term, using the Fubini-Tonelli integral theorem and standard primitive rules, we obtain

$$\begin{aligned}&\int_1^{\theta}\left\{\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)g(y)dxdy\right\}d\delta\\&=\int_0^{+\infty}\int_0^{+\infty}\left\{\int_1^{\theta}\frac{1}{[xy+\delta\max(x,y)]^2}d\delta\right\}f(x)g(y)dxdy\\&=\int_0^{+\infty}\int_0^{+\infty}\left[-\frac{1}{\max(x,y)[xy+\delta\max(x,y)]}\right]_{\delta=1}^{\delta=\theta}f(x)g(y)dxdy\\&=\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\max(x,y)}\left[\frac{1}{xy+\max(x,y)}-\frac{1}{xy+\theta\max(x,y)}\right]f(x)g(y)dxdy\\&=(\theta-1)\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\max(x,y)][xy+\theta\max(x,y)]}f(x)g(y)dxdy.\end{aligned}\tag{30}$$

Combining Eqs. (29) and (30), we have

$$\begin{aligned}&(\theta-1)\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\max(x,y)][xy+\theta\max(x,y)]}f(x)g(y)dxdy\\&\leq\frac{2(\theta-1)}{\theta}\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}.\end{aligned}$$

Since $\theta>1$, this is equivalent to

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\max(x,y)][xy+\theta\max(x,y)]}f(x)g(y)dxdy\leq\frac{2}{\theta}\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}.$$

The proof of the proposition is concluded. □

A sophisticated HH-type integral inequality with a logarithmic-power-max kernel function is shown in the proposition below.

Proposition 9. *Let $v>1$ and $f,g:(0,+\infty)\mapsto(0,+\infty)$ be two functions. Then we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\max^2(x,y)}\left\{\log\left[\frac{xy+v\max(x,y)}{xy+\max(x,y)}\right]+\frac{(1-v)xy\max(x,y)}{[xy+\max(x,y)][xy+v\max(x,y)]}\right\}f(x)g(y)dxdy$$

$$\leq 2\log(v)\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q},$$

under the condition that the integrals defining the upper bound converge.

Proof of Proposition 9. It follows from Proposition 7 that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)g(y)dxdy \leq \frac{2}{\delta^2}\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q},$$

with $\delta > 0$. Since $\delta > 0$, this can be rewritten as follows:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{\delta}{[xy+\delta\max(x,y)]^2}f(x)g(y)dxdy \leq \frac{2}{\delta}\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}.$$

Let us now consider δ as a variable. Integrating both sides with respect to δ with $\delta \in (1, v)$ and expressing only the upper bound, we have

$$\begin{aligned}
\int_1^v\left\{\int_0^{+\infty}\int_0^{+\infty}\frac{\delta}{[xy+\delta\max(x,y)]^2}f(x)g(y)dxdy\right\}d\delta &\leq \int_1^v\left\{\frac{2}{\delta}\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}\right\}d\delta\\
&= \left(\int_1^v\frac{2}{\delta}d\delta\right)\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}\\
&= [2\log(\delta)]_{\delta=1}^{\delta=v}\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}\\
&= 2\log(v)\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}.
\end{aligned} \tag{31}$$

For the left-hand side term, using the Fubini-Tonelli integral theorem and standard primitive rules, we get

$$\begin{aligned}
&\int_1^v\left\{\int_0^{+\infty}\int_0^{+\infty}\frac{\delta}{[xy+\delta\max(x,y)]^2}f(x)g(y)dxdy\right\}d\delta\\
&= \int_0^{+\infty}\int_0^{+\infty}\left\{\int_1^v\frac{\delta}{[xy+\delta\max(x,y)]^2}d\delta\right\}f(x)g(y)dxdy\\
&= \int_0^{+\infty}\int_0^{+\infty}\frac{1}{\max(x,y)}\left[\int_1^v\left\{\frac{1}{xy+\delta\max(x,y)}-\frac{xy}{[xy+\delta\max(x,y)]^2}\right\}d\delta\right]\times f(x)g(y)dxdy\\
&= \int_0^{+\infty}\int_0^{+\infty}\frac{1}{\max^2(x,y)}\left[\log[xy+\delta\max(x,y)]+\frac{xy}{xy+\delta\max(x,y)}\right]_{\delta=1}^{\delta=v}\\
&= \int_0^{+\infty}\int_0^{+\infty}\frac{1}{\max^2(x,y)}\left\{\log[xy+v\max(x,y)]-\log[xy+\max(x,y)]\right.\\
&\quad\left.+xy\left[\frac{1}{xy+v\max(x,y)}-\frac{1}{xy+\max(x,y)}\right]\right\}f(x)g(y)dxdy\\
&= \int_0^{+\infty}\int_0^{+\infty}\frac{1}{\max^2(x,y)}\left\{\log\left[\frac{xy+v\max(x,y)}{xy+\max(x,y)}\right]\right.\\
&\quad\left.+\frac{(1-v)xy\max(x,y)}{[xy+\max(x,y)][xy+v\max(x,y)]}\right\}f(x)g(y)dxdy.
\end{aligned} \tag{32}$$

Combining Eqs. (31) and (32), we obtain

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\max^2(x,y)}\left\{\log\left[\frac{xy+v\max(x,y)}{xy+\max(x,y)}\right]+\frac{(1-v)xy\max(x,y)}{[xy+\max(x,y)][xy+v\max(x,y)]}\right\}f(x)g(y)dxdy$$

$$\leq 2\log(v)\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}.$$

The proof of the proposition is concluded. □

We can rewrite this inequality using only positive terms, as follows:

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\max^2(x,y)}\log\left[\frac{xy+v\max(x,y)}{xy+\max(x,y)}\right]f(x)g(y)dxdy\\&\leq 2\log(v)\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}\\&\quad+(v-1)\int_0^{+\infty}\int_0^{+\infty}\frac{xy}{\max(x,y)[xy+\max(x,y)][xy+v\max(x,y)]}f(x)g(y)dxdy.\end{aligned}$$

Another HH-type integral inequality is given below, again derived from Proposition 7. It has the property of involving a kernel function defined with a special function.

Proposition 10. *Let $\nu > 0$ and $f,g:(0,+\infty)\mapsto(0,+\infty)$ be two functions. Then we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\max^2(x,y)}\mathcal{H}\left(\frac{xy}{\nu^2\max^2(x,y)}\right)f(x)g(y)dxdy\leq\frac{\pi^2}{3}\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q},$$

where, for any $z>0$,

$$\mathcal{H}(z)=\sum_{n=1}^{+\infty}\frac{1}{(n+z)^2},$$

which is a special case of the Hurwitz zeta function, under the condition that the integrals defining the upper bound converge.

Proof of Proposition 10. It follows from Proposition 7 that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+\delta\max(x,y)]^2}f(x)g(y)dxdy\leq\frac{2}{\delta^2}\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q},$$

with $\delta>0$. Let us now consider δ as a variable by taking $\delta=n\nu$ with $\nu>0$ and $n\in\mathbb{N}\backslash\{0\}$. So we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+n\nu\max(x,y)]^2}f(x)g(y)dxdy\leq\frac{2}{n^2\nu^2}\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}.$$

Summing both sides with respect to n with $n\in\mathbb{N}\backslash\{0\}$ and expressing only the upper bound via the well-known formula $\sum_{n=1}^{+\infty}1/n^2=\pi^2/6$, we have

$$\begin{aligned}\sum_{n=1}^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+n\nu\max(x,y)]^2}f(x)g(y)dxdy&\leq\sum_{n=1}^{+\infty}\left\{\frac{2}{n^2\nu^2}\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}\right\}\\&=\frac{2}{\nu^2}\left(\sum_{n=1}^{+\infty}\frac{1}{n^2}\right)\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}\\&=\frac{2}{\nu^2}\times\frac{\pi^2}{6}\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}\\&=\frac{\pi^2}{3\nu^2}\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}.\end{aligned}\tag{33}$$

For the left-hand side term, using the Fubini-Tonelli integral theorem and standard primitive rules, we obtain

$$\begin{aligned}
&\sum_{n=1}^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[xy+n\nu\max(x,y)]^2}f(x)g(y)dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty}\left\{\sum_{n=1}^{+\infty}\frac{1}{[xy+n\nu\max(x,y)]^2}\right\}f(x)g(y)dxdy\\
&=\frac{1}{\nu^2}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\max^2(x,y)}\left\{\sum_{n=1}^{+\infty}\frac{1}{\{n+xy/[\nu\max(x,y)]\}^2}\right\}f(x)g(y)dxdy\\
&=\frac{1}{\nu^2}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\max^2(x,y)}\mathcal{H}\left(\frac{xy}{\nu^2\max^2(x,y)}\right)f(x)g(y)dxdy. \qquad (34)
\end{aligned}$$

Combining Eqs. (33) and (34), we get

$$\frac{1}{\nu^2}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\max^2(x,y)}\mathcal{H}\left(\frac{xy}{\nu^2\max^2(x,y)}\right)f(x)g(y)dxdy\le\frac{\pi^2}{3\nu^2}\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}.$$

This is equivalent to

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\max^2(x,y)}\mathcal{H}\left(\frac{xy}{\nu^2\max^2(x,y)}\right)f(x)g(y)dxdy\le\frac{\pi^2}{3}\left[\int_0^{+\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}.$$

The proof of the proposition is completed. □

To the best of our knowledge, it is the first HH-type integral inequality defined with a "Hurwitz zeta power-max" kernel function, which is also non-homogeneous.

The theorem below can be seen as a functional generalization of the item 3 of Theorem 3. It follows the framework of Theorem 2.

Theorem 4. *Let $a,b\in\mathbb{R}\cup\{-\infty,+\infty\}$ with $a<b$, $c,d\in\mathbb{R}\cup\{-\infty,+\infty\}$ with $c<d$, $\alpha>1$, $\delta>0$, $p>1$, $q=p/(p-1)$, $f:(a,b)\mapsto(0,+\infty)$ and $g:(c,d)\mapsto(0,+\infty)$ be two functions, and $\phi:(a,b)\mapsto(0,+\infty)$ and $\psi:(c,d)\mapsto(0,+\infty)$ be two differentiable non-decreasing functions such that $\lim_{z\to a}\phi(z)=\beta$, $\lim_{z\to b}\phi(z)=+\infty$, $\lim_{z\to c}\psi(z)=\beta$ and $\lim_{z\to d}\psi(z)=+\infty$. Then we have*

$$\begin{aligned}
&\int_c^d\int_a^b\frac{1}{\{\phi(x)\psi(y)+\delta\max[\phi(x),\psi(y)]\}^2}f(x)g(y)dxdy\\
&\le\frac{1}{\delta}\left\{\int_a^b\frac{2\delta+\phi(x)}{\phi(x)[\phi(x)+\delta]^2}f^p(x)\frac{1}{[\phi'(x)]^{p-1}}dx\right\}^{1/p}\left\{\int_c^d\frac{2\delta+\psi(y)}{\psi(y)[\psi(y)+\delta]^2}g^q(y)\frac{1}{[\psi'(y)]^{q-1}}dy\right\}^{1/q},
\end{aligned}$$

under the condition that the integrals defining the upper bound converge.

Proof of Theorem 4. Applying the changes of variables $u=\phi(x)$ and $v=\psi(y)$, so that $x=\phi^{-1}(u)$ and $y=\psi^{-1}(v)$, with $dx=[1/\phi'(\phi^{-1}(u))]\,du$ and $dy=[1/\psi'(\psi^{-1}(v))]\,dv$, and paying attention to the integration domain, we have

$$\begin{aligned}
&\int_c^d\int_a^b\frac{1}{\{\phi(x)\psi(y)+\delta\max[\phi(x),\psi(y)]\}^2}f(x)g(y)dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[uv+\delta\max(u,v)]^2}f(\phi^{-1}(u))g(\psi^{-1}(v))\frac{1}{\phi'(\phi^{-1}(u))\psi'(\psi^{-1}(v))}dudv\\
&=\int_0^{+\infty}\int_0^{+\infty}\frac{1}{[uv+\delta\max(u,v)]^2}f_\dagger(u)g_\dagger(v)dudv, \qquad (35)
\end{aligned}$$

where

$$f_\dagger(u)=f(\phi^{-1}(u))\frac{1}{\phi'(\phi^{-1}(u))},\quad g_\dagger(v)=g(\psi^{-1}(v))\frac{1}{\psi'(\psi^{-1}(v))}.$$

Applying the item 3 of Theorem 3 to the functions $f_\dagger$ and $g_\dagger$, we obtain

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{1}{[uv+\delta\max(u,v)]^2} f_\dagger(u) g_\dagger(v) du dv \\
&\leq \frac{1}{\delta}\left[\int_0^{+\infty} \frac{2\delta+u}{u(u+\delta)^2} f_\dagger^p(u) du\right]^{1/p}\left[\int_0^{+\infty} \frac{2\delta+v}{v(v+\delta)^2} g_\dagger^q(v) dv\right]^{1/q} \\
&= \frac{1}{\delta}\left\{\int_0^{+\infty} \frac{2\delta+u}{u(u+\delta)^2} f^p(\phi^{-1}(u)) \frac{1}{[\phi'(\phi^{-1}(u))]^p} du\right\}^{1/p} \times \left\{\int_0^{+\infty} \frac{2\delta+v}{v(v+\delta)^2} g^q(\psi^{-1}(v)) \frac{1}{[\psi'(\psi^{-1}(v))]^q} dv\right\}^{1/q}.
\end{aligned} \tag{36}$$

Now, reversing the changes of variables process in the upper bound by setting $u=\phi(x)$ and $v=\psi(y)$, we get

$$\begin{aligned}
&\frac{1}{\delta}\left\{\int_0^{+\infty} \frac{2\delta+u}{u(u+\delta)^2} f^p(\phi^{-1}(u)) \frac{1}{[\phi'(\phi^{-1}(u))]^p} du\right\}^{1/p} \times \left\{\int_0^{+\infty} \frac{2\delta+v}{v(v+\delta)^2} g^q(\psi^{-1}(v)) \frac{1}{[\psi'(\psi^{-1}(v))]^q} dv\right\}^{1/q} \\
&= \frac{1}{\delta}\left\{\int_a^b \frac{2\delta+\phi(x)}{\phi(x)[\phi(x)+\delta]^2} f^p(x) \frac{1}{[\phi'(x)]^p}\phi'(x) dx\right\}^{1/p} \times \left\{\int_c^d \frac{2\delta+\psi(y)}{\psi(y)[\psi(y)+\delta]^2} g^q(y) \frac{1}{[\psi'(y)]^q}\psi'(y) dy\right\}^{1/q} \\
&= \frac{1}{\delta}\left\{\int_a^b \frac{2\delta+\phi(x)}{\phi(x)[\phi(x)+\delta]^2} f^p(x) \frac{1}{[\phi'(x)]^{p-1}} dx\right\}^{1/p} \times \left\{\int_c^d \frac{2\delta+\psi(y)}{\psi(y)[\psi(y)+\delta]^2} g^q(y) \frac{1}{[\psi'(y)]^{q-1}} dy\right\}^{1/q}.
\end{aligned} \tag{37}$$

Combining Eqs. (35), (36) and (37), we find that

$$\begin{aligned}
&\int_c^d\int_a^b \frac{1}{\{\phi(x)\psi(y)+\delta\max[\phi(x),\psi(y)]\}^2} f(x)g(y) dx dy \\
&\leq \frac{1}{\delta}\left\{\int_a^b \frac{2\delta+\phi(x)}{\phi(x)[\phi(x)+\delta]^2} f^p(x) \frac{1}{[\phi'(x)]^{p-1}} dx\right\}^{1/p}\left\{\int_c^d \frac{2\delta+\psi(y)}{\psi(y)[\psi(y)+\delta]^2} g^q(y) \frac{1}{[\psi'(y)]^{q-1}} dy\right\}^{1/q}.
\end{aligned}$$

This concludes the proof of the theorem. □

As an example, if we take $a=0$, "$b=+\infty$", $c=0$, "$d=+\infty$", $\phi(x)=x^\tau$ with $\tau>0$ and $\psi(y)=y^\iota$ with $\iota>0$, since $\phi'(x)=\tau x^{\tau-1}$ and $\psi'(y)=\iota y^{\iota-1}$, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{1}{[x^\tau y^\iota+\delta\max(x^\tau,y^\iota)]^2} f(x)g(y) dx dy \\
&\leq \frac{1}{\delta}\left[\int_0^{+\infty} \frac{2\delta+x^\tau}{x^\tau(x^\tau+\delta)^2} f^p(x) \frac{1}{(\tau x^{\tau-1})^{p-1}} dx\right]^{1/p}\left[\int_0^{+\infty} \frac{2\delta+y^\iota}{y^\iota(y^\iota+\delta)^2} g^q(y) \frac{1}{(\iota y^{\iota-1})^{q-1}} dy\right]^{1/q} \\
&= \frac{1}{\delta\tau^{1-1/p}\iota^{1-1/q}}\left[\int_0^{+\infty} \frac{2\delta+x^\tau}{x^{p(\tau-1)+1}(x^\tau+\delta)^2} f^p(x) dx\right]^{1/p}\left[\int_0^{+\infty} \frac{2\delta+y^\iota}{y^{q(\iota-1)+1}(y^\iota+\delta)^2} g^q(y) dy\right]^{1/q}.
\end{aligned}$$

Especially, when $\tau=1$ and $\iota=1$, we obtain the inequality of the item 3 of Theorem 3. Many more functional examples of this kind can be presented, extending the scope of this theorem.

Functional generalizations of the items 1 and 2 of Theorem 3 can be extended in a similar way. The details are omitted for reasons of redundancy.

4. Conclusion

In this article, we have established new HH-type integral inequalities based on a less studied class of kernel functions, called non-homogeneous power-max kernel functions. They complement the most standard homogeneous case, as studied in [2,5,18,19,25,26]. Our theory is centered on two main theorems, from which several propositions are derived. They are rigorously proved and provide a deeper understanding of these new inequalities. They are supported by simple but relevant numerical examples. Like all HH-type integral inequalities, our results can find applications in several mathematical and applied fields, including functional analysis, integral operators, and mathematical physics. Further investigations may explore further generalizations, possible connections with other integral inequalities and the consideration of non-homogeneous power-max kernel functions. As an example of the last point, we can think of investigating HH-type integral inequalities based on the kernel function

$$k(x,y) = \frac{1}{[\min(xy,1)+\max(x,y)]^{\alpha}},$$

with $\alpha > 1$. We leave these directions for future research.

Conflicts of Interest: "The author declares no conflict of interest."

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Appendix: R codes of the numerical work

R codes associated with Subsection 2.2

```
library(pracma)
```

Listing 1: Use of the library pracma containing the functions dblquad and integral

```
alpha = 2.5
beta = 1
p = 2

f1 <- function(x, y){ 1 / (x * y + max(x, y))^alpha}
dblquad(f1, beta, Inf, beta, Inf)


fun1 <- function(x) { (1 / x^alpha) * (1 / ((beta + 1)^(alpha - 1)) -1 / ((x + 1)^alpha))}
int1 = integral(fun1, beta, Inf)

fun2 <- function(x) { (1 / x^alpha) * (1 / ((beta + 1)^(alpha - 1)) -1 / ((x + 1)^alpha))}
int2 = integral(fun2, beta, Inf)

(1 / (alpha - 1)) * (int1^(1 / p)) * (int2^((p - 1) / p))
```

Listing 2: Example 1

```
alpha = 1.1
beta = 2
p = 3

f1 <- function(x, y){ exp(-y) / ((x * y + max(x, y))^alpha)}
dblquad(f1, beta, Inf, beta, Inf)


fun1 <- function(x) { (1 / x^alpha) * (1 / ((beta + 1)^(alpha - 1)) -1 / ((x + 1)^alpha))}
int1 = integral(fun1, beta, Inf)

fun2 <- function(x) { (1 / x^alpha) * (1 / ((beta + 1)^(alpha - 1)) -1 / ((x + 1)^alpha))
* exp(-(p / (p - 1)) * x)}
int2 = integral(fun2, beta, Inf)

(1 / (alpha - 1)) * (int1^(1 / p)) * (int2^((p - 1) / p))
```

Listing 3: Example 2

```
alpha = 2
beta = 0
p = 4 / 3

f1 <- function(x, y){ exp(-1 / x^2) * exp(-1 / y) / ((x * y + max(x, y))^alpha)}
dblquad(f1, beta, Inf, beta, Inf)


fun1 <- function(x) { (1 / x^alpha) * (1 / ((beta + 1)^(alpha - 1)) -1 / ((x + 1)^alpha))
* exp(-p * (1/x^2))}
int1 = integral(fun1, beta, Inf)

fun2 <- function(x) { (1 / x^alpha) * (1 / ((beta + 1)^(alpha - 1)) -1 / ((x + 1)^alpha))
* exp(-(p / (p - 1)) * (1/x))}
int2 = integral(fun2, beta, Inf)

(1 / (alpha - 1)) * (int1^(1 / p)) * (int2^((p - 1) / p))
```

Listing 4: Example 3

R codes associated with Subsection 3.2

```
delta = 2
p = 2

f1 <- function(x, y){ 1 / (x * y + delta * max(x, y))^2}
dblquad(f1, 1, Inf, 1, Inf)


fun1 <- function(x) { ((2 * x - 1) * delta + x^2)/(x^2 * (x + delta)^2) }
int1 = integral(fun1, 1, Inf)

fun2 <- function(x) { ((2 * x - 1) * delta + x^2)/(x^2 * (x + delta)^2)}
int2 = integral(fun2, 1, Inf)

(1 / (delta + 1)) * (int1^(1 / p)) * (int2^((p - 1) / p))
```

Listing 5: Example 4

```
delta = 1.5
p = 3

f1 <- function(x, y){ exp(-y) / (x * y + delta * max(x, y))^2}
dblquad(f1, 1, Inf, 1, Inf)


fun1 <- function(x) { ((2 * x - 1) * delta + x^2)/(x^2 * (x + delta)^2) }
int1 = integral(fun1, 1, Inf)

fun2 <- function(x) { ((2 * x - 1) * delta + x^2)/(x^2 * (x + delta)^2)
* exp(-(p / (p - 1)) * x)}
int2 = integral(fun2, 1, Inf)

(1 / (delta + 1)) * (int1^(1 / p)) * (int2^((p - 1) / p))
```

Listing 6: Example 5

```
delta = 3
p = 4 / 3

f1 <- function(x, y){ exp(-1 / x^2) * exp(-1 / y) / ((x * y + delta * max(x, y))^2)}
dblquad(f1, 0, Inf, 0, Inf)


fun1 <- function(x) { (2 * delta + x) / (x * (x + delta)^2) * exp(-p * (1 / x^2))}
int1 = integral(fun1, 0, Inf)

fun2 <- function(x) { (2 * delta + x) / (x * (x + delta)^2)* exp(-(p / (p-1)) * (1 / x))}
int2 = integral(fun2, 0, Inf)

(1 / delta) * (int1^(1 / p)) * (int2^((p - 1) / p))
```

Listing 7: Example 6