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Summation formulae for generalized Legendre-Gould Hopper polynomials

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Abstract: In this paper, we derive summation formulae for the generalized Legendre-Gould Hopper polynomials (gLeGHP) ${}_SH_n^{(m)}(x,y,z,w)$ and $\frac{{}_RH_n^{(m)}(x,y,z,w)}{n!}$ by using different analytical means on their respective generating functions. Further, we derive the summation formulae for polynomials related to ${}_SH_n^{(m)}(x,y,z,w)$ and $\frac{{}_RH_n^{(m)}(x,y,z,w)}{n!}$ as applications of main results. Some concluding remarks are also given.

Keywords: Generalized Legendre-Gould-Hopper polynomials, generalized Gould-Hopper polynomials, summation formulae

MSC: 33C45, 33C47, 33E20.

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1. Introduction

Recently, Hassan, [1] interoduced the Generalized Legendre-Gould-Hopper polynomials (gLeGHP) ${}_SH_n^{(m)}(x,y,z,w)$ and $\frac{{}_RH_n^{(m)}(x,y,z,w)}{n!}$ and studied their properties. These polynomials are defined by the following expansion series:

$$ {}_SH_n^{(m)}(x,y,z,w) \;=\; n!\sum_{k=0}^{[\frac{n}{m}]}\sum_{r=0}^{[\frac{k}{2}]} \frac{x^r w^{k-2r} y^{n+kz-mk}}{(r!)^2\,(k-2r)!\,(n-mk)!}, \tag{1}$$

$$ \frac{{}_RH_n^{(m)}(x,y,z,w)}{n!} \;=\; n!\sum_{k=0}^{[\frac{n}{m}]}\sum_{r=0}^{k} \frac{(-1)^{k-r} w^r x^{k-r} y^{n+kz-mk}}{(r!)^2\,[(k-r)!]^2\,(n-mk)!}, \tag{2}$$

and are defined by the following generating functions:

$$ \exp(yt+wy^z t^m)C_0(-y^{2z}xt^{2m}) = \sum_{n=0}^{\infty} {}_SH_n^{(m)}(x,y,z,w)\frac{t^n}{n!}, \tag{3}$$

$$ \exp(yt)C_0(y^z xt^m)C_0(-y^z wt^m) = \sum_{n=0}^{\infty} \frac{{}_RH_n^{(m)}(x,y,z,w)}{n!}\frac{t^n}{n!}. \tag{4}$$

The polynomials ${}_SH_n^{(m)}(x,y,z,w)$ and $\frac{{}_RH_n^{(m)}(x,y,z,w)}{n!}$ are defined by the operational definitions:

$$ {}_SH_n^{(m)}(x,y,z,w) = \exp\left(D_x^{-1}\frac{\partial^2}{\partial w^2}\right)H_n^{(m)}(y,z,w), \tag{5}$$

$$ \frac{{}_RH_n^{(m)}(x,y,z,w)}{n!} \;=\; \exp\Big(-D_x^{-1}\frac{\partial}{\partial D_w^{-1}}\Big)H_n^{(m)}(y,z,D_w^{-1}), \tag{6}$$

where $H_n^{(m)}(y,z,w)$ are the Generalized Gould-Hopper polynomials defined by the generating function [2] :

$$\exp(yt + wy^z t^m) = \sum_{n=0}^{\infty} H_n^{(m)}(y,z,w)\,\frac{t^n}{n!}, \tag{7}$$

and these polynomials are the solutions of the generalized heat equation:

$$\frac{\partial}{\partial w} H_n^{(m)}(y,z,w) = \frac{\partial^m}{\partial y^m} H_n^{(m)}(y,z,w), \;\; H_n^{(m)}(y,z,0) = y^n. \tag{8}$$

In particular, we note that:

$$H_n^{(2)}(y,z,w) = H_n(y,z,w), \tag{9}$$

where $H_n(y,z,w)$ denotes the 3-variable Hermite polynomials (3VHP), that can be defined by means of the generating function:

$$\exp(yt + wy^z t^2) = \sum_{n=0}^{\infty} H_n(y,z,w)\,\frac{t^n}{n!}. \tag{10}$$

Also, we note the following special case for these polynomials:

$$H_n(y,0,w) = H_n(y,w), \tag{11}$$

where $H_n(y,w)$ are 2-variable Hermite-Kamp*é* de F*é*riet polynomials (2VHKdFP) [3], that can be defined by means of the generating function:

$$\exp(yt + wt^2) = \sum_{n=0}^{\infty} H_n(y,w)\,\frac{t^n}{n!}. \tag{12}$$

Further, in particular, we note that:

$${}_SH_n^{(2)}(x,y,z,w) = {}_SH_n(x,y,z,w), \tag{13}$$

$$\frac{{}_RH_n^{(2)}(x,y,z,w)}{n!} = \frac{{}_RH_n(x,y,z,w)}{n!}, \tag{14}$$

where ${}_SH_n(x,y,z,w)$ and $\frac{{}_RH_n(x,y,z,w)}{n!}$ are denoted the 4-variable Legendre-Hermite polynomials (4VLeHP), that can be defined by means of the generating functions:

$$\exp(yt + wy^z t^2) C_0(-y^{2z} x t^4) = \sum_{n=0}^{\infty} {}_SH_n(x,y,z,w)\frac{t^n}{n!}, \tag{15}$$

$$\exp(yt) C_0(y^z x t^2) C_0(-y^z w t^2) = \sum_{n=0}^{\infty} \frac{{}_RH_n(x,y,z,w)}{n!}\,\frac{t^n}{n!}. \tag{16}$$

Also, we note the following special cases for these polynomials:

$${}_SH_n(x,y,0,w) = {}_SH_n(x,y,w), \tag{17}$$

$$\frac{{}_RH_n(x,y,0,w)}{n!} = \frac{{}_RH_n(x,y,w)}{n!}, \tag{18}$$

where ${}_SH_n(x,y,w)$ and $\frac{{}_RH_n(x,y,w)}{n!}$ are denoted the 3-variable Legendre-Hermite polynomials (3VLeHP), that can be defined by means of the generating functions:

$$\exp(yt + wt^2) C_0(-xt^4) = \sum_{n=0}^{\infty} {}_SH_n(x,y,w)\frac{t^n}{n!}, \tag{19}$$

$$\exp(yt)C_0(xt^2)C_0(-wt^2) = \sum_{n=0}^{\infty} \frac{{}_RH_n(x,y,w)}{n!}\frac{t^n}{n!}. \tag{20}$$

The study of the properties of multi-variable generalized special functions has provided new means of analysis for the solution of large classes of partial differential equations often encountered in physical problems. The relevance of the special functions in physics is well established. Most of the special functions of mathematical physics as well as their generalizations have been suggested by physical problems. The importance of multi-variable Hermite polynomials has been recognized [4–6] and these polynomials have been exploited to deal with quantum mechanical and optical beam transport problems.

It happens very often that the solution of a given problem in physics or applied mathematics requires the evalution of infinite sums, involving special functions. Problems of this type arise, for example, in the computation of the higher-order moments of a distribution or to evaluate transition matrix elements in quantum mechanics. In Ref. [4], it has been shown that the summation formulae of special functions, often encountered in applications ranging from electromagnetic processes to combinatorics, can be written in terms of Hermite polynomials of more than one variable. The work of this paper is motivated by the results on summation formulae for Hermite, Gould-Hopper and Laguerre-Gould Hopper polynomials due to [4,7–9]. In this paper we establish summation formulae for the gLeGHP ${}_SH_n^{(m)}(x,y,z,w)$ and $\frac{{}_RH_n^{(m)}(x,y,z,w)}{n!}$ by using different analytical means on their respective generating functions. Further, we derive the summation formulae for polynomials related to ${}_SH_n^{(m)}(x,y,z,w)$ and $\frac{{}_RH_n^{(m)}(x,y,z,w)}{n!}$ as applications. Finally, by combining operational and series rearrangement techniques, we derive certain other forms summation formulae for ${}_SH_n^{(m)}(x,y,z,w)$ and $\frac{{}_RH_n^{(m)}(x,y,z,w)}{n!}$.

2. Summation formulae for the generalized Legendre-Gould Hopper polynomials

First, we prove the following result involving the gLeGHP ${}_SH_n^{(m)}(x,y,z,w)$:

Theorem 1. *The following Summation formula for the gLeGHP ${}_SH_n^{(m)}(x,y,z,w)$ holds true:*

$${}_SH_{n+r}^{(m)}(x,y,z,v) = \sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}(-y)^{l+q}H_{k+p-l-q}^{(m)}(y,z,v-w)\,{}_SH_{n+r-k-p}^{(m)}(x,y,z,w). \tag{21}$$

Proof. Replacing t by t+u in Eq. (3) and using the formula ([10], p.52 (2)):

$$\sum_{n=0}^{\infty} f(n)\frac{(x+y)^n}{n!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}, \tag{22}$$

in the resultant equation, we find the following generating function for the gLeGHP ${}_SH_n^{(m)}(x,y,z,w)$:

$$\exp(y(t+u)+wy^z(t+u)^m)C_0(-y^{2z}x(t+u)^{2m}) = \sum_{n,r=0}^{\infty} {}_SH_{n+r}^{(m)}(x,y,z,w)\frac{t^n}{n!}\frac{u^r}{r!}, \tag{23}$$

which can be written as

$$C_0(-y^{2z}x(t+u)^{2m}) = \exp(-y(t+u)-wy^z(t+u)^m)\sum_{n,r=0}^{\infty} {}_SH_{n+r}^{(m)}(x,y,z,w)\frac{t^n}{n!}\frac{u^r}{r!}. \tag{24}$$

Replacing w by v in Eq. (24) and equating the resultant equation to itself, we find

$$\sum_{n,r=0}^{\infty} {}_SH_{n+r}^{(m)}(x,y,z,v)\frac{t^n}{n!}\frac{u^r}{r!} = \exp(-y(t+u)\exp(y(t+u)+(v-w)y^z(t+u)^m)\sum_{n,r=0}^{\infty} {}_SH_{n+r}^{(m)}(x,y,z,w)\frac{t^n}{n!}\frac{u^r}{r!}, \tag{25}$$

which on expanding the first exponential in series and using the generating function (7) in the second exponential on the r.h.s., becomes

$$\sum_{n,r=0}^{\infty} {}_SH_{n+r}^{(m)}(x,y,z,v)\frac{t^n}{n!}\frac{u^r}{r!} = \sum_{l=0}^{\infty}(-1)^l y^l\,\frac{(t+u)^l}{l!}\sum_{k=0}^{\infty}H_k^{(m)}(y,z,v-w)\,\frac{(t+u)^k}{k!}\sum_{n,r=0}^{\infty}{}_SH_{n+r}^{(m)}(x,y,z,w)\frac{t^n}{n!}\frac{u^r}{r!}. \tag{26}$$

Again, using formula (22) in the first and second summation on the r.h.s., we have

$$\sum_{n,r=0}^{\infty} {}_SH_{n+r}^{(m)}(x,y,z,v)\frac{t^n}{n!}\frac{u^r}{r!} = \sum_{l,q=0}^{\infty}(-1)^{l+q}y^{l+q}\,\frac{t^l}{l!}\frac{u^q}{q!}\sum_{k,p=0}^{\infty}H_{k+p}^{(m)}(y,z,v-w)\frac{t^k}{k!}\frac{u^p}{p!}\sum_{n,r=0}^{\infty}{}_SH_{n+r}^{(m)}(x,y,z,w)\frac{t^n}{n!}\frac{u^r}{r!}. \tag{27}$$

Now, replacing k by k-l, p by p-q, n by n-k and r by r-p in the r.h.s. of Eq. (27) and using the lemma ([10], p.100 (1))

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k), \tag{28}$$

in the resultant equation, we find

$$\sum_{n,r=0}^{\infty} {}_SH_{n+r}^{(m)}(x,y,z,v)\,\frac{t^n}{n!}\frac{u^r}{r!} = \sum_{n,r=0}^{\infty}\sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\frac{(-y)^{l+q}\,H_{k+P-l-q}^{(m)}(y,z,v-w)\,{}_SH_{n+r-k-p}^{(m)}(x,y,z,w)}{l!q!(n-k)!(r-p)!(k-l)!(p-q)!}\,t^n u^r. \tag{29}$$

Finally, on equating the coefficients of like powers of t and u in Eq. (29), we get assertion (21) of Theorem 1. □

Remark 1. Replacing v by w in assertion (21) of Theorem 1 and using relation (8), we deduce the following consequence of Theorem 1.

Corollary 1. *The following Summation formula for the gLeGHP ${}_SH_n^{(m)}(x,y,z,w)$ holds true:*

$${}_SH_{n+r}^{(m)}(x,y,z,w) = \sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}(-1)^{l+q}\,y^{k+p}\,{}_SH_{n+r-k-p}^{(m)}(x,y,z,w). \tag{30}$$

Next, we prove the following result involving products of the gLeGHP ${}_SH_n^{(m)}(x,y,z,w)$:

Theorem 2. *The following Summation formula involving products of the gLeGHP ${}_SH_n^{(m)}(x,y,z,w)$ holds true:*

$$\begin{aligned}{}_SH_n^{(m)}(x,y,z,v)\,{}_SH_r^{(m)}(X,Y,Z,V) &= \sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}(-y)^{k-l}(-Y)^{p-q}\\ &\times H_l^{(m)}(y,z,v-w)H_q^{(m)}(Y,Z,V-W)\,{}_SH_{n-k}^{(m)}(x,y,z,w)\,{}_SH_{r-p}^{(m)}(X,Y,Z,W).\end{aligned} \tag{31}$$

Proof. Consider the product of the gLeGHP ${}_SH_n^{(m)}(x,y,z,w)$ generating function (3) in the following form:

$$\begin{aligned}&\exp(yt+YT+wy^zt^m+WY^ZT^m)C_0(-y^{2z}xt^{2m})C_0(-Y^{2Z}XT^{2m})\\ &=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}{}_SH_n^{(m)}(x,y,z,w)\,{}_SH_r^{(m)}(X,Y,Z,W)\frac{t^n}{n!}\frac{T^r}{r!}.\end{aligned} \tag{32}$$

Replacing w by v and W by V in Eq. (32) and equating the resultant equation to itself, we find

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}{}_SH_n^{(m)}(x,y,z,v)\,{}_SH_r^{(m)}(X,Y,Z,V)\frac{t^n}{n!}\frac{T^r}{r!} &= \exp(-yt-YT)\exp(yt+(v-w)y^zt^m)\\ \times\exp(YT+(V-W)Y^ZT^m)\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}{}_SH_n^{(m)}(x,y,z,w)\,{}_SH_r^{(m)}(X,Y,Z,W)\frac{t^n}{n!}\frac{T^r}{r!},\end{aligned} \tag{33}$$

which on expanding the first exponential in series and using the generating function (7) in the second and third exponential on the r.h.s., becomes

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} {}_SH_n^{(m)}(x,y,z,v)\,{}_SH_r^{(m)}(X,Y,Z,V)\frac{t^n}{n!}\frac{T^r}{r!} = \sum_{k=0}^{\infty}(-1)^k\frac{(yt+YT)^k}{k!}$$
$$\times\sum_{l,q=0}^{\infty} H_l^{(m)}(y,z,v-w)H_q^{(m)}(Y,Z,V-W)\frac{t^l}{l!}\frac{T^q}{q!}\times\sum_{n,r=0}^{\infty} {}_SH_n^{(m)}(x,y,z,w){}_SH_r^{(m)}(X,Y,Z,W)\frac{t^n}{n!}\frac{T^r}{r!}. \quad (34)$$

Now, using formula (22) in the first summation on the r.h.s., we have

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} {}_SH_n^{(m)}(x,y,z,v)\,{}_SH_r^{(m)}(X,Y,Z,V)\frac{t^n}{n!}\frac{T^r}{r!} = \sum_{k,l=0}^{\infty}\sum_{p,q=0}^{\infty}(-y)^kH_l^{(m)}(y,z,v-w)$$
$$\times\frac{t^{k+l}}{k!l!}(-Y)^pH_q^{(m)}(Y,Z,V-W)\frac{T^{p+q}}{p!q!}\times\sum_{n,r=0}^{\infty} {}_SH_n^{(m)}(x,y,z,w){}_SH_r^{(m)}(X,Y,Z,W)\frac{t^n}{n!}\frac{T^r}{r!}. \quad (35)$$

Finally, replacing k by k-l, p by p-q, n by n-k and r by r-p and using (28) in the r.h.s. of the above equation and then equating the coefficients of like powers of t and T , we get assertion (31) of Theorem 2. □

Remark 2. Replacing v by w and V by W in assertion (31) of Theorem 2 and using relation (8), we deduce the following consequence of Theorem 2.

Corollary 2. *The following Summation formula involving products of the gLeGHP* ${}_SH_n^{(m)}(x,y,z,w)$ *holds true:*

$${}_SH_n^{(m)}(x,y,z,w){}_SH_r^{(m)}(X,Y,Z,W) = \sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}(-1)^{l+q}(-y)^k(-Y)^p$$
$$\times\,{}_SH_{n-k}^{(m)}(x,y,z,w){}_SH_{r-p}^{(m)}(X,Y,Z,W). \quad (36)$$

Further, we prove the following result involving the gLeGHP $\frac{{}_RH_n^{(m)}(x,y,z,w)}{n!}$:

Theorem 3. *The following Summation formula for the gLeGHP* $\frac{{}_RH_n^{(m)}(x,y,z,w)}{n!}$ *holds true:*

$$\frac{{}_RH_{n+r}^{(m)}(-w,y,z,-x)}{(n+r)!} = \sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}H_{l+q}^{(m)}(y,2z,w)H_{k+p-l-q}^{(m)}(-y,2z,-w)$$
$$\times\frac{{}_RH_{n+r-k-p}^{(m)}(x,y,z,w)}{(n+r-k-p)!}. \quad (37)$$

Proof. Replacing t by $t+u$ in Eq. (4) and using the formula (22) and then making use the same previous process given in the proof of Theorem 1, we get assertion (37) of Theorem 3. □

Also, we prove the following result involving products of the gLeGHP $\frac{{}_RH_n^{(m)}(x,y,z,w)}{n!}$:

Theorem 4. *The following summation formula involving products of the gLeGHP* $\frac{{}_RH_n^{(m)}(x,y,z,w)}{n!}$ *holds true:*

$$\frac{{}_RH_n^{(m)}(-w,y,z,-x)}{n!}\frac{{}_RH_r^{(m)}(-W,Y,Z,-X)}{r!}$$
$$=\sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}H_l^{(m)}(y,2z,w)H_q^{(m)}(Y,2Z,W)H_{k-l}^{(m)}(-y,2z,-w)$$
$$\times H_{P-q}^{(m)}(-Y,2Z,-W)\frac{{}_RH_{n-k}^{(m)}(x,y,z,w)}{(n-k)!}\frac{{}_RH_{r-p}^{(m)}(X,Y,Z,W)}{(r-p)!}. \quad (38)$$

Proof. Consider the product of the gLeGHP $\frac{{}_RH_n^{(m)}(x,y,z,w)}{n!}$ generating function (4) in the following form:

$$\begin{aligned}&\exp(yt+YT)C_0(y^z x t^m)C_0(-y^z w t^m)C_0(Y^Z X T^m)C_0(-Y^Z W T^m)\\&=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{{}_RH_n^{(m)}(x,y,z,w)}{n!}\,\frac{{}_RH_r^{(m)}(X,Y,Z,W)}{r!}\,\frac{t^n}{n!}\,\frac{T^r}{r!}.\end{aligned} \tag{39}$$

Replacing x by $-w$, w by $-x$, X by $-W$ and W by $-X$ in Eq. (39) and equating the resultant equation to itself and then using Eqs. (7) and (28) respectively, we get assertion (38) of Theorem 4. □

3. Applications

I. Taking $r=0$ in Eq. (21) and replacing v by $v+w$ in the resultant equation, we get

$${}_SH_n^{(m)}(x,y,z,v+w)=\sum_{k,l=0}^{n,k}\binom{n}{k}\binom{k}{l}(-y)^l H_{k-l}^{(m)}(y,z,v)\,{}_SH_{n-k}^{(m)}(x,y,z,w), \tag{40}$$

which on taking $v=0$ and using relation (8), yields

$${}_SH_n^{(m)}(x,y,z,w)=\sum_{k,l=0}^{n,k}\binom{n}{k}\binom{k}{l}(-1)^l y^k\,{}_SH_{n-k}^{(m)}(x,y,z,w). \tag{41}$$

Next, taking $m=2$ in Eqs. (21), (30), (40) and (41) , we get the following summation formulae for the 4VLeHP ${}_SH_n(x,y,z,w)$:

$${}_SH_{n+r}(x,y,z,v)=\sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}(-y)^{l+q}H_{k+p-l-q}(y,z,v-w)\,{}_SH_{n+r-k-p}(x,y,z,w), \tag{42}$$

$${}_SH_{n+r}(x,y,z,w)=\sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}(-1)^{l+q}\,y^{k+p}\,{}_SH_{n+r-k-p}(x,y,z,w), \tag{43}$$

$${}_SH_n(x,y,z,v+w)=\sum_{k,l=0}^{n,k}\binom{n}{k}\binom{k}{l}(-y)^l\,H_{k-l}(y,z,v)\,{}_SH_{n-k}(x,y,z,w), \tag{44}$$

$${}_SH_n(x,y,z,w)=\sum_{k,l=0}^{n,k}\binom{n}{k}\binom{k}{l}(-1)^l y^k\,{}_SH_{n-k}(x,y,z,w). \tag{45}$$

Again, taking $z=0$ in Eqs. (42), (43), (44) and (45), we get the following summation formulae for the 3VLeHP ${}_SH_n(x,y,w)$:

$${}_SH_{n+r}(x,y,v)=\sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}(-y)^{l+q}H_{k+p-l-q}(y,v-w)\,{}_SH_{n+r-k-p}(x,y,w), \tag{46}$$

$${}_SH_{n+r}(x,y,w)=\sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}(-1)^{l+q}\,y^{k+p}\,{}_SH_{n+r-k-p}(x,y,w), \tag{47}$$

$${}_SH_n(x,y,v+w)=\sum_{k,l=0}^{n,k}\binom{n}{k}\binom{k}{l}(-y)^l\,H_{k-l}(y,v)\,{}_SH_{n-k}(x,y,w), \tag{48}$$

$${}_SH_n(x,y,w)=\sum_{k,l=0}^{n,k}\binom{n}{k}\binom{k}{l}(-1)^l y^k\,{}_SH_{n-k}(x,y,w). \tag{49}$$

II. Taking m=2 in Eqs. (31) and (36), we get the following summation formulae involving products of the 4VLeHP ${}_SH_n(x,y,z,w)$:

$$
{}_SH_n(x,y,z,v)\,{}_SH_r(X,Y,Z,V)=\sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}(-y)^{k-l}(-Y)^{p-q}
\times H_l(y,z,v-w)H_q(Y,Z,V-W)\,{}_SH_{n-k}(x,y,z,w)\,{}_SH_{r-p}(X,Y,Z,W), \tag{50}
$$

$$
{}_SH_n(x,y,z,w)\,{}_SH_r(X,Y,Z,W)=\sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}(-1)^{l+q}(-y)^{k}(-Y)^{p}
\times {}_SH_{n-k}(x,y,z,w)\,{}_SH_{r-p}(X,Y,Z,W). \tag{51}
$$

Again, taking $z=Z=0$ in Eqs. (50) and (51), we get the following summation formulae involving products of the 3VLeHP ${}_SH_n(x,y,z)$:

$$
{}_SH_n(x,y,v)\,{}_SH_r(X,Y,V)=\sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}(-y)^{k-l}(-Y)^{p-q}
\times H_l(y,v-w)H_q(Y,V-W)\,{}_SH_{n-k}(x,y,w)\,{}_SH_{r-p}(X,Y,W), \tag{52}
$$

$$
{}_SH_n(x,y,w)\,{}_SH_r(X,Y,W)=\sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}(-1)^{l+q}(-y)^{k}(-Y)^{p}
\times {}_SH_{n-k}(x,y,w)\,{}_SH_{r-p}(X,Y,W). \tag{53}
$$

III. Taking r=0 in Eqs. (37), we get the following Summation formula for the gLeGHP $\frac{{}_RH_n^{(m)}(x,y,z,w)}{n!}$:

$$
\frac{{}_RH_n^{(m)}(-w,y,z,-x)}{n!}=\sum_{k,l=0}^{n,k}\binom{n}{k}\binom{k}{l}H_l^{(m)}(y,2z,w)H_{k-l}^{(m)}(-y,2z,-w)\times\frac{{}_RH_{n-k}^{(m)}(x,y,z,w)}{(n-k)!}. \tag{54}
$$

Next, taking $m=2$ in Eqs. (37) and (54), we get the following Summation formulae for the 4VLeHP $\frac{{}_RH_n(x,y,z,w)}{n!}$:

$$
\frac{{}_RH_{n+r}(-w,y,z,-x)}{(n+r)!}=\sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}H_{l+q}(y,2z,w)H_{k+p-l-q}(-y,2z,-w)
\times\frac{{}_RH_{n+r-k-p}(x,y,z,w)}{(n+r-k-p)!}, \tag{55}
$$

$$
\frac{{}_RH_n(-w,y,z,-x)}{n!}=\sum_{k,l=0}^{n,k}\binom{n}{k}\binom{k}{l}H_l(y,2z,w)H_{k-l}(-y,2z,-w)\times\frac{{}_RH_{n-k}(x,y,z,w)}{(n-k)!}. \tag{56}
$$

Again, taking $z=0$ in Eqs. (55) and (56), we get the following summation formulae for the 3VLeHP $\frac{{}_RH_n(x,y,z)}{n!}$:

$$
\frac{{}_RH_{n+r}(-w,y,-x)}{(n+r)!}=\sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}H_{l+q}(y,w)H_{k+p-l-q}(-y,-w)\times\frac{{}_RH_{n+r-k-p}(x,y,w)}{(n+r-k-p)!}, \tag{57}
$$

$$\frac{{}_RH_n(-w,y,-x)}{n!} = \sum_{k,l=0}^{n,k}\binom{n}{k}\binom{k}{l}H_l(y,w)H_{k-l}(-y,-w) \times \frac{{}_RH_{n-k}(x,y,w)}{(n-k)!}. \tag{58}$$

IV. Taking $m = 2$ in Eq. (38), we get the following summation formula involving products of the 4VLeHP $\frac{{}_RH_n(x,y,z,w)}{n!}$:

$$\begin{aligned}\frac{{}_RH_n(-w,y,z,-x)}{n!}\frac{{}_RH_r(-W,Y,Z,-X)}{r!}\\ = \sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}H_l(y,2z,w)H_q(Y,2Z,W)H_{k-l}(-y,2z,-w)\\ \times H_{P-q}(-Y,2Z,-W)\frac{{}_RH_{n-k}(x,y,z,w)}{(n-k)!}\frac{{}_RH_{r-p}(X,Y,Z,W)}{(r-p)!}.\end{aligned} \tag{59}$$

Also, taking $z = Z = 0$ in Eq. (59), we get the following summation formula involving products of the 3VLeHP $\frac{{}_RH_n(x,y,z)}{n!}$:

$$\begin{aligned}\frac{{}_RH_n(-w,y,-x)}{n!}\frac{{}_RH_r(-W,Y,-X)}{r!} = \sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}H_l(y,w)H_q(Y,W)H_{k-l}(-y,-w)\\ \times H_{P-q}(-Y,-W)\frac{{}_RH_{n-k}(x,y,w)}{(n-k)!}\frac{{}_RH_{r-p}(X,Y,W)}{(r-p)!}.\end{aligned} \tag{60}$$

4. Concluding Remarks

In view of Eqs. (5), (6) and (7) , we can combin operational and series rearrangement techniques to derive certain other forms summation formulae for ${}_SH_n^{(m)}(x,y,z,w)$ and $\frac{{}_RH_n^{(m)}(x,y,z,w)}{n!}$ respectively.

Replacing t by t+u in Eq. (7) and using the formula (22) and then making use the same previous process given in the proof of Theorem 1, we get

$$H_{n+r}^{(m)}(y,z,v) = \sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}(-y)^{l+q}H_{k+p-l-q}^{(m)}(y,z,v-w)H_{n+r-k-p}^{(m)}(y,z,w). \tag{61}$$

Next, operating $\exp\left(D_x^{-1}\frac{\partial^2}{\partial v^2}\right)$ on Eq. (61) and then using operational definition (5), we get the following summation formula for ${}_SH_n^{(m)}(x,y,z,w)$:

$${}_SH_{n+r}^{(m)}(x,y,z,v) = \sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}(-y)^{l+q}H_{n+r-k-p}^{(m)}(y,z,w){}_SH_{k+p-l-q}^{(m)}(x,y,z,v-w). \tag{62}$$

Also, replacing v by D_v^{-1} in Eq. (61) and then operating $\exp\left(-D_x^{-1}\frac{\partial}{\partial D_v^{-1}}\right)$ on the both side and using operational definition (6), we get the following summation formula for $\frac{{}_RH_n^{(m)}(x,y,z,w)}{n!}$:

$$\frac{{}_RH_{n+r}^{(m)}(x,y,z,v)}{(n+r)!} = \sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}(-y)^{l+q}H_{n+r-k-p}^{(m)}(y,z,w) \times \frac{{}_RH_{k+p-l-q}^{(m)}(x,y,z,v-w)}{(k+p-l-q)!}. \tag{63}$$

Further, taking w=0 in Eqs. (62) and (63), we get respectively

$${}_SH_{n+r}^{(m)}(x,y,z,v) = \sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}(-y)^{l+q}\,(\frac{1}{y})^{k+p}\,y^{n+r}\,{}_SH_{k+p-l-q}^{(m)}(x,y,z,v), \tag{64}$$

$$\frac{{}_RH^{(m)}_{n+r}(x,y,z,v)}{(n+r)!} = \sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}(-y)^{l+q}\,(\frac{1}{y})^{k+p}\,y^{n+r}\times\frac{{}_RH^{(m)}_{k+p-l-q}(x,y,z,v)}{(k+p-l-q)!}. \tag{65}$$

Now, Consider the product of the Generalized Gould-Hopper polynomials $H^{(m)}_n(y,z,w)$ generating function (7) in the following form:

$$\exp(yt + wy^z t^m + YT + WY^Z T^m) = \sum_{n,r=0}^{\infty} H^{(m)}_n(y,z,w)\,H^{(m)}_r(Y,Z,W)\,\frac{t^n}{n!}\frac{T^r}{r!}. \tag{66}$$

Replacing w by v and W by V in Eq. (66) and equating the resultant equation to itself and then following the same process given in the proof of Theorem 2, we get

$$\begin{aligned}H^{(m)}_n(y,z,v)\,H^{(m)}_r(Y,Z,V) = &\sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}(-y)^{k-l}(-Y)^{p-q}\\ &\times H^{(m)}_{n-k}(y,z,w)H^{(m)}_{r-p}(Y,Z,W)\,H^{(m)}_l(y,z,v-w)\,H^{(m)}_q(Y,Z,V-W).\end{aligned} \tag{67}$$

Operating $\exp\left(D_x^{-1}\frac{\partial^2}{\partial v^2}\right)\exp\left(D_X^{-1}\frac{\partial^2}{\partial V^2}\right)$ on Eq. (67) and then using operational definition (5), we get

$$\begin{aligned}{}_SH^{(m)}_n(x,y,z,v)\,{}_SH^{(m)}_r(X,Y,Z,V) = &\sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}(-y)^{k-l}(-Y)^{p-q}\\ &\times H^{(m)}_{n-k}(y,z,w)H^{(m)}_{r-p}(Y,Z,W)\,{}_SH^{(m)}_l(x,y,z,v-w)\,{}_SH^{(m)}_q(X,Y,Z,V-W).\end{aligned} \tag{68}$$

Also, replacing v by D_v^{-1} and V by D_V^{-1} in Eq. (67) and then operating $\exp\left(-D_x^{-1}\frac{\partial}{\partial D_v^{-1}}\right)\exp\left(-D_X^{-1}\frac{\partial}{\partial D_V^{-1}}\right)$ on the both side and using operational definition (6), we get

$$\begin{aligned}\frac{{}_RH^{(m)}_n(x,y,z,v)}{n!}\frac{{}_RH^{(m)}_r(X,Y,Z,V)}{r!} = &\sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}(-y)^{k-l}(-Y)^{p-q}\\ &\times H^{(m)}_{n-k}(y,z,w)H^{(m)}_{r-p}(Y,Z,W)\frac{{}_RH^{(m)}_l(x,y,z,v-w)}{l!}\frac{{}_RH^{(m)}_q(X,Y,Z,V-W)}{q!}.\end{aligned} \tag{69}$$

Further, taking $w = W = 0$ in Eqs. (68) and (69), we get respectively

$$\begin{aligned}{}_SH^{(m)}_n(x,y,z,v)\,{}_SH^{(m)}_r(X,Y,Z,V) = &\sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}(-y)^{k-l}(-Y)^{p-q}\\ &\times y^{n-k}\,Y^{r-p}\,{}_SH^{(m)}_l(x,y,z,v)\,{}_SH^{(m)}_q(X,Y,Z,V),\end{aligned} \tag{70}$$

$$\begin{aligned}\frac{{}_RH^{(m)}_n(x,y,z,v)}{n!}\frac{{}_RH^{(m)}_r(X,Y,Z,V)}{r!} = &\sum_{k,p=0}^{n,r}\sum_{l,q=0}^{k,p}\binom{n}{k}\binom{r}{p}\binom{k}{l}\binom{p}{q}(-y)^{k-l}(-Y)^{p-q}\\ &\times y^{n-k}\,Y^{r-p}\,\frac{{}_RH^{(m)}_l(x,y,z,v)}{l!}\frac{{}_RH^{(m)}_q(X,Y,Z,V)}{q!}.\end{aligned} \tag{71}$$

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