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A new extended one-parameter Hardy-Hilbert integral inequality

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Abstract: This paper studies a natural one-parameter extension of the Hardy-Hilbert integral inequality. The proposed generalization introduces a parameter that interpolates between different forms. This allows us to establish a hierarchy among a family of related double integrals. We provide sharp upper bounds expressed in terms of the integral norms of the functions involved. In doing so, we extend a classical result while maintaining the optimality of the constant in the original inequality.

Keywords: Hardy-Hilbert integral inequality; power function; Heinz inequality; Hölder integral inequality; optimal constant

MSC: 26D15.

1. Introduction

The study of integral inequalities has long been at the heart of modern analysis. One of the most celebrated examples is the Hardy-Hilbert integral inequality. This inequality captures the boundedness of a bilinear integral form defined by a specific ratio integrand. Its precise formulation is given below. Let $p \in (1, +\infty)$, q be the conjugate exponent of p , i.e., $1/p + 1/q = 1$, and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions such that

$$\int_{(0,+\infty)} f^p(x)dx < +\infty, \quad \int_{(0,+\infty)} g^q(y)dy < +\infty.$$

Then we have

$$\iint_{(0,+\infty)^2} \frac{f(x)g(y)}{x+y} dx dy \leq \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left(\int_{(0,+\infty)} f^p(x)dx \right)^{1/p} \left(\int_{(0,+\infty)} g^q(y)dy \right)^{1/q}. \quad (1)$$

The constant factor $\pi/\sin(\pi/p)$ is sharp and cannot be improved. The appearance of the L^p - and L^q -norms of f and g highlights the functional-analytic nature of the inequality and its connection with the classical theory of Lebesgue spaces.

This fundamental result has inspired a vast body of literature. Various problems in harmonic analysis and functional inequalities have been solved by developing weighted variants, discrete analogues, and operator-theoretic interpretations. The Hardy-Hilbert integral inequality also appears in the spectral theory of integral operators with homogeneous kernel functions, demonstrating its versatility across several branches of analysis. Over the last few decades, a considerable number of refinements and extensions have been obtained. These are documented in monographs such as [1–3] and in survey papers such as [4]. More recent progress, including new approaches and applications, can be found in [5–10].

In this paper, we investigate a natural Hardy-Hilbert-type integral inequality depending on an adjustable parameter. The central object is the double integral

$$\iint_{(0,+\infty)^2} \frac{f(x)g(y)}{x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha} dx dy,$$

where $f, g : [0, +\infty) \mapsto [0, +\infty)$ are two functions and α denotes a parameter such that $\alpha \in [0, 1/2]$. The presence of α governs the symmetry between the variables x and y , and different values of this parameter interpolate between distinct integral forms. In particular, when $\alpha = 0$, the denominator reduces to $x + y$, and the classical double integral form is obtained. In this case, the corresponding inequality coincides with the Hardy-Hilbert integral inequality recalled in Eq. (1). The general case $\alpha \in (0, 1/2]$ yields new integral forms, for which we derive sharp upper bounds in terms of the L^p - and L^q -norms of f and g . This approach shows that a simple structural modification parametrized by α leads to an inequality that extends the classical result.

The remainder of the paper is organized as follows: Section 2 is devoted to the statement and proof of the main theorems. Section 3 provides concluding remarks and perspectives for future research.

2. Theorems

2.1. First observation

The theorem below establishes a hierarchy between the classical Hardy-Hilbert double integral, as recalled in Eq. (1), and the generalized integral considered in this study. This ordering is obtained by applying the well-known Heinz inequality.

Theorem 1. *Let $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then, for any $\alpha \in [0, 1/2]$, we have*

$$\iint_{(0,+\infty)^2} \frac{f(x)g(y)}{x+y} dx dy \leq \iint_{(0,+\infty)^2} \frac{f(x)g(y)}{x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha} dx dy,$$

assuming that

$$\iint_{(0,+\infty)^2} \frac{f(x)g(y)}{x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha} dx dy < +\infty.$$

Proof of Theorem 1. The Heinz inequality implies that, for any $\alpha \in [0, 1/2]$ and any $x, y \in (0, +\infty)$,

$$x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha \leq x + y.$$

Therefore, we have

$$\frac{1}{x+y} \leq \frac{1}{x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha},$$

so that

$$\iint_{(0,+\infty)^2} \frac{f(x)g(y)}{x+y} dx dy \leq \iint_{(0,+\infty)^2} \frac{f(x)g(y)}{x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha} dx dy,$$

(by the non-negativity of f and g , which will be implicitly used throughout the paper). This concludes the proof. $\square$

In view of this result and the framework of the classical Hardy-Hilbert integral inequality stated in Eq. (1), it is natural to expect that the constant

$$\frac{\pi}{\sin\left(\frac{\pi}{p}\right)},$$

is difficult to improve. This situation is clarified in the subsection below.

2.2. Main theorem

The theorem below provides an expression for the upper bound of the considered one-parameter double integral, stated entirely in terms of the L^p - and L^q -norms of f and g .

Theorem 2. Let $p \in (1, +\infty)$, q be the conjugate exponent of p , i.e., $1/p + 1/q = 1$, and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions such that

$$\int_{(0,+\infty)} f^p(x)dx < +\infty, \quad \int_{(0,+\infty)} g^q(y)dy < +\infty.$$

Then, for any $\alpha \in [0, \min(1/2, 1/p, 1/q))$, we have

$$\iint_{(0,+\infty)^2} \frac{f(x)g(y)}{x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha} dx dy \leq C(\alpha, p) \left(\int_{(0,+\infty)} f^p(x)dx \right)^{1/p} \left(\int_{(0,+\infty)} g^q(y)dy \right)^{1/q},$$

where

$$C(\alpha, p) = \frac{\pi}{(1-2\alpha) \sin\left(\frac{\pi(1/p - \alpha)}{1-2\alpha}\right)}.$$

Proof of Theorem 2. Based on the main double integral, making the change of variables $y = tx$ with $t \in (0, +\infty)$ and applying the Fubini-Tonelli integral theorem, we get

$$\begin{aligned} \iint_{(0,+\infty)^2} \frac{f(x)g(y)}{x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha} dx dy &= \iint_{(0,+\infty)^2} \frac{f(x)g(tx)}{x^\alpha (tx)^{1-\alpha} + x^{1-\alpha} (tx)^\alpha} dx (xdt) \\ &= \iint_{(0,+\infty)^2} \frac{f(x)g(tx)}{t^{1-\alpha} + t^\alpha} dx dt \\ &= \int_{(0,+\infty)} \frac{1}{t^{1-\alpha} + t^\alpha} \left(\int_{(0,+\infty)} f(x)g(tx) dx \right) dt. \end{aligned}$$

The Hölder integral inequality and the change of variables $z = tx$ with $z \in (0, +\infty)$ give

$$\begin{aligned} \int_{(0,+\infty)} f(x)g(tx) dx &\leq \left(\int_{(0,+\infty)} f^p(x) dx \right)^{1/p} \left(\int_{(0,+\infty)} g^q(tx) dx \right)^{1/q} \\ &= \left(\int_{(0,+\infty)} f^p(x) dx \right)^{1/p} \left(\int_{(0,+\infty)} g^q(z) t^{-1} dz \right)^{1/q} \\ &= t^{-1/q} \left(\int_{(0,+\infty)} f^p(x) dx \right)^{1/p} \left(\int_{(0,+\infty)} g^q(y) dy \right)^{1/q}. \end{aligned}$$

Combining the inequalities above, we obtain

$$\begin{aligned} \iint_{(0,+\infty)^2} \frac{f(x)g(y)}{x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha} dx dy &\leq \int_{(0,+\infty)} \frac{1}{t^{1-\alpha} + t^\alpha} \left(t^{-1/q} \left(\int_{(0,+\infty)} f^p(x) dx \right)^{1/p} \left(\int_{(0,+\infty)} g^q(y) dy \right)^{1/q} \right) dt \\ &\leq \left(\int_{(0,+\infty)} \frac{t^{-1/q}}{t^{1-\alpha} + t^\alpha} dt \right) \left(\int_{(0,+\infty)} f^p(x) dx \right)^{1/p} \left(\int_{(0,+\infty)} g^q(y) dy \right)^{1/q}. \end{aligned}$$

Now, we need to evaluate the first integral. Using the identity $-1/q = 1/p - 1$, we can write

$$\int_{(0,+\infty)} \frac{t^{-1/q}}{t^{1-\alpha} + t^\alpha} dt = \int_{(0,+\infty)} \frac{t^{-1/q-\alpha}}{1 + t^{1-2\alpha}} dt = \int_{(0,+\infty)} \frac{t^{s-1}}{1 + t^m} dt,$$

where $s = 1/p - \alpha$ and $m = 1 - 2\alpha$ with $s \in (0, m)$ because $\alpha \in [0, \min(1/2, 1/p, 1/q))$. In this setting, the following well-known integral formula holds:

$$\int_{(0,+\infty)} \frac{t^{s-1}}{1 + t^m} dt = \frac{\pi}{m \sin\left(\frac{\pi s}{m}\right)}. \quad (2)$$

See [11, 3.241 2]. Hence, by substitution, we derive

$$\int_{(0,+\infty)} \frac{t^{-1/q}}{t^{1-\alpha} + t^\alpha} dt = \frac{\pi}{(1-2\alpha) \sin\left(\frac{\pi(1/p-\alpha)}{1-2\alpha}\right)} = C(\alpha, p). \quad (3)$$

Therefore, we establish that

$$\iint_{(0,+\infty)^2} \frac{f(x)g(y)}{x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha} dx dy \leq C(\alpha, p) \left(\int_{(0,+\infty)} f^p(x) dx \right)^{1/p} \left(\int_{(0,+\infty)} g^q(y) dy \right)^{1/q}.$$

This completes the proof. $\square$

In particular, when $p = 2$ (and hence $q = 2$), we have

$$\iint_{(0,+\infty)^2} \frac{f(x)g(y)}{x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha} dx dy \leq C(\alpha, p) \left(\int_{(0,+\infty)} f^2(x) dx \right)^{1/2} \left(\int_{(0,+\infty)} g^2(y) dy \right)^{1/2},$$

where

$$C(\alpha, p) = \frac{\pi}{(1-2\alpha) \sin\left(\frac{\pi(1/2-\alpha)}{1-2\alpha}\right)} = \frac{\pi}{1-2\alpha},$$

with $\alpha \in [0, 1/2)$. Hence, we have the simplified inequality

$$\iint_{(0,+\infty)^2} \frac{f(x)g(y)}{x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha} dx dy \leq \frac{\pi}{1-2\alpha} \left(\int_{(0,+\infty)} f^2(x) dx \right)^{1/2} \left(\int_{(0,+\infty)} g^2(y) dy \right)^{1/2}.$$

In this case, the constant factor is therefore independent of the sine term. A more general discussion on Theorem 2 is given in the next subsection.

2.3. Discussion

Note that the condition $\alpha \in [0, \min(1/2, 1/p, 1/q))$ has a natural theoretical justification, as it arises directly from the application of Eq. (2).

Taking $\alpha = 0$ yields

$$C(\alpha, p) = \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}.$$

As expected, in this case, Theorem 2 coincides with the Hardy-Hilbert integral inequality as given in Eq. (1). The other values of $\alpha \in (0, \min(1/2, 1/p, 1/q))$ lead to new integral inequalities of the Hardy-Hilbert type. In the extreme case $\alpha \rightarrow 1/2^-$, we have $C(\alpha, p) \rightarrow +\infty$, which leads to

$$\frac{1}{2} \left(\int_{(0,+\infty)} \frac{f(x)}{\sqrt{x}} dx \right) \left(\int_{(0,+\infty)} \frac{g(y)}{\sqrt{y}} dy \right) = \iint_{(0,+\infty)^2} \frac{f(x)g(y)}{x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha} dx dy \leq +\infty,$$

which is always true and not of interest.

Note that the kernel function associated with the inequality in Theorem 2 is

$$K(x, y) = \frac{1}{x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha}.$$

It is homogeneous of degree -1 , i.e., $K(tx, ty) = t^{-1}K(x, y)$ for any $x, y, t \in (0, +\infty)$. On this basis, an alternative proof of Theorem 2 can be obtained by the use of general results of the type presented in [4]. Our proof, however, has the advantage of being self-contained and readily reproducible for the interested reader. This transparency can also motivate new extensions.

2.4. Optimality of the constant factor

The theorem below establishes the optimality of the constant factor $C(\alpha, p)$ appearing in the inequality stated in Theorem 2.

Theorem 3. *In the framework of Theorem 2, the constant factor $C(\alpha, p)$ is sharp and cannot be improved.*

Proof of Theorem 3. Suppose that $C(\alpha, p)$ can be improved. Then there exists a constant

$$\sigma \in (0, C(\alpha, p))$$

satisfying, for any $f, g : [0, +\infty) \mapsto [0, +\infty)$,

$$\iint_{(0,+\infty)^2} \frac{f(x)g(y)}{x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha} dx dy \leq \sigma \left(\int_{(0,+\infty)} f^p(x) dx \right)^{1/p} \left(\int_{(0,+\infty)} g^q(y) dy \right)^{1/q}. \quad (4)$$

To reach a contradiction, we consider special extremal functions depending on a certain parameter $\eta > 0$, and analyze the behavior of both sides of Equation (4) as $\eta \rightarrow 0^+$. Specifically, for any $\eta > 0$, we set

$$f_\eta(x) = \begin{cases} 0 & \text{if } x \in [0, 1), \\ x^{-(1+\eta)/p} & \text{if } x \in [1, +\infty), \end{cases} \quad g_\eta(y) = \begin{cases} 0 & \text{if } y \in [0, 1), \\ y^{-(1+\eta)/q} & \text{if } y \in [1, +\infty). \end{cases}$$

Then we have

$$\int_{(0,+\infty)} f_\eta^p(x) dx = \int_{(1,+\infty)} (x^{-(1+\eta)/p})^p dx = \int_{(1,+\infty)} x^{-1-\eta} dx = \frac{1}{\eta},$$

and, similarly,

$$\int_{(0,+\infty)} g_\eta^q(y) dy = \frac{1}{\eta}.$$

This, with the identity $1/p + 1/q = 1$ and Eq. (4), gives

$$\begin{aligned} \sigma &= \sigma \eta \times \frac{1}{\eta^{1/p}} \times \frac{1}{\eta^{1/q}} \\ &= \eta \left(\sigma \left(\int_{(0,+\infty)} f_\eta^p(x) dx \right)^{1/p} \left(\int_{(0,+\infty)} g_\eta^q(y) dy \right)^{1/q} \right) \\ &\geq \eta \iint_{(0,+\infty)^2} \frac{f_\eta(x)g_\eta(y)}{x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha} dx dy. \end{aligned} \quad (5)$$

Performing the change of variables $x = uy$, applying the Fubini-Tonelli integral theorem and using the identity $1/p + 1/q = 1$, we obtain

$$\begin{aligned} \iint_{(0,+\infty)^2} \frac{f_\eta(x)g_\eta(y)}{x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha} dx dy &= \iint_{(1,+\infty)^2} \frac{x^{-(1+\eta)/p} y^{-(1+\eta)/q}}{x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha} dx dy \\ &= \int_{(1,+\infty)} \left(\int_{(1,+\infty)} \frac{x^{-(1+\eta)/p}}{x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha} dx \right) y^{-(1+\eta)/q} dy \\ &= \int_{(1,+\infty)} \left(\int_{(1/y,+\infty)} \frac{(uy)^{-(1+\eta)/p}}{(uy)^\alpha y^{1-\alpha} + (uy)^{1-\alpha} y^\alpha} (y du) \right) y^{-(1+\eta)/q} dy \\ &= \int_{(1,+\infty)} \left(\int_{(1/y,+\infty)} \frac{u^{-(1+\eta)/p}}{u^\alpha + u^{1-\alpha}} du \right) y^{-(1+\eta)} dy. \end{aligned} \quad (6)$$

Using the Chasles integral relation, the Fubini-Tonelli integral theorem and the identity $1/p + 1/q = 1$, we get

$$\begin{aligned}
 & \int_{(1,+\infty)} \left(\int_{(1/y,+\infty)} \frac{u^{-(1+\eta)/p}}{u^\alpha + u^{1-\alpha}} du \right) y^{-(1+\eta)} dy \\
 &= \int_{(1,+\infty)} \left(\int_{(1/y,1)} \frac{u^{-(1+\eta)/p}}{u^\alpha + u^{1-\alpha}} du \right) y^{-(1+\eta)} dy + \int_{(1,+\infty)} \left(\int_{(1,+\infty)} \frac{u^{-(1+\eta)/p}}{u^\alpha + u^{1-\alpha}} du \right) y^{-(1+\eta)} dy \\
 &= \int_{(0,1)} \left(\int_{(1/u,+\infty)} y^{-(1+\eta)} dy \right) \frac{u^{-(1+\eta)/p}}{u^\alpha + u^{1-\alpha}} du + \left(\int_{(1,+\infty)} \frac{u^{-(1+\eta)/p}}{u^\alpha + u^{1-\alpha}} du \right) \left(\int_{(1,+\infty)} y^{-(1+\eta)} dy \right) \\
 &= \int_{(0,1)} \left(\frac{1}{\eta} u^\eta \right) \frac{u^{-(1+\eta)/p}}{u^\alpha + u^{1-\alpha}} du + \frac{1}{\eta} \left(\int_{(1,+\infty)} \frac{u^{-(1+\eta)/p}}{u^\alpha + u^{1-\alpha}} du \right) \\
 &= \frac{1}{\eta} \left(\int_{(0,1)} \frac{u^{-1/p}}{u^\alpha + u^{1-\alpha}} u^{\eta/q} du + \int_{(1,+\infty)} \frac{u^{-1/p}}{u^\alpha + u^{1-\alpha}} u^{-\eta/p} du \right). \tag{7}
 \end{aligned}$$

Combining Eqs (5), (6) and (7), we obtain

$$\sigma \geq \int_{(0,1)} \frac{u^{-1/p}}{u^\alpha + u^{1-\alpha}} u^{\eta/q} du + \int_{(1,+\infty)} \frac{u^{-1/p}}{u^\alpha + u^{1-\alpha}} u^{-\eta/p} du.$$

Now, let us introduce the inferior limit based on $\eta \rightarrow 0^+$, denoted as $\underline{\lim}$. It follows from the Fatou integral lemma (which does not require a specific condition), $\underline{\lim} u^{\eta/q} = 1$ for $u \in (0, 1)$, $\underline{\lim} u^{-\eta/p} = 1$ for $u \in [1, +\infty)$, and the Chasles integral relation that

$$\begin{aligned}
 \sigma &\geq \underline{\lim} \int_{(0,1)} \frac{u^{-1/p}}{u^\alpha + u^{1-\alpha}} u^{\eta/q} du + \underline{\lim} \int_{(1,+\infty)} \frac{u^{-1/p}}{u^\alpha + u^{1-\alpha}} u^{-\eta/p} du \\
 &\geq \int_{(0,1)} \frac{u^{-1/p}}{u^\alpha + u^{1-\alpha}} \left(\underline{\lim} u^{\eta/q} \right) du + \int_{(1,+\infty)} \frac{u^{-1/p}}{u^\alpha + u^{1-\alpha}} \left(\underline{\lim} u^{-\eta/p} \right) du \\
 &= \int_{(0,1)} \frac{u^{-1/p}}{u^\alpha + u^{1-\alpha}} du + \int_{(1,+\infty)} \frac{u^{-1/p}}{u^\alpha + u^{1-\alpha}} du \\
 &= \int_{(0,+\infty)} \frac{u^{-1/p}}{u^\alpha + u^{1-\alpha}} du.
 \end{aligned}$$

Making the change of variables $v = 1/u$, and using Eq. (3) and the identity $1/p + 1/q = 1$, we get

$$\begin{aligned}
 \int_{(0,+\infty)} \frac{u^{-1/p}}{u^\alpha + u^{1-\alpha}} du &= \int_{(+\infty,0)} \frac{(1/v)^{-1/p}}{(1/v)^\alpha + (1/v)^{1-\alpha}} \left(-\frac{1}{v^2} dv \right) \\
 &= \int_{(0,+\infty)} \frac{v^{1/p-1}}{v^{1-\alpha} + v^\alpha} dv \\
 &= \int_{(0,+\infty)} \frac{v^{-1/q}}{v^{1-\alpha} + v^\alpha} dv = C(\alpha, p).
 \end{aligned}$$

As a result, we have $\sigma \geq C(\alpha, p)$, so that

$$\sigma \notin (0, C(\alpha, p)).$$

The initial assumption is contradicted, implying that $C(\alpha, p)$ can not be improved. This concludes the proof. $\square$

This result demonstrates the sharpness of the upper bound determined in Theorem 2, and the constant factor in particular.

3. Conclusion

In summary, we have introduced a natural, one-parameter extension to the Hardy-Hilbert integral inequality. In particular, we have established a sharp upper bound that generalizes this classical inequality while retaining the optimality of the constant. This can be presented concisely as follows:

$$\iint_{(0,+\infty)^2} \frac{f(x)g(y)}{x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha} dx dy \leq \frac{\pi}{(1-2\alpha) \sin\left(\frac{\pi(1/p-\alpha)}{1-2\alpha}\right)} \left(\int_{(0,+\infty)} f^p(x) dx\right)^{1/p} \left(\int_{(0,+\infty)} g^q(y) dy\right)^{1/q},$$

with $\alpha \in [0, \min(1/2, 1/p, 1/q))$, the classical case corresponding to $\alpha = 0$.

Further refinements to be explored in future work may include weighted or logarithmic variants, as well as applications to operator theory and other areas of functional analysis.

Conflicts of Interest: The author declares that he has no competing interests.

References

- [1] Hardy, G. H., Littlewood, J. E., & Polya, G. (1934). *Inequalities*. Cambridge University Press, Cambridge.
- [2] Yang, B. C. (2009). *The Norm of Operator and Hilbert-Type Inequalities*. Science Press, Beijing.
- [3] Yang, B. (2009). *Hilbert-Type Integral Inequalities*. Bentham Science Publishers.
- [4] Chen, Q., & Yang, B. (2015). A survey on the study of Hilbert-type inequalities. *Journal of Inequalities and Applications*, 2015(1), 302.
- [5] Chesneau, C. (2024). Study of two three-parameter non-homogeneous variants of the Hilbert integral inequality. *Lobachevskii Journal of Mathematics*, 45(10), 4931-4953.
- [6] Huang, Z., & Yang, B. (2011). A New Hilbert-type Integral Inequality with the Combination Kernel. In *International Mathematical Forum* (Vol. 6, No. 28, pp. 1363-1369).
- [7] Li, Y. J., Wu, J., & He, B. (2006). A new Hilbert-type integral inequality and the equivalent form. *International Journal of Mathematics and Mathematical Sciences*, 2006, Article ID 45378, Pages 1-6.
- [8] Sun, B. (2006). Best generalization of a Hilbert type inequality. *JIPAM. Journal of Inequalities in Pure & Applied Mathematics*, 7(3), 1-7.
- [9] Bicheng, Y. (2001). On Hardy-Hilbert's integral inequality. *Journal of Mathematical Analysis and Applications*, 261(1), 295-306.
- [10] Yang, B. C. (2010). Hilbert-type integral inequality with the homogeneous kernel of 0-degree. *Journal of Shanghai University (English Edition)*, 14(6), 391-395.
- [11] Gradshteyn, I. S., & Ryzhik, I. M. (2007). *Table of Integrals, Series, and Products*. 7th Edition. Academic Press.



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