Open Journal of
Mathematical Analysis

PSR Press

Article

An elliptic population system with multiple functions

Joon Hyuk Kang

Department of Mathematics, Andrews University, Berrien Springs, MI. 49104, USA; kang@andrews.edu

Received: 10 October 2025; Accepted: 08 November 2025; Published: 24 December 2025.

Abstract: The purpose of this paper is to give sufficient conditions for the existence and uniqueness of positive solutions to a rather general type of elliptic system of the Dirichlet problem on a bounded domain Ω in R^n. Also considered are the effects of perturbations on the coexistence state and uniqueness. The techniques used in this paper are super-sub solutions method, eigenvalues of operators, maximum principles, spectrum estimates, inverse function theory, and general elliptic theory. The arguments also rely on some detailed properties for the solution of logistic equations. These results yield an algebraically computable criterion for the positive coexistence of species of animals with predator-prey relation in many biological models.

Keywords: predator-prey system, coexistence state, existence, uniqueness, perturbation

MSC: 35J57, 35J67

1. Introduction

One of the prominent subjects of study and analysis in mathematical biology concerns the survival of two or more species of animals in the same environment. Especially, pertinent areas of investigation include the conditions under which the species can coexist, as well as the conditions under which any one of the species becomes extinct, that is, one of the species is excluded by the others. In this paper, we focus on the general predator-prey model to better understand the competitive interactions between multiple species. Specifically, we investigate the conditions needed for the coexistence of species when the factors affecting them are fixed or perturbed.

In this paper, we focus on the existence and uniqueness of the positive steady state solution of the general predator-prey model for arbitrary N species,

$$\begin{cases} (u_i)_t(x,t) = \Delta u_i(x,t) + u_i(x,t)g_i(u_1(x,t),...,u_i(x,t),...,u_N(x,t)) & \text{in } \Omega \times R^+, \\ u_i(x,t)|_{\partial\Omega} = 0, & i = 1,...,N, \end{cases}$$

or, equivalently, the positive solution to

$$\begin{cases} \Delta u_i(x) + u_i(x)g_i(u_1(x),...,u_i(x),...,u_N(x)) = 0 & \text{in } \Omega, \\ u_i|_{\partial\Omega} = 0, & i = 1,...,N, \end{cases} \tag{1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$, $g_i \in C^1$ are relative growth rates such that $(g_1)_{u_j} < 0, j = 1,...,N$ and for each $i = 2,...,N$,

$$(g_i)_{u_1} > 0,$$
$$(g_i)_{u_j} < 0, j = 2,...,N.$$

Because of its broader applicability, the general predator-prey model has become a more popular subject of research within the mathematical community over the past few years.

The functions g_i describe how species u_i interact among themselves and with each other.

The followings are questions raised in the general model with nonlinear growth rates.

Problem 1. What are the sufficient conditions for existence of positive solutions?

Problem 2. What are the sufficient conditions for uniqueness of positive solutions?

Problem 3. What is the effect of perturbation for existence and uniqueness? In our analysis, we focus on the conditions required for the maintenance of the coexistence state of the model when interaction rates $(g_1, ..., g_N)$ are slightly perturbed. Biologically, our conclusion implies that the species may slightly relax ecologically and yet continue to coexist at unique densities.

In §3, we establish sufficient conditions for the existence and non-existence of positive solution of the system. We also achieve solution estimates in §4 to prove the uniqueness and the invertibility of linearization in §5, §6 and §7, where we investigate the effect of perturbation for existence and uniqueness. In §8, we study an easy concrete example to illustrate abstract mathematical conditions and the practical relevance, and to discuss of how the mathematical results could inform ecological understanding and management.

An especially significant aspect of the global uniqueness result is the stability of the positive steady state solution, which has become an important subject of mathematical study. Indeed, researchers have obtained several stability results for the Lotka-Volterra model with constant rates (see [1], [2].) The research presented in this paper therefore begins the mathematical community's discussion on the stability of the steady state solution for the general predator-prey model.

2. Preliminaries

Before entering into our primary arguments and results, we must first present a few preliminary items that we later employ throughout the proofs detailed in this paper. The following definition and lemmas are established and accepted throughout the literature on our topic.

Definition 1. (Super and Sub solutions). The vector functions $(\bar{u}^1, ..., \bar{u}^N), (\underline{u}^1, ..., \underline{u}^N)$ form a super/sub solution pair for the system

$$\begin{cases} \Delta u^i + g^i(u^1, ..., u^N) = 0 & \text{in } \Omega, \\ u^i = 0 & \text{on } \partial\Omega, \end{cases}$$

if for $i = 1, ..., N$,

$$\left\{ \begin{array}{l} \Delta \bar{u}^i + g^i(u^1, ..., u^{i-1}, \bar{u}^i, u^{i+1}, ..., u^N) \leq 0, \\ \Delta \underline{u}^i + g^i(u^1, ..., u^{i-1}, \underline{u}^i, u^{i+1}, ..., u^N) \geq 0, \end{array} \right. \quad \text{in } \Omega \text{ for } \underline{u}^j \leq u^j \leq \bar{u}^j, j \neq i,$$

and

$$\begin{cases} \underline{u}^i \leq \bar{u}^i & \text{on } \Omega, \\ \underline{u}^i \leq 0 \leq \bar{u}^i & \text{on } \partial\Omega. \end{cases}$$

Lemma 1. *If g^i in the Definition 1 are in C^1 and the system admits a super/sub solution pair $(\underline{u}^1, ..., \underline{u}^N), (\bar{u}^1, ..., \bar{u}^N)$, respectively, then there is a solution $(u_1, ..., u_N)$ to the system in Definition 1 with $\underline{u}^i \leq u^i \leq \bar{u}^i$ in $\bar{\Omega}$. If*

$$\begin{array}{l} \Delta \bar{u}^i + g^i(\bar{u}^1, ..., \bar{u}^N) \neq 0, \\ \Delta \underline{u}^i + g^i(\underline{u}^1, ..., \underline{u}^N) \neq 0, \end{array}$$

in Ω for $i = 1, ..., N$, then $\underline{u}^i < u^i < \bar{u}^i$ in Ω.

Lemma 2. *(The first eigenvalue). Consider*

$$\left\{ \begin{array}{l} -\Delta u + q(x)u = \lambda u \quad in \quad \Omega, \\ u|_{\partial\Omega} = 0, \end{array} \right. \tag{2}$$

where $q(x)$ is a smooth function from Ω to R and Ω is a bounded domain.

(A) *The first eigenvalue $\lambda_1(q)$ of (2), denoted by simply λ_1 when $q \equiv 0$, is simple with a positive eigenfunction ϕ_q.*

(B) *If $q_1(x) < q_2(x)$ for all $x \in \Omega$, then $\lambda_1(q_1) < \lambda_1(q_2)$.*

(C) *(Variational Characterization of the first eigenvalue)*

$$\lambda_1(q) = \min_{\phi \in W_0^1(\Omega), \phi \neq 0} \frac{\int_\Omega (|\nabla \phi|^2 + q\phi^2)dx}{\int_\Omega \phi^2 dx}.$$

In our proof, we also employ accepted conclusions concerning the solutions of the following logistic equations.

Lemma 3. *Consider*

$$\begin{cases} \Delta u + uf(u) = 0 & in \;\; \Omega, \\ u|_{\partial\Omega} = 0, & u > 0, \end{cases}$$

where f is a decreasing C^1 function such that there exists $c_0 > 0$ such that $f(u) \leq 0$ for $u \geq c_0$ and Ω is a bounded domain.

(A) *If $f(0) > \lambda_1$, then the above equation has a unique positive solution. We denote this unique positive solution as θ_f.*

(B) *If $f(0) \leq \lambda_1$, then $u \equiv 0$ is the only nonnegative solution to the above equation.*

The main property about this positive solution is that θ_f is increasing as f is increasing.

Especially, for $\alpha > \lambda_1$, we denote θ_α as the unique positive solution of

$$\begin{cases} \Delta u + u(\alpha - u) = 0 & \text{in} \;\; \Omega, \\ u|_{\partial\Omega} = 0, & u > 0. \end{cases}$$

Hence, θ_α is increasing as $\alpha > 0$ is increasing.

Having established these preliminaries, we now commence our investigation of the general predator-prey model.

3. Existence, nonexistence

We establish the following existence result for (1):

Theorem 1. *If for each $i = 2, ..., N$,*

(A)

$$\sup(g_1)_{u_1} \sup(g_i)_{u_i} + (N-1)\inf(g_1)_{u_i} \sup(g_i)_{u_1} > 0,$$

and

(B)

$$g_1(0, ..., 0) > \frac{\sup(g_1)_{u_1} \sup(g_i)_{u_i} \left(\lambda_1 + \frac{(N-1)\inf(g_1)_{u_i} g_i(0,...,0)}{\sup(g_i)_{u_i}}\right)}{\sup(g_1)_{u_1} \sup(g_i)_{u_i} + (N-1)\inf(g_1)_{u_i} \sup(g_i)_{u_1}},$$

and

$$g_i(0, ..., 0) > \lambda_1 - (N-2) \inf_{j=2,...,N, j \neq i} \{\bar{u}_j \inf(g_i)_{u_j}\},$$

where $\bar{u}_j$ is defined below, then (1) has a positive solution.

Proof. Let $\underline{u_i} = \gamma_i \omega$, $i = 1, ..., N$, $\bar{u}_1 = -\frac{g_1(0,...,0)}{\sup(g_1)_{u_1}}$, $\bar{u}_i = -\frac{1}{\sup(g_i)_{u_i}} \left(g_i(0, ..., 0) - \frac{g_1(0,...,0)\sup(g_i)_{u_1}}{\sup(g_1)_{u_1}}\right)$, $i = 2, ..., N$, where $\gamma_i > 0$ are constants and ω is the eigenfunction of (2) with $q(x) \equiv 0$ corresponding to the first eigenvalue λ_1.

Then by the Mean Value Theorem, for all u_i such that $\underline{u_i} \leq u_i \leq \bar{u}_i, i = 2, ..., N$,

$$\begin{aligned} \Delta \bar{u}_1 + \bar{u}_1 g_1(\bar{u}_1, u_2, ..., u_N) =& \bar{u}_1 [g_1(0, ..., 0) + g_1(\bar{u}_1, u_2, ..., u_N) - g_1(0, u_2, ..., u_N) \\ &+ g_1(0, u_2, ..., u_N) - g_1(0, 0, u_3, ..., u_N)] + g_1(0, 0, u_3, ..., u_N) - g_1(0, 0, 0, u_4, ..., u_N) \\ &+ ... + g_1(0, ..., 0, u_N) - g_1(0, ..., 0)] \end{aligned}$$

$$
\begin{aligned}
&\leq \bar{u}_1[g_1(0,...,0) + \bar{u}_1 \sup(g_1)_{u_1} + u_2 \sup(g_1)_{u_2} + ... + u_N \sup(g_1)_{u_N}] \\
&= \bar{u}_1[u_2 \sup(g_1)_{u_2} + ..., + u_N \sup(g_1)_{u_N} < 0,
\end{aligned}
$$

and for all u_1 such that $\underline{u_1} \leq u_1 \leq \bar{u}_1$ and for all $i = 2, ..., N$,

$$
\begin{aligned}
\Delta\bar{u}_i + \bar{u}_i g_i(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N) =& \bar{u}_i[g_i(0,...,0) + g_i(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N) \\
&- g_i(0, u_2, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N) \\
&+ g_i(0, u_2, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N) - g_i(0, 0, u_3, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N) \\
&+ g_i(0, 0, u_3, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N) \\
&- g_i(0, 0, 0, u_4, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N) + ... + \\
&+ g_i(0, ..., 0, \bar{u}_i, u_{i+1}, ..., u_N) - g_i(0, ..., 0, u_{i+1}, ..., u_N) \\
&+ g_i(0, ..., 0, u_{i+1}, ..., u_N) - g_i(0, ..., 0, u_{i+2}, ..., u_N) \\
&+ ... + g_i(0, ..., 0, u_N) - g_i(0, ..., 0)] \\
\leq& \bar{u}_i[g_i(0,...,0) + u_1 \sup(g_i)_{u_1} + ... u_{i-1} \sup(g_i)_{u_{i-1}} + \bar{u}_i \sup(g_i)_{u_i} \\
&+ u_{i+1} \sup(g_i)_{u_{i+1}} + ... + u_N \sup(g_i)_{u_N}] \\
\leq& \bar{u}_i[g_i(0,...,0) + \bar{u}_1 \sup(g_i)_{u_1} + \bar{u}_i \sup(g_i)_{u_i}] = 0.
\end{aligned}
$$

By the condition and the Mean Value Theorem again, for all u_i such that $\underline{u_i} \leq u_i \leq \bar{u}_i, i = 2, ..., N$,

$$
\begin{aligned}
&\Delta\underline{u_1} + \underline{u_1} g_1(\underline{u_1}, u_2, ..., u_N) \\
&= \Delta(\gamma_1\omega) + \gamma_1\omega g_1(\gamma_1\omega, u_2, ..., u_N) \\
&= \Delta(\gamma_1\omega) + \gamma_1\omega[g_1(0,...,0) + g_1(\gamma_1\omega, u_2, ..., u_N) - g_1(0, u_2, ..., u_N) + g_1(0, u_2, ..., u_N) - g_1(0, 0, u_3, ..., u_N) \\
&\quad + g_1(0, 0, u_3, ..., u_N) - g_1(0, 0, 0, u_4, ..., u_N) + ... + g_1(0, ..., 0, u_N) - g_1(0, ..., 0)] \\
&\geq -\gamma_1\lambda_1\omega + \gamma_1\omega\left[g_1(0,...,0) + \gamma_1\omega \inf(g_1)_{u_1} + \sum_{i=2}^{N} u_i \inf(g_1)_{u_i}\right] \\
&\geq -\gamma_1\lambda_1\omega + \gamma_1\omega\left[g_1(0,...,0) + (N-1)\inf_{i=2,...,N}\{\bar{u}_i \inf(g_1)_{u_i}\} + \gamma_1\omega \inf(g_1)_{u_1}\right] \\
&= -\gamma_1\lambda_1\omega + \gamma_1\omega\Bigg[g_1(0,...,0) + (N-1)\inf_{i=2,...,N}\Bigg\{-\frac{1}{\sup(g_i)_{u_i}}(g_i(0,...,0) \\
&\quad -\frac{g_1(0,...,0)\sup(g_i)_{u_1}}{\sup(g_1)_{u_1}})\inf(g_1)_{u_i}\Bigg\} + \gamma_1\omega \inf(g_1)_{u_1}\Bigg] \\
&= -\gamma_1\lambda_1\omega + \gamma_1\omega\Bigg[g_1(0,...,0) + (N-1)\inf_{i=2,...,N}\Bigg\{\inf(g_1)_{u_i}\Bigg(-\frac{g_i(0,...,0)}{\sup(g_i)_{u_i}} \\
&\quad +\frac{g_1(0,...,0)\sup(g_i)_{u_1}}{\sup(g_1)_{u_1}\sup(g_i)_{u_i}}\Bigg)\Bigg\} + \gamma_1\omega \inf(g_1)_{u_1}\Bigg] \\
&= -\gamma_1\lambda_1\omega + \gamma_1\omega\Bigg[\inf_{i=2,...,N}\Bigg\{g_1(0,...,0) - \frac{(N-1)\inf(g_1)_{u_i} g_i(0,...,0)}{\sup(g_i)_{u_i}} \\
&\quad +\frac{(N-1)g_1(0,...,0)\inf(g_1)_{u_i}\sup(g_i)_{u_1}}{\sup(g_1)_{u_1}\sup(g_i)_{u_i}}\Bigg\} + \gamma_1\omega \inf(g_1)_{u_1}\Bigg] \\
&= -\gamma_1\lambda_1\omega + \gamma_1\omega\Bigg[\inf_{i=2,...,N}\Bigg\{g_1(0,...,0)\left(1 + \frac{(N-1)\inf(g_1)_{u_i}\sup(g_i)_{u_1}}{\sup(g_1)_{u_1}\sup(g_i)_{u_i}}\right) \\
&\quad -\frac{(N-1)\inf(g_1)_{u_i} g_i(0,...,0)}{\sup(g_i)_{u_i}}\Bigg\} + \gamma_1\omega \inf(g_1)_{u_1}\Bigg] \\
&= \gamma_1\omega\Bigg[-\lambda_1 + \inf_{i=2,...,N}\Bigg\{g_1(0,...,0)\left(1 + \frac{(N-1)\inf(g_1)_{u_i}\sup(g_i)_{u_1}}{\sup(g_1)_{u_1}\sup(g_i)_{u_i}}\right) \\
&\quad -\frac{(N-1)\inf(g_1)_{u_i} g_i(0,...,0)}{\sup(g_i)_{u_i}}\Bigg\} + \gamma_1\omega \inf(g_1)_{u_1}\Bigg] \\
&> 0
\end{aligned}
$$

with small enough $\gamma_1 > 0$, and for all $i = 2, ..., N$, $j = 1, ..., N$, $j \neq i$ and u_j such that $\underline{u_j} \leq u_j \leq \bar{u}_j$,

$$\begin{aligned}
\Delta\underline{u_i} + \underline{u_i}g_i(u_1, ..., u_{i-1}, \underline{u_i}, u_{i+1}, ..., u_N) =& - \gamma_i\lambda_1\omega + \gamma_i\omega g_i(u_1, ..., u_{i-1}, \gamma_i\omega, u_{i+1}, ..., u_N) \\
=& - \gamma_i\lambda_1\omega + \gamma_i\omega[g_i(0, ..., 0) + g_i(u_1, ..., u_{i-1}, \gamma_i\omega, u_{i+1}, ..., u_N) \\
&- g_i(0, u_2, ..., u_{i-1}, \gamma_i\omega, u_{i+1}, ..., u_N) + g_i(0, u_2, ..., u_{i-1}, \gamma_i\omega, u_{i+1}, ..., u_N) \\
&- g_i(0, 0, u_3, ..., u_{i-1}, \gamma_i\omega, u_{i+1}, ..., u_N)] + ...+ \\
&+ g_i(0, ..., 0, \gamma_i\omega, u_{i+1}, ..., u_N) - g_i(0, ..., 0, u_{i+1}, ..., u_N) \\
&+ g_i(0, ..., 0, u_{i+1}, ..., u_N) - g_i(0, ..., 0, u_{i+2}, ..., u_N) \\
&+ ... + g_i(0, ..., 0, u_N) - g_i(0, ..., 0)] \\
\geq& - \gamma_i\lambda_1\omega + \gamma_i\omega\left[g_i(0, ..., 0) + \sum_{j=1, j\neq i}^{N} u_j \inf(g_i)_{u_j} + \gamma_i\omega \inf(g_i)_{u_i}\right] \\
\geq&\gamma_i\omega\left[g_i(0, ..., 0) - \lambda_1 + \gamma_i\omega \inf(g_i)_{u_i} + \sum_{j=2, j\neq i}^{N} \bar{u}_j \inf(g_i)_{u_j}\right] > 0
\end{aligned}$$

with small enough $\gamma_i > 0$. Furthermore,

$$\underline{u_i} = \bar{u}_i = 0 \text{ on } \partial\Omega,$$

and

$$\underline{u_i} \leq \bar{u}_i,$$

with small enough $\gamma_i > 0$. Hence, by the Lemma 1, there is a solution $(u_1, ..., u_N)$ to (1) with

$$\underline{u_i} \leq u_i \leq \bar{u}_i.$$

□

We also establish the following nonexistence results.

Theorem 2. *Suppose $g_i(0, ..., 0) \leq \lambda_1, i = 1, ..., N$. Then $u_i \equiv 0, i = 1, ..., N$ is the only nonnegative solution to (1).*

Proof. Let $(u_1, ..., u_N)$ be a nonnegative solution to (8). By the Mean Value Theorem, for $i = 2, ..., N$, there are $\tilde{u}_i$ such that

$$\begin{aligned}
g_1(u_1, 0, ..., 0, u_i, 0, ..., 0) - g_1(u_1, 0, ..., 0) &= (g_1)_{u_i}(u_1, 0, ..., 0, \tilde{u}_i, 0, ..., 0)u_i, \\
g_i(u_1, 0, ..., 0, u_i, 0, ..., 0) - g_i(0, ..., 0, u_i, 0, ..., 0) &= (g_i)_{u_1}(\tilde{u}_1, 0, ..., 0, u_i, 0, ..., 0)u_1.
\end{aligned}$$

Hence, by the monotonicity, for $i = 2, ..., N$, (1) implies that

$$\begin{aligned}
\Delta u_1 + &u_1[g_1(u_1, 0, ..., 0) + (g_1)_{u_i}(u_1, 0, ..., 0, \tilde{u}_i 0, ..., 0)u_i] \\
&= \Delta u_1 + u_1[g_1(u_1, 0, ..., 0) + g_1(u_1, 0, ..., 0, u_i, 0, ..., 0) - g_1(u_1, 0, ..., 0)] \\
&\geq \Delta u_1 + u_1[g_1(u_1, 0, ..., 0) + g_1(u_1, ..., u_N) - g_1(u_1, 0, ..., 0)] \\
&= \Delta u_1 + u_1 g_1(u_1, ..., u_N) \\
&= 0 \text{ in } \Omega,
\end{aligned}$$

and

$$\begin{aligned}
\Delta u_i + &u_i[g_i(0, ..., 0, u_i, 0, ..., 0) + (g_i)_{u_1}(\tilde{u}_1, 0, ..., 0, u_i, 0, ..., 0)u_1] \\
&= \Delta u_i + u_i[g_i(0, ..., 0, u_i, 0, ..., 0) + g_i(u_1, 0, ..., 0, u_i, 0, ..., 0) \\
&\quad - g_i(0, ..., 0, u_i, 0, ..., 0)] \\
&= \Delta u_i + u_i g_i(u_1, 0, ..., 0, u_i, 0, ..., 0) \\
&\geq \Delta u_i + u_i g_i(u_1, ..., u_N) = 0 \text{ in } \Omega.
\end{aligned}$$

Hence, for $i = 2, ..., N$,

$$\Delta u_1 + u_1[g_1(u_1, 0, ..., 0) + \sup(g_1)_{u_i} u_i] \geq 0 \text{ in } \Omega,$$
$$\Delta u_i + u_i[g_i(0, ..., 0, u_i, 0, ..., 0) + \sup(g_i)_{u_1} u_1] \geq 0 \text{ in } \Omega.$$

Therefore, for $i = 2, ..., N$,

$$\sup(g_i)_{u_1}\phi_1\Delta u_1 + \sup(g_i)_{u_1}\phi_1 u_1[g_1(u_1, 0, ..., 0) + \sup(g_1)_{u_i} u_i] \geq 0 \text{ in } \Omega,$$
$$-\sup(g_1)_{u_i}\phi_1\Delta u_i - \sup(g_1)_{u_i}\phi_1 u_i[g_i(0, ..., 0, u_i, 0, ..., 0) + \sup(g_i)_{u_1} u_1] \geq 0 \text{ in } \Omega.$$

So, for $i = 2, ..., N$,

$$\int_\Omega -\sup(g_i)_{u_1}\phi_1\Delta u_1 dx \leq \int_\Omega [g_1(u_1, 0, ..., 0)\sup(g_i)_{u_1} u_1 + \sup(g_1)_{u_i}\sup(g_i)_{u_1} u_1 u_i]\phi_1 dx$$
$$\int_\Omega \sup(g_1)_{u_i}\phi_1\Delta u_i dx \leq \int_\Omega [-g_i(0, ..., 0, u_i, 0, ..., 0)\sup(g_1)_{u_i} u_i - \sup(g_1)_{u_i}\sup(g_i)_{u_1} u_1 u_i]\phi_1 dx.$$

Hence, by the Green's Identity, we have

$$\int_\Omega \sup(g_i)_{u_1}\lambda_1\phi_1 u_1 dx \leq \int_\Omega [g_1(u_1, 0, ..., 0)\sup(g_i)_{u_1} u_1 + \sup(g_1)_{u_i}\sup(g_i)_{u_1} u_1 u_i]\phi_1 dx$$
$$\int_\Omega -\sup(g_1)_{u_i}\lambda_1\phi_1 u_i dx \leq \int_\Omega [-g_i(0, ..., 0, u_i, 0, ..., 0)\sup(g_1)_{u_i} u_i - \sup(g_1)_{u_i}\sup(g_i)_{u_1} u_1 u_i]\phi_1 dx.$$

Therefore,

$$\int_\Omega \sup(g_i)_{u_1}([\lambda_1 - g_1(u_1, 0, ..., 0)]u_1\phi_1 - \sup(g_1)_{u_i}[\lambda_1 - g_i(0, ..., 0, u_i, 0, ..., 0)]u_i\phi_1 dx \leq 0.$$

Since the left hand side is nonnegative from

$$g_1(u_1, 0, ..., 0) \leq g_1(0, ..., 0) \leq \lambda_1,$$
$$g_i(0, ..., 0, u_i, 0, ..., 0) \leq g_i(0, ..., 0) \leq \lambda_1,$$

we conclude that $u_i \equiv 0, i = 1, ..., N$. □

Theorem 3. *Let $u_i \geq 0, i = 1, ..., N$ be a solution to (1). If $g_1(0, ..., 0) \leq \lambda_1$, then $u_1 \equiv 0$.*

Proof. Proceeding as in the proof of the Theorem 2, we obtain

$$0 \leq \int_\Omega [\lambda_1 - g_1(u_1, 0, ..., 0)]u_1\phi_1 dx \leq \int_\Omega \sup(g_1)_{u_i} u_1 u_i\phi_1 dx \leq 0,$$

and so, $u_1 \equiv 0$. □

4. Solution estimate

In order to prove further results, we will need the following solution estimate.

For the rest of this section, we assume the following additional growth condition:

$$\lim_{x_i\to\infty} g_i(x_1, ..., x_i, ..., x_N) = -\infty, i = 1, ..., N.$$

Lemma 4. *Let $(u_1, ..., u_N), u_i \geq 0, i = 1, ..., N$ be a solution of the problem*

$$\begin{cases} -\Delta u_i = t u_i g_i(u_1, , ..., u_N) & \text{in } \Omega, \\ u_i|_{\partial\Omega} = 0, & i = 1, ..., N, \end{cases} \tag{3}$$

where $t \in [0, 1]$. Then

(A)

$$u_1 \le M_1, u_i \le M_i,$$

where

$$M_1 = -\frac{g_1(0,...,0)}{\sup(g_1)_{u_1}}, \qquad M_i = \frac{\sup(g_i)_{u_1}(M_1) + g_i(0,...,0)}{-\sup(g_i)_{u_i}}, i = 2,...,N.$$

(B) *For* $t = 1$, *if* $u_i > 0$,

$$\sup(g_1)_{u_1}\sup(g_i)_{u_i} + \inf(g_1)_{u_i}\sup(g_i)_{u_1} > 0,$$

$$g_1(0, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N) + \inf(g_1)_{u_i}\left[\frac{g_1(0,...,0)\sup(g_i)_{u_1} - g_i(0,...,0)\sup(g_1)_{u_1}}{\sup(g_1)_{u_1}\sup(g_i)_{u_i}}\right] > \lambda_1,$$

and

$$g_i(0, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N) > \lambda_1, i = 2, ..., N,$$

then

$$\theta_{M(g_1,g_i)} \le u_1 \le \theta_{g_1(\cdot,0,...,0)},$$
$$\theta_{g_i(0,M_2,...,M_{i-1},\cdot,M_{i+1},...,M_N)} \le u_i \le \theta_{g_i(0,...,0,\cdot,0,...,0) - \frac{g_1(0,...,0)\sup(g_i)_{u_1}}{\sup(g_1)_{u_1}}}, i = 2, ..., N,$$

where

$$\begin{aligned} M(g_1, g_i) &= g_1(\cdot, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N) \\ &\quad + \inf(g_1)_{u_i}\left[\frac{g_1(0,...,0)\sup(g_i)_{u_1} - g_i(0,...,0)\sup(g_1)_{u_1}}{\sup(g_1)_{u_1}\sup(g_i)_{u_i}}\right], i = 2, ..., N. \end{aligned}$$

Proof. (A) Since $(u_1, ..., u_N)$ is a solution to (3), by the Mean Value Theorem,

$$\begin{aligned} \Delta u_1 + &u_1[g_1(0,...,0) + \sup(g_1)_{u_1}u_1 + ... + \sup(g_1)_{u_N}u_N] \\ &\ge \Delta u_1 + u_1[g_1(0,...,0) + g_1(u_1,...,u_N) - g_1(0,u_2,...,u_N) \\ &\quad + g_1(0,u_2,...,u_N) - g_1(0,0,u_3,...,u_N) + ... + g_1(0,...,0,u_N) - g_1(0,...,0)] \\ &= \Delta u_1 + u_1 g_1(u_1,...,u_N) \\ &= 0, \end{aligned}$$

and so,

$$\Delta u_1 + u_1[g_1(0,...,0) + \sup(g_1)_{u_1}u_1] \ge -u_1\sum_{i=2}^{N}\sup(g_1)_{u_i}u_i \ge 0.$$

Hence, by the Maximum Principles,

$$g_1(0,...,0) + \sup(g_1)_{u_1}u_1 \ge 0,$$

equivalently,

$$u_1 \le -\frac{g_1(0,...,0)}{\sup(g_1)_{u_1}}.$$

Since $(u_1, ..., u_N)$ is a solution to (3), by the Mean Value Theorem and above estimation, for $i = 2, ..., N$,

$$\begin{aligned} \Delta u_i + &u_i\left[g_i(0,...,0) + \sup(g_i)_{u_i}u_i + \sup(g_i)_{u_1}\left(-\frac{g_1(0,...,0)}{\sup(g_1)_{u_1}}\right)\right] \\ &\ge \Delta u_i + u_i[g_i(0,...,0) + \sup(g_i)_{u_i}u_i + \sup(g_i)_{u_1}u_1] \\ &\ge \Delta u_i + u_i[g_i(0,...,0) + g_i(u_1,...,u_N) - g_i(0,u_2,...,u_N)] \\ &\quad + g_i(0,u_2,...,u_N) - g_i(0,0,u_3,...,u_N) + ... + g_i(0,...,0,u_N) - g_i(0,...,0)] \\ &= 0. \end{aligned}$$

Hence, by the Maximum Principles again,

$$g_i(0,...,0)+\sup(g_i)_{u_i}u_i+\sup(g_i)_{u_1}\left[-\frac{g_1(0,...,0)}{\sup(g_1)_{u_1}}\right]\geq 0,$$

equivalently,

$$u_i\leq\frac{\sup(g_i)_{u_1}\left[-\frac{g_1(0,...,0)}{\sup(g_1)_{u_1}}\right]+g_i(0,...,0)}{-\sup(g_i)_{u_i}}, i=2,...,N.$$

(B) By the monotonicity of g_1,

$$\Delta u_1+u_1g_1(u_1,0,...,0)=u_1[g_1(u_1,0,...,0)-g_1(u_1,...,u_N)]\geq 0,$$

and so, u_1 is a subsolution to

$$\begin{aligned}&\Delta Z+Zg_1(Z,0,...,0)=0 \text{ in } \Omega,\\ &Z|_{\partial\Omega}=0.\end{aligned}$$

But, since any sufficiently large positive constant is a supersolution to

$$\begin{aligned}&\Delta Z+Zg_1(Z,0,...,0)=0 \text{ in } \Omega,\\ &Z|_{\partial\Omega}=0,\end{aligned}$$

by the Lemmas 1 and 3, we conclude that

$$u_1\leq\theta_{g_1(\cdot,0,...,0)}. \tag{4}$$

Since

$$\begin{aligned}
&\Delta u_1+u_1\left[g_1(u_1,M_2,...,M_{i-1},0,M_{i+1},...,M_N)+\inf(g_1)_{u_i}\left(\frac{\sup(g_i)_{u_1}\left[-\frac{g_1(0,...,0)}{\sup(g_1)_{u_1}}\right]+g_i(0,...,0)}{-\sup(g_i)_{u_i}}\right)\right]\\
&=u_1\Bigg[-g_1(u_1,...,u_N)+g_1(u_1,M_2,...,M_{i-1},0,M_{i+1},...,M_N)\\
&\quad+\inf(g_1)_{u_i}\left(\frac{\sup(g_i)_{u_1}\left[-\frac{g_1(0,...,0)}{\sup(g_1)_{u_1}}\right]+g_i(0,...,0)}{-\sup(g_i)_{u_i}}\right)\Bigg]\\
&\leq u_1\Bigg[-g_1(u_1,M_2,...,M_{i-1},u_i,M_{i+1},...,M_N)+g_1(u_1,M_2,...,M_{i-1},0,M_{i+1},...,M_N)\\
&\quad+\inf(g_1)_{u_i}\left(\frac{\sup(g_i)_{u_1}\left[-\frac{g_1(0,...,0)}{\sup(g_1)_{u_1}}\right]+g_i(0,...,0)}{-\sup(g_i)_{u_i}}\right)\Bigg]\\
&\leq u_1\left[-\inf(g_1)_{u_i}u_i+\inf(g_1)_{u_i}\left(\frac{\sup(g_i)_{u_1}\left[-\frac{g_1(0,...,0)}{\sup(g_1)_{u_1}}\right]+g_i(0,...,0)}{-\sup(g_i)_{u_i}}\right)\right]\\
&=-u_1\inf(g_1)_{u_i}\left[u_i-\frac{\sup(g_i)_{u_1}[-\frac{g_1(0,...,0)}{\sup(g_1)_{u_1}}]+g_i(0,...,0)}{-\sup(g_i)_{u_i}}\right]\\
&\leq 0,\qquad i=2,...,N,
\end{aligned}$$

by the Mean Value Theorem, monotonicity and (A), u_1 is a supersolution to

$$\begin{cases} \Delta Z + Z\left[g_1(Z, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N) + \inf(g_1)_{u_i}\left(\frac{\sup(g_i)_{u_1}\left[-\frac{g_1(0,...,0)}{\sup(g_1)_{u_1}}\right]+g_i(0,...,0)}{-\sup(g_i)_{u_i}}\right)\right] = 0 & \text{in } \Omega, \\ Z|_{\partial\Omega} = 0, & i = 2, ..., N. \end{cases}$$

But, by the continuity of g_1 and the condition, for sufficiently small $\epsilon > 0$,

$$\begin{aligned} &\Delta\epsilon\phi_1 + \epsilon\phi_1\left[g_1(\epsilon\phi_1, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N) + \inf(g_1)_{u_i}\left(\frac{\sup(g_i)_{u_1}\left[-\frac{g_1(0,...,0)}{\sup(g_1)_{u_1}}\right] + g_i(0, ..., 0)}{-\sup(g_i)_{u_i}}\right)\right] \\ &=\epsilon\phi_1\left[-\lambda_1 + g_1(\epsilon\phi_1, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N) + \inf(g_1)_{u_i}\left(\frac{\sup(g_i)_{u_1}\left[-\frac{g_1(0,...,0)}{\sup(g_1)_{u_1}}\right] + g_i(0, ..., 0)}{-\sup(g_i)_{u_i}}\right)\right] \\ &>0, \qquad i = 2, ..., N, \end{aligned}$$

and so, $\epsilon\phi_1$ is a subsolution to

$$\begin{cases} \Delta Z + Z\left[g_1(Z, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N) + \inf(g_1)_{u_i}\left(\frac{\sup(g_i)_{u_1}\left[-\frac{g_1(0,...,0)}{\sup(g_1)_{u_1}}\right]+g_i(0,...,0)}{-\sup(g_i)_{u_i}}\right)\right] = 0 & \text{in } \Omega, \\ Z|_{\partial\Omega} = 0, & i = 2, ..., N. \end{cases}$$

Therefore, by the Lemmas 1 and 3, we have

$$\theta_{g_1(\cdot,M_2,...,M_{i-1},0,M_{i+1},...,M_N)+\inf(g_1)_{u_i}\left(\frac{\sup(g_i)_{u_1}\left[-\frac{g_1(0,...,0)}{\sup(g_1)_{u_1}}\right]+g_i(0,...,0)}{-\sup(g_i)_{u_i}}\right)} \leq u_1, \qquad i = 2, ..., N. \tag{5}$$

By the monotonicity of g_i,

$$\begin{aligned} \Delta u_i + u_i g_i(0, M_2, ..., M_{i-1}, u_i, M_{i+1}, ..., M_N) =&u_i[g_i(0, M_2, ..., M_{i-1}, u_i, M_{i+1}, ..., M_N) - g_i(u_1, ..., u_N)] \\ <&0, \qquad i = 2, ..., N, \end{aligned}$$

and so, u_i is a supersolution to

$$\begin{cases} \Delta Z + Zg_i(0, M_2, ..., M_{i-1}, Z, M_{i+1}, ..., M_N) = 0 & \text{in } \Omega, \\ Z|_{\partial\Omega} = 0, & i = 2, ..., N. \end{cases}$$

But, by the continuity of g_i and the condition, for sufficiently small $\epsilon > 0$,

$$\begin{aligned} \Delta\epsilon\phi_1 + \epsilon\phi_1 g_i(0, M_2, ..., M_{i-1}, \epsilon\phi_1, M_{i+1}, ..., M_N) =&\epsilon\phi_1[-\lambda_1 + g_i(0, M_2, ..., M_{i-1}, \epsilon\phi_1, M_{i+1}, ..., M_N)] \\ >&0, \qquad i = 2, ..., N, \end{aligned}$$

and so, $\epsilon\phi_1$ is a subsolution to

$$\begin{cases} \Delta Z + Zg_i(0, M_2, ..., M_{i-1}, Z, M_{i+1}, ..., M_N) = 0 & \text{in } \Omega, \\ Z|_{\partial\Omega} = 0, & i = 2, ..., N. \end{cases}$$

Hence, by the Lemmas 1 and 3, we have

$$\theta_{g_i(0,M_2,...,M_{i-1},\cdot,M_{i+1},...,M_N)} \leq u_i, \qquad i = 2, ..., N. \tag{6}$$

Since $(u_1,...,u_N)$ is a solution to (3), by the Mean Value Theorem and (A),

$$\begin{aligned}
\Delta u_i + u_i &\left[g_i(0,...,0,u_i,0,...,0) - \frac{\sup(g_i)_{u_1} g_1(0,...,0)}{\sup(g_1)_{u_1}}\right] \\
=& u_i \left[-g_i(u_1,...,u_N) + g_i(0,...,0,u_i,0,...,0) - \frac{\sup(g_i)_{u_1} g_1(0,...,0)}{\sup(g_1)_{u_1}}\right] \\
=& u_i \Bigg[- g_i(u_1,...,u_N) + g_i(0,u_2,...,u_N) - g_i(0,u_2,...,u_N) + g_i(0,0,u_3,...,u_N) \\
& - g_i(0,0,u_3,...,u_N) + g_i(0,0,0,u_4,...,u_N) - ... - g_i(0,...,0,u_{i-1},u_i,...,u_N) + g_i(0,...,0,u_i,...,u_N) \\
& - g_i(0,...,0,u_i,...,u_N) + g_i(0,...,0,u_i,0,u_{i+2},...,u_N) - g_i(0,...,0,u_i,0,u_{i+2},...,u_N) \\
& + g_i(0,...,0,u_i,0,0,u_{i+3},...,u_N) - ... - g_i(0,...,0,u_i,0,...,0,u_N) \\
& + g_i(0,...,0,u_i,0,...,0) - \frac{\sup(g_i)_{u_1} g_1(0,...,0)}{\sup(g_1)_{u_1}}\Bigg] \\
\geq& u_i \left[- \sup(g_i)_{u_1} u_1 - \sum_{j=2, j\neq i}^{N} u_j \sup(g_i)_{u_j} - \frac{\sup(g_i)_{u_1} g_1(0,...,0)}{\sup(g_1)_{u_1}}\right] \\
\geq& u_i \left[-u_1 \sup(g_i)_{u_1} - \frac{\sup(g_i)_{u_1} g_1(0,...,0)}{\sup(g_1)_{u_1}}\right] \\
=& \sup(g_i)_{u_1} u_i \left[-u_1 - \frac{g_1(0,...,0)}{\sup(g_1)_{u_1}}\right] \\
\geq& 0, \qquad i = 2,...,N.
\end{aligned}$$

Therefore, u_i is a subsolution to

$$\begin{cases} \Delta Z + Z\left[g_i(0,...,0,Z,0,...,0) - \frac{\sup(g_i)_{u_1} g_1(0,...,0)}{\sup(g_1)_{u_1}}\right] = 0 & \text{in } \Omega, \\ Z|_{\partial\Omega} = 0, & i = 2,...,N. \end{cases}$$

But, since any sufficiently large constant is a supersolution to

$$\begin{cases} \Delta Z + Z[g_i(0,...,0,Z,0,...,0) - \frac{\sup(g_i)_{u_1} g_1(0,...,0)}{\sup(g_1)_{u_1}}] = 0 & \text{in } \Omega, \\ Z|_{\partial\Omega} = 0, & i = 2,...,N, \end{cases}$$

by the Lemmas 1 and 3, we have

$$u_i \leq \theta_{g_i(0,...,0,\cdot,0,...,0) - \frac{g_1(0,...,0)\sup(g_i)_{u_1}}{\sup(g_1)_{u_1}}}, \qquad i = 2,...,N. \tag{7}$$

By (4), (5), (6) and (7), we establish the desired inequalities. □

5. Uniqueness

In this section, we prove the uniqueness of positive solution to (1) with the following additional growth condition:

$$\lim_{x_i \to \infty} g_i(x_1,...,x_i,...,x_N) = -\infty, \qquad i = 1,...,N$$

.

We have the following uniqueness result.

Theorem 4. *In addition to the Theorem 1, if*
(A)

$$g_1(0,M_2,...,M_{i-1},0,M_{i+1},...,M_N) + \inf(g_1)_{x_i}\left[\frac{g_1(0,...,0)\sup(g_i)_{x_1} - g_i(0,...,0)\sup(g_1)_{x_1}}{\sup(g_1)_{x_1}\sup(g_i)_{x_i}}\right]$$

$$>\lambda_1, \; g_i(0, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N) > \lambda_1, \qquad i = 2, ..., N,$$

and

(B)

$$2\sup(g_1)_{x_1} + \sum_{i=2}^{N} \sup(g_i)_{x_1} \sup \frac{\theta_{g_i(0,...,0,\cdot,0,...,0) - \frac{g_1(0,...,0)\sup(g_i)_{x_1}}{\sup(g_1)_{u_1}}}}{\theta_{M(g_1,g_i)}}$$
$$< \sum_{j=2}^{N} \left[\inf(g_1)_{x_j} + \inf(g_j)_{x_1} \inf \frac{\theta_{g_j(0,M_2,...,M_{j-1},\cdot,M_{j+1},...,M_N)}}{\theta_{g_1(\cdot,0,...,0)}} \right],$$

and

$$2\sup(g_i)_{x_i} + \sup(g_i)_{x_1} < \sum_{j=2, j\neq i}^{N} \left[\inf(g_i)_{x_j} + \inf(g_j)_{x_i} \sup \frac{\theta_{g_j(0,...,0,\cdot,0,...,0) - \frac{g_1(0,...,0)\sup(g_j)_{x_1}}{\sup(g_1)_{x_1}}}}{\theta_{g_i(0,M_2,...,M_{i-1},\cdot,M_{i+1},...,M_N)}} \right] \qquad \textit{for } i = 2, ..., N,$$

then (1) has a unique posotive solution.

The conditions imply that species 1 interacts strongly among themselves and weakly with species 2. Similarly for species 2, they interact more strongly among themselves than they do with species 1.

Proof. The existence was already proved in the last section. We prove the uniqueness. Let $(u_1, ..., u_N), (v_1, ..., v_N)$ be positive solutions to (1), and let $p_i = u_i - v_i, i = 1, ..., N$. We want to show that $p_i \equiv 0, i = 1, ..., N$.

Since $(u_1, ..., u_N), (v_1, ..., v_N)$ are solutions to (1),

$$\begin{aligned} \Delta p_i + p_i g_i(u_1, ..., u_N) =& \Delta u_i - \Delta v_i + (u_i - v_i) g_i(u_1, ..., u_N) \\ =& -\Delta v_i - v_i g_i(u_1, ..., u_N) \\ =& -\Delta v_i - v_i g_i(v_1, ..., v_N) + v_i g_i(v_1, ..., v_N) - v_i g_i(u_1, ..., u_N) \\ =& -v_i[g_i(u_1, ..., u_N) - g_i(v_1, ..., v_N)], \qquad i = 1, ..., N. \end{aligned}$$

So,

$$\Delta p_i + p_i g_i(u_1, ..., u_N) - v_i[g_i(v_1, ..., v_N) - g_i(u_1, ..., u_N)] = 0, \qquad i = 1, ..., N.$$

So, for $i = 1, ..., N$,

$$-p_i \Delta p_i - (p_i)^2 g_i(u_1, ..., u_N) + v_i p_i [g_i(v_1, ..., v_N) - g_i(u_1, ..., u_N)] = 0.$$

Since

$$\Delta u_i + u_i g_i(u_1, .., u_N) = 0, \qquad i = 1, ..., N,$$

by the Lemma 2, we have

$$\textstyle\int_\Omega -p_i \Delta p_i - g_i(u_1, ..., u_N)(p_i)^2 dx \geq 0, \qquad i = 1, ..., N,$$

and so,

$$\int_\Omega v_i p_i [g_i(v_1, ..., v_N) - g_i(u_1, ..., u_N)] dx \leq 0, \qquad i = 1, ..., N,$$

and so,

$$\int_\Omega \sum_{i=1}^{N} v_i p_i [g_i(v_1, ..., v_N) - g_i(u_1, ..., u_N)] dx \leq 0.$$

But, by the Mean Value Theorem, for $i = 1, ..., N$, there are z_{ij} such that z_{ij} is between u_j and v_j, $j = 1, ..., N$, and

$$\begin{aligned} g_i(v_1, ..., v_N) - g_i(u_1, ..., u_N) =& g_i(v_1, v_2, ..., v_N) - g_i(u_1, v_2, ..., v_N) + g_i(u_1, v_2, v_3, ..., v_N) - g_i(u_1, u_2, v_3, ..., v_N) \\ &+ ... - g_i(u_1, u_2, ..., u_{N-1}, v_N) - g_i(u_1, u_2, ..., u_{N-1}, u_N) \\ =& \sum_{j=1}^{N} (g_i)_{x_j}(u_1, u_2, ..., u_{j-1}, z_{ij}, v_{j+1}, ..., v_N)(-p_j). \end{aligned}$$

Therefore,

$$\int_\Omega \sum_{i=1}^{N} v_i p_i \left[\sum_{j=1}^{N} (g_i)_{x_j}(u_1, u_2, ..., u_{j-1}, z_{ij}, v_{j+1}, ..., v_N)(-p_j) \right] dx \leq 0.$$

Hence,

$$\int_\Omega \sum_{i=1}^{N} \sum_{j=1}^{N} (g_i)_{x_j}(u_1, ..., u_{j-1}, z_{ij}, v_{j+1}, ..., v_N)(-v_i p_i p_j) dx \leq 0.$$

So,

$$\int_\Omega \sum_{i=1}^{N} \left\{ (g_i)_{x_i}(u_1, ..., u_{j-1}, z_{ii}, v_{i+1}, ..., v_N)[-v_i(p_i)^2] + \sum_{j=1, j\neq i}^{N} (g_i)_{x_j}(u_1, ..., u_{j-1}, z_{ij}, v_{j+1}, ..., v_N)(-v_i p_i p_j) \right\} dx \leq 0.$$

So,

$$\begin{aligned} &\int_\Omega (g_1)_{x_1}(z_{11}, v_2, ..., v_N)[-v_1(p_1)^2] + \sum_{j=2}^{N} (g_1)_{x_j}(u_1, ..., u_{j-1}, z_{1j}, v_{j+1}, ..., v_N)(-v_1 p_1 p_j) \\ &+ \sum_{i=2}^{N} \{ (g_i)_{x_i}(u_1, ..., u_{i-1}, z_{ii}, v_{i+1}, ..., v_N)[-v_i(p_i)^2] + \sum_{j=1, j\neq i}^{N} (g_i)_{x_j}(u_1, ..., u_{j-1}, z_{ij}, v_{j+1}, ..., v_N)(-v_i p_i p_j) \} dx \\ &\quad \leq 0, \end{aligned}$$

so,

$$\begin{aligned} &\int_\Omega (g_1)_{x_1}(z_{11}, v_2, ..., v_N)[-v_1(p_1)^2] + \sum_{j=2}^{N} (g_1)_{x_j}(u_1, ..., u_{j-1}, z_{1j}, v_{j+1}, ..., v_N)(-v_1 p_1 p_j) \\ &+ \sum_{i=2}^{N} \left\{ (g_i)_{x_i}(u_1, ..., u_{i-1}, z_{ii}, v_{i+1}, ..., v_N)[-v_i(p_i)^2] + \sum_{j=2, j\neq i}^{N} (g_i)_{x_j}(u_1, ..., u_{j-1}, z_{ij}, v_{j+1}, ..., v_N)(-v_i p_i p_j) \right. \\ &\left. + (g_i)_{x_1}(z_{i1}, v_2, ..., v_N)(-v_i p_i p_1) \right\} dx \leq 0. \end{aligned}$$

But, since $(g_i)_{x_1}(-v_i) < 0$ and $p_i p_1 \leq \frac{(p_i)^2}{2} + \frac{(p_1)^2}{2}$ for $i = 2, ..., N$,

$$\begin{aligned} &\int_\Omega (g_1)_{x_1}(z_{11}, v_2, ..., v_N)[-v_1(p_1)^2] + \sum_{j=2}^{N} (g_1)_{x_j}(u_1, ..., u_{j-1}, z_{1j}, v_{j+1}, ..., v_N)(-v_1 p_1 p_j) \\ &+ \sum_{i=2}^{N} \left\{ (g_i)_{x_i}(u_1, ..., u_{i-1}, z_{ii}, v_{i+1}, ..., v_N)[-v_i(p_i)^2] + \sum_{j=2, j\neq i}^{N} (g_i)_{x_j}(u_1, ..., u_{j-1}, z_{ij}, v_{j+1}, ..., v_N)(-v_i p_i p_j) \right. \\ &\left. + (g_i)_{x_1}(z_{i1}, v_2, ..., v_N)(-v_i) \left(\frac{(p_i)^2}{2} + \frac{(p_1)^2}{2} \right) \right\} dx \leq 0. \end{aligned}$$

Hence,

$$\int_\Omega \left[(g_1)_{x_1}(z_{11},v_2,...,v_N)(-v_1)+\sum_{i=2}^N \frac{(g_i)_{x_1}(z_{i1},v_2,...,v_N)(-v_i)}{2}\right](p_1)^2$$
$$+\sum_{j=2}^N (g_1)_{x_j}(u_1,...,u_{j-1},z_{1j},v_{j+1},...,v_N)(-v_1p_1p_j)$$
$$+\sum_{i=2}^N\left\{\left[(g_i)_{x_i}(u_1,...,u_{i-1},z_{ii},v_{i+1},...,v_N)(-v_i)+\frac{(g_i)_{x_1}((z_{i1},v_2,...,v_N)(-v_i)}{2}\right][(p_i)^2]\right.$$
$$\left.+\sum_{j=2,j\neq i}^N (g_i)_{x_j}(u_1,...,u_{j-1},z_{ij},v_{j+1},...,v_N)(-v_ip_ip_j)\right\}dx\le 0.$$

If the integrand on the left side is positive definite, then $p_i\equiv 0, i=1,...,N$, which means the uniqueness. But,

$$\begin{aligned}(g_1)_{x_j}(-v_1p_1p_j) \le&(g_1)_{x_j}(-v_1)\left(\frac{(p_1)^2}{2}+\frac{(p_j)^2}{2}\right),\\ (g_i)_{x_j}(-v_ip_ip_j) \le&(g_i)_{x_j}(-v_i)\left(\frac{(p_i)^2}{2}+\frac{(p_j)^2}{2}\right),\end{aligned}$$

for $j=2,...,N, j\neq i$, and so, the result follows if the condition is satisfied by the solution estimates in the Lemma 4. □

6. Uniqueness with perturbation

Define $B=\{(g_1,...,g_N)\in[C^1]^N|\lim_{x_i\to\infty}g_i(x_1,...,x_i,...,x_N)=-\infty, i=1,...,N\}$ with $\|(g_1,...,g_N)\|_B=\sum_{i=1}^N|g_i(0,...,0)|+\sum_{i,j=1}^N\sup|(g_i)_{x_j}|$ for all $(g_1,...,g_N)\in B$.

Then by the functional analysis theory, $(B,\|\cdot\|_B)$ is a Banach space.

The following theorem is our main result about the perturbation of uniqueness.

Theorem 5. *Suppose $(g_1,...,g_N)\in B$ is such that*

(A)
$$\sup(g_1)_{u_1}\sup(g_i)_{u_i}+\inf(g_1)_{u_i}\sup(g_i)_{u_1}>0, i=2,...,N,$$

(B)
$$g_1(0,M_2,...,M_{i-1},0,M_{i+1},...,M_N)+\inf(g_1)_{u_i}\left[\frac{g_1(0,...,0)\sup(g_i)_{u_1}-g_i(0,...,0)\sup(g_1)_{u_1}}{\sup(g_1)_{u_1}\sup(g_i)_{u_i}}\right]>\lambda_1,$$
$$g_1(M_1,...,M_{i-1},0,M_{i+1},...,M_N)>\lambda_1,$$
$$g_i(0,M_2,...,M_N)>\lambda_1, i=2,...,N,$$

(C) *(1) has a unique coexistence state $(u_1,...,u_N)$,*

(D) *the Frechet derivative of (1) at $(u_1,...,u_N)$ is invertible.*

Then there is a neighborhood V of $(g_1,...,g_N)$ in B such that if $(\bar{g_1},...,\bar{g_N})\in V$, then (1) with $(\bar{g_1},...,\bar{g_N})$ has a unique positive solution.

Biologically, the first two conditions in Theorem 5 indicates that the rates of reproduction are relatively large, and the birth rate of the prey must be larger than that of predator. Furthermore, comparing the two conditions. Similarly, the fourth condition, which requires the invertibility of the Frechet derivative, signifies that the rates of self-limitation are relatively larger than the rates of competition, a relationship that will be established in Lemma 5. When these conditions are fulfilled, the conclusion of our theorem asserts that small perturbations of the rates do not affect the existence and uniqueness of the positive steady state. That is, the two species implied can continue to coexist even if the factors determining the population densities vary slightly.

Now, at first glance, Theorem 5 may appear to be a consequence of the Implicit Function Theorem. However, the Implicit Function Theorem only guarantees local uniqueness. In contrast, our result in Theorem 5 guarantees global uniqueness. The techniques we will use in the proof of Theorem 5 include the Implicit Function Theorem and a priori estimates on solutions of (1).

Proof. Since the Frechet derivative of (1) at $(u_1, ..., u_N)$ is invertible, by the Implicit Function Theorem, there is a neighborhood V of $(g_1, ..., g_N)$ in B and a neighborhood W of $(u_1, ..., u_N)$ in $[C_0^{2,\alpha}(\bar{\Omega})]^N$ such that for all $(\bar{g_1}, ..., \bar{g_N}) \in V$, there is a unique positive solution $(\bar{u_1}, ..., \bar{u_N}) \in W$ of (1) with $(\bar{g_1}, ..., \bar{g_N})$. Thus, the local uniqueness of the solution is guaranteed.

To prove global uniqueness, suppose that the conclusion of Theorem 5 is false. Then, there are sequences $(g_{1n}, g_{2n}, ..., g_{Nn}, u_{1n}, u_{2n}, ..., u_{Nn})$ and $(g_{1n}, g_{2n}, ..., g_{Nn}, u^*_{1n}, u^*_{2n}, ..., u^*_{Nn})$ in $V \times [C_0^{2,\alpha}(\bar{\Omega})]^N$ such that $(u_{1n}, ..., u_{Nn})$ and $(u^*_{1n}, ..., u^*_{Nn})$ are positive solutions of (1) with $(g_{1n}, ..., g_{Nn})$,
$(u_{1n}, ..., u_{Nn}) \neq (u^*_{1n}, ..., u^*_{Nn})$ and $(g_{1n}, ..., g_{Nn}) \to (g_1, ..., g_N)$. By Schauder's estimate in elliptic theory and the solution estimate in the Lemma 4, there are uniformly convergent subsequences of $\{u_{1n}\}, ..., \{u_{Nn}\}$, which again will be denoted by $\{u_{1n}\}, ..., \{u_{Nn}\}$.

Thus, let

$$(u_{1n}, ..., u_{Nn}) \to (\bar{u_1}, ..., \bar{u_N}),$$
$$(u^*_{1n}, ..., u^*_{Nn}) \to (u^*_1, ..., u^*_N).$$

Then $(\bar{u_1}, ..., \bar{u_N}), (u^*_1, ..., u^*_N) \in (C^{2,\alpha})^N$ are also solutions to (1) with $(g_1, ..., g_N)$. We claim that $\bar{u_1} > 0, ..., \bar{u_N} > 0, u^*_1 > 0, ..., u^*_N > 0$. By the Maximum Principles, it suffices to claim $\bar{u_1}, ..., \bar{u_N}, u^*_1, ..., u^*_N$ are not identically zero.

Suppose that it is not true. Then by the Maximum Principles again, either one of the followings will hold:
(1) $\bar{u_1} \equiv 0$ and $\bar{u_i} \equiv 0$ for all $i = 2, ..., N$.
(2) $\bar{u_1} \equiv 0$ and $\bar{u_i} > 0$ for some $i = 2, ..., N$.
(3) $\bar{u_1} > 0$ and $\bar{u_i} \equiv 0$ for some $i = 2, ..., N$.
First, suppose $\bar{u_1} \equiv 0$.
Let $\tilde{u_{1n}} = \frac{u_{1n}}{\|u_{1n}\|_\infty}$ and $\tilde{u_{in}} = u_{in}$ for $i = 2, ..., N$.
Then

$$\Delta\tilde{u_{1n}} + \tilde{u_{1n}} g_{1n}(u_{1n}, \tilde{u_{2n}}, ..., \tilde{u_{Nn}}) = 0,$$
$$\Delta\tilde{u_{in}} + \tilde{u_{in}} g_{in}(u_{1n}, \tilde{u_{2n}}, ..., \tilde{u_{Nn}}) = 0, i = 2, ..., N.$$

By the elliptic theory again, there is $\tilde{u_1}$ such that $\tilde{u_{1n}} \to \tilde{u_1}$, and so,

$$\Delta\tilde{u_1} + \tilde{u_1} g_1(0, \bar{u_2}, ..., \bar{u_N}) = 0,$$
$$\Delta\bar{u_i} + \bar{u_i} g_i(0, \bar{u_2}, ..., \bar{u_N}) = 0, i = 2, ..., N.$$

Hence, $\lambda_1[-g_1(0, \bar{u_2}, ..., \bar{u_N})] = 0$.

If $\bar{u_i} \equiv 0$ for all $i = 2, ..., N$, then $\lambda_1 - g_1(0, ..., 0) = \lambda_1[-g_1(0, ..., 0)] = 0$, which is a contradiction to our assumption. If $\bar{u_i} > 0$ for some $i = 2, ..., N$, then by the monotonicity,

$$\lambda_1 - g_i(0, M_2, ..., M_N) = \lambda_1[-g_i(0, M_2, ..., M_N)] \geq \lambda_1[-g_i(0, \bar{u_2}, ..., \bar{u_N})] = 0,$$

which is a contradiction.

Suppose $\bar{u_1} > 0$ and $\bar{u_i} \equiv 0$ for some $i = 2, ..., N$.
Then

$$\Delta u_{1n} + u_{1n} g_{1n}(u_{1n}, u_{2n}, ..., u_{Nn}) = 0.$$

So,

$$\Delta\bar{u_1} + \bar{u_1} g_1(\bar{u_1}, ..., \bar{u_{i-1}}, 0, \bar{u_{i+1}}, ..., \bar{u_N}) = 0.$$

Therefore,

$$\begin{aligned}\lambda_1 - g_1(M_1, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N) =&\lambda_1[-g_1(M_1, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N)]\\ \geq&\lambda_1[-g_1(\bar{u_1}, ..., \bar{u_{i-1}}, 0, \bar{u_{i+1}}, ..., \bar{u_N})]\\ =&0,\end{aligned}$$

which is a contradiction.

Consequently, $(\bar{u_1}, ..., \bar{u_N})$ and $(u_1^*, ..., u_N^*)$ are positive solutions to (1) with $(g_1, ..., g_N)$, and so $(\bar{u_1}, ..., \bar{u_N}) = (u_1^*, ..., u_N^*) = (u_1, ..., u_N)$ by the uniqueness condition. But, this is a contradiction to the Implicit Function Theorem, since $(u_{1n}, ..., u_{Nn}) \neq (u_{1n}^*, ..., u_{Nn}^*)$. □

In biological terms, the proof of our theorem indicates that if one of two species living in the same domain becomes extinct, that is, if one species is excluded by the other, then the reproduction rates of both must be small. In other words, the region condition of reproduction rates (A) is reasonable.

Now, the condition (C) in Theorem 5 requiring the invertibility of the Frechet derivative is too artificial to have any direct biological implications. We therefore turn our attention to more applicable conditions that will guarantee the invertibility of the Frechet derivative. We then obtain the following relationship:

Lemma 5. *Suppose $(u_1, ..., u_N)$ is a positive solution to (1). If*

$$2\sup(g_1)_{u_1} + \sum_{i=2}^{N} \sup(g_i)_{u_1}\frac{u_i}{u_1} < \sum_{j=2}^{N}\left[\inf(g_1)_{u_j} + \inf(g_j)_{u_1}\frac{u_j}{u_1}\right],$$

$$2\sup(g_i)_{u_i} + \sup(g_i)_{u_1} < \sum_{j=2, j\neq i}^{N}\left[\inf(g_i)_{u_j} + \inf(g_j)_{u_i}\frac{u_j}{u_i}\right], i = 2, ..., N,$$

then the Frechet derivative of (1) at $(u_1, ..., u_N)$ is invertible.

Proof. The Frechet derivative of (1) at $(u_1, ..., u_N)$ is $A = (A_{ij})$, where

$$A_{ij} = \begin{cases} -\Delta - g_i(u_1, ..., u_N) - u_i\frac{\partial g_i(u_1,...,u_N)}{\partial u_i}, & i = j\\ -u_i\frac{\partial g_i(u_1,...,u_N)}{\partial u_j}, & i \neq j\end{cases}$$

for $i, j = 1, ..., N$. We need to show that $N(A) = \{0\}$ by the Fredholm Alternative, where $N(A)$ is the null space of A. In fact, from the equations

$$\begin{aligned}&\int_\Omega |\nabla\phi_1|^2 - \left(g_1(u_1, ..., u_N) + u_1\frac{\partial g_1(u_1, ..., u_N)}{\partial u_1}\right)\phi_1^2\\ &\quad - \left(\frac{\partial g_1(u_1, ..., u_N)}{\partial u_2}\phi_2 + ... + \frac{\partial g_1(u_1, ..., u_N)}{\partial u_N}\phi_N\right)u_1\phi_1 dx = 0,\\ &\int_\Omega |\nabla\phi_2|^2 - \left(g_2(u_1, ..., u_N) + u_2\frac{\partial g_2(u_1, ..., u_N)}{\partial u_2}\right)\phi_2^2\\ &\quad - \left(\frac{\partial g_2(u_1, ..., u_N)}{\partial u_1}\phi_1 + \frac{\partial g_2(u_1, ..., u_N)}{\partial u_3}\phi_3 + ... + \frac{\partial g_2(u_1, ..., u_N)}{\partial u_N}\phi_N\right)u_2\phi_2 dx = 0,\\ &\qquad\vdots,\\ &\int_\Omega |\nabla\phi_N|^2 - \left(g_N(u_1, ..., u_N) + u_N\frac{\partial g_N(u_1, ..., u_N)}{\partial u_N}\right)\phi_N^2\\ &\quad - \left(\frac{\partial g_N(u_1, ..., u_N)}{\partial u_1}\phi_1 + ... + \frac{\partial g_N(u_1, ..., u_N)}{\partial u_{N-1}}\phi_{N-1}\right)u_N\phi_N dx = 0.\end{aligned}$$

Since $\lambda_1(-g_i(u_1, ..., u_N)) = 0$ for $i = 1, ..., N$, we see that

$$\int_\Omega |\nabla\phi_i|^2 - g_i(u_1, ..., u_N)\phi_i^2 dx \geq 0, i = 1, ..., N.$$

Hence,

$$\int_\Omega -u_1\frac{\partial g_1(u_1,...,u_N)}{\partial u_1}\phi_1^2-\left(\frac{\partial g_1(u_1,...,u_N)}{\partial u_2}\phi_2+...+\frac{\partial g_1(u_1,...,u_N)}{\partial u_N}\phi_N\right)u_1\phi_1dx\le 0,$$

$$\int_\Omega -u_2\frac{\partial g_2(u_1,...,u_N)}{\partial u_2}\phi_2^2-\left(\frac{\partial g_2(u_1,...,u_N)}{\partial u_1}\phi_1+\frac{\partial g_2(u_1,...,u_N)}{\partial u_3}\phi_3+...+\frac{\partial g_2(u_1,...,u_N)}{\partial u_N}\phi_N\right)u_2\phi_2dx\le 0,$$

$$\vdots,$$

$$\int_\Omega -u_N\frac{\partial g_N(u_1,...,u_N)}{\partial u_N}\phi_N^2-\left(\frac{\partial g_N(u_1,...,u_N)}{\partial u_1}\phi_1+...+\frac{\partial g_N(u_1,...,u_N)}{\partial u_{N-1}}\phi_{N-1}\right)u_N\phi_Ndx\le 0.$$

Therefore,

$$\int_\Omega -\sum_{i=1}^N u_i\frac{\partial g_i(u_1,...,u_N)}{\partial u_i}\phi_i^2-\sum_{i=1}^N u_i\phi_i\sum_{j=1,j\neq i}^N\frac{\partial g_i(u_1,...,u_N)}{\partial u_j}\phi_jdx\le 0.$$

It implies that

$$\int_\Omega\sum_{i=1}^N(-u_i\frac{\partial g_i(u_1,...,u_N)}{\partial u_i}\phi_i^2-\sum_{j=1,j\neq i}^N\frac{\partial g_i(u_1,...,u_N)}{\partial u_j}u_i\phi_j\phi_i)dx\le 0.$$

Hence,

$$\int_\Omega -u_1(g_1)_{u_1}(u_1,...,u_N)(\phi_1)^2-\sum_{j=2}^N(g_1)_{u_j}(u_1,...,u_N)u_1\phi_j\phi_1+\sum_{i=2}^N[-u_i(g_i)_{u_i}(u_1,...,u_N)(\phi_i)^2$$
$$-(g_i)_{u_1}(u_1,...,u_N)u_i\phi_1\phi_i-\sum_{j=2,j\neq i}^N(g_i)_{u_j}(u_1,...,u_N)u_i\phi_j\phi_i]dx\le 0.$$

But, since $-(g_i)_{u_1}(u_1,...,u_N)u_i\phi_1\phi_i\ge -(g_i)_{u_1}(u_1,...,u_N)u_i(\frac{(\phi_1)^2}{2}+\frac{(\phi_i)^2}{2})$, for all $i=2,...,N$,

$$\int_\Omega -u_1(g_1)_{u_1}(u_1,...,u_N)(\phi_1)^2-\sum_{j=2}^N(g_1)_{u_j}(u_1,...,u_N)u_1\phi_j\phi_1+\sum_{i=2}^N\Bigg[-u_i(g_i)_{u_i}(u_1,...,u_N)(\phi_i)^2$$
$$-(g_i)_{u_1}(u_1,...,u_N)u_i(\frac{(\phi_1)^2}{2}+\frac{(\phi_i)^2}{2})-\sum_{j=2,j\neq i}^N(g_i)_{u_j}(u_1,...,u_N)u_i\phi_j\phi_i\Bigg]dx\le 0.$$

Hence,

$$\int_\Omega\left[-u_1(g_1)_{u_1}(u_1,...,u_N)-\sum_{i=2}^N\frac{(g_i)_{u_1}(u_1,...,u_N)}{2}u_i\right](\phi_1)^2-\sum_{j=2}^N(g_1)_{u_j}(u_1,...,u_N)u_1\phi_j\phi_1$$
$$+\sum_{i=2}^N\left[(-u_i(g_i)_{u_i}(u_1,...,u_N)-\frac{(g_i)_{u_1}(u_1,...,u_N)}{2})(\phi_i)^2-\sum_{j=2,j\neq i}^N(g_i)_{u_j}(u_1,...,u_N)u_i\phi_j\phi_i\right]dx\le 0.$$

Since

$$-(g_1)_{u_j}(u_1,...,u_N)u_1\phi_j\phi_1\le-(g_1)_{u_j}(u_1,...,u_N)u_1[\frac{(\phi_j)^2}{2}+\frac{(\phi_1)^2}{2}],j=2,...,N,$$
$$-(g_i)_{u_j}(u_1,...,u_N)u_i\phi_j\phi_i\le-(g_i)_{u_j}(u_1,...,u_N)u_i[\frac{(\phi_j)^2}{2}+\frac{(\phi_i)^2}{2}],i,j=2,...,N,i\neq j,$$

if

$$-u_1(g_1)_{u_1}(u_1,...,u_N)-\frac{1}{2}\sum_{i=2}^N(g_i)_{u_1}(u_1,...,u_N)u_i>-\frac{1}{2}\sum_{j=2}^N\left[(g_1)_{u_j}(u_1,...,u_N)u_1+(g_j)_{u_1}(u_1,...,u_N)u_j\right],$$

$$-u_i(g_i)_{u_i}(u_1,...,u_N)-\frac{(g_i)_{u_1}(u_1,...,u_N)}{2}u_i>-\frac{1}{2}\sum_{j=2,j\neq i}^{N}\Big[(g_i)_{u_j}(u_1,...,u_N)u_i$$
$$+(g_j)_{u_i}(u_1,...,u_N)u_j\Big], i=2,...,N,$$

then the integrand in above inequality is positive definite, which means $(\phi_1,...,\phi_N)$ is trivial. But, it holds if the conditions are satisfied. □

Combining Lemma 4, Theorem 4, Theorem 5, and Lemma 5, we obtain the following corollary.

Corollary 1. *If $(g_1,...,g_N)\in B$ is such that*

(A)
$$\sup(g_1)_{u_1}\sup(g_i)_{u_i}+(N-1)\inf(g_1)_{u_i}\sup(g_i)_{u_1}>0,\ \ i=2,...,N,$$

(B)
$$g_1(0,...,0)>\frac{\sup(g_1)_{u_1}\sup(g_i)_{u_i}\left(\lambda_1+\frac{(N-1)\inf(g_1)_{u_i}g_i(0,...,0)}{\sup(g_i)_{u_i}}\right)}{\sup(g_1)_{u_1}\sup(g_i)_{u_i}+(N-1)\inf(g_1)_{u_i}\sup(g_i)_{u_1}},$$

and

$$g_i(0,...,0)>\lambda_1-(N-2)\inf_{j=2,...,N,j\neq i}\{\bar{u_j}\inf(g_i)_{u_j}\},\ \ i=2,...,N,$$

(C)
$$g_1(0,M_2,...,M_{i-1},0,M_{i+1},...,M_N)+\inf(g_1)_{x_i}\left[\frac{g_1(0,...,0)\sup(g_i)_{x_1}-g_i(0,...,0)\sup(g_1)_{x_1}}{\sup(g_1)_{x_1}\sup(g_i)_{x_i}}\right]>\lambda_1,$$
$$g_1(M_1,...,M_{i-1},0,M_{i+1},...,M_N)>\lambda_1,\qquad \textit{and}$$
$$g_i(0,M_2,...,M_{i-1},0,M_{i+1},...,M_N)>\lambda_1, i=2,...,N,$$

and

(D)
$$2\sup(g_1)_{x_1}+\sum_{i=2}^{N}\sup(g_i)_{x_1}\sup\frac{\theta_{g_i(0,...,0,\cdot,0,...,0)-\frac{g_1(0,...,0)\sup(g_i)_{x_1}}{\sup(g_1)_{u_1}}}}{\theta_{M(g_1,g_i)}}$$
$$<\sum_{j=2}^{N}\left[\inf(g_1)_{x_j}+\inf(g_j)_{x_1}\inf\frac{\theta_{g_j(0,M_2,...,M_{j-1},\cdot,M_{j+1},...,M_N)}}{\theta_{g_1(\cdot,0,...,0)}}\right],$$

and

$$2\sup(g_i)_{x_i}+\sup(g_i)_{x_1}$$
$$<\sum_{j=2,j\neq i}^{N}\left[\inf(g_i)_{x_j}+\inf(g_j)_{x_i}\sup\frac{\theta_{g_j(0,...,0,\cdot,0,...,0)-\frac{g_1(0,...,0)\sup(g_j)_{x_1}}{\sup(g_1)_{x_1}}}}{\theta_{g_i(0,M_2,...,M_{i-1},\cdot,M_{i+1},...,M_N)}}\right]\ \textit{for } i=2,...,N,$$

then there is a neighborhood V of $(g_1,...,g_N)$ such that if $(\bar{g_1},...,\bar{g_N})\in V$, then (1) with $(\bar{g_1},...,\bar{g_N})$ has a unique positive solution.

In biological terms, the result obtained in Corollary 1 confirms that under certain conditions, two species who relax ecologically can continue to coexist at fixed rates. The requirements given in (A) and (B) simply state that each species must interact strongly with itself and weakly with the other species.

7. Uniqueness with perturbation of region

The following Theorem is the main result.

Theorem 6. *Suppose $\Gamma \subseteq B$ be a closed, bounded, convex region in B such that*

(A) *for all $(g_1, ..., g_N) \in \Gamma$,*

$$\sup(g_1)_{u_1} \sup(g_i)_{u_i} + \inf(g_1)_{u_i} \sup(g_i)_{u_1} > 0, i = 2, ..., N,$$

(B) *for all $(g_1, ..., g_N) \in \Gamma$,*

$$g_1(0, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N) + \inf(g_1)_{u_i} \left[\frac{g_1(0, ..., 0) \sup(g_i)_{u_1} - g_i(0, ..., 0) \sup(g_1)_{u_1}}{\sup(g_1)_{u_1} \sup(g_i)_{u_i}} \right] > \lambda_1,$$
$$g_1(M_1, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N) > \lambda_1,$$
$$g_i(0, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N) > \lambda_1, i = 2, ..., N,$$

(C) *for all $(g_1, ..., g_N) \in \partial_L \Gamma$, (1) has a unique positive solution, where $\partial_L \Gamma = \{(\lambda_{g_2,...,g_N}, g_2, ..., g_N) \in \Gamma |$ for any fixed $g_2, ..., g_N, \lambda_{g_2,...,g_N} = \inf\{\| g_1 \| |(g_1, g_2, ..., g_N) \in \Gamma\}.\}$,*

(D) *for all $(g_1, ..., g_N) \in \Gamma$, the Frechet derivative of (1) at every positive solution $(u_1, ..., u_N)$ is invertible.*

Then for all $(g_1, ..., g_N) \in \Gamma$, (1) has a unique positive solution. Furthermore, there is an open set W in B such that $\Gamma \subseteq W$ and for every $(g_1, ..., g_N) \in W$, (1) has a unique positive solution.

Theorem 6 goes even further than Theorem 5 which states uniqueness in the whole region whenever we have uniqueness on the left boundary and invertibility of the linearized operator at any particular solution inside the domain.

Proof. For each fixed $g_2, ..., g_N$, consider $(g_1, g_2, ..., g_N) \in \partial_L \Gamma$ and $(\bar{g_1}, g_2, ..., g_N) \in \Gamma$. We need to show that for all $0 \leq t \leq 1$, (1) with $(1-t)(g_1, ..., g_N) + t(\bar{g_1}, g_2, ..., g_N)$ has a unique positive solution. Since (1) with $(g_1, ..., g_N)$ has a unique positive solution $(u_1, ..., u_N)$ and the Frechet derivative of (1) at $(u_1, ..., u_N)$ is invertible, Theorem 5 implies that there is an open neighborhood V of $(g_1, ..., g_N)$ in B such that if $(g_{10}, ..., g_{N0}) \in V$, then (1) with $(g_{10}, ..., g_{N0})$ has a unique positive solution.

Let $\lambda_s = \sup\{0 \leq \lambda \leq 1|$ (1) with $(1-t)(g_1, ..., g_N) + t(\bar{g_1}, g_2, ..., g_N)$ has a unique coexistence state for $0 \leq t \leq \lambda.\}$. We need to show that $\lambda_s = 1$. Suppose $\lambda_s < 1$. From the definition of λ_s, there is a sequence $\{\lambda_n\}$ such that $\lambda_n \to \lambda_s^-$ and there is a sequence $(u_{1n}, ..., u_{Nn})$ of the unique positive solutions of (1) with $(1-\lambda_n)(g_1, ..., g_N) + \lambda_n(\bar{g_1}, g_2, ..., g_N)$. Then by elliptic theory, there is $(u_{10}..., , u_{N0})$ such that $(u_{1n}, ..., u_{Nn})$ converges to $(u_{10}, ..., u_{N0})$ uniformly and $(u_{10}, ..., u_{N0})$ is a solution of (1) with $(1-\lambda_s)(g_1, ..., g_N) + \lambda_s(\bar{g_1}, g_2, ..., g_N)$.

But, by the same proof as in the §7, $u_{10} > 0, ..., u_{N0} > 0$.

We claim that (1) has a unique coexistence state with $(1-\lambda_s)(g_1, ..., g_N) + \lambda_s(\bar{g_1}, g_2, ..., g_N)$. In fact, if not, assume that $(\bar{u}_{10}, ..., \bar{u}_{N0}) \neq (u_{10}, ..., u_{N0})$ is another coexistence state. By the Implicit Function Theorem, there exists $0 < \tilde{a} < \lambda_s$ and very close to λ_s such that (1) with $(1-\tilde{a})(g_1, ..., g_N) + \tilde{a}(\bar{g_1}, g_2, ..., g_N)$ has a coexistence state very close to $(\bar{u}_{10}, ..., \bar{u}_{N0})$, which means that (1) with $(1-\tilde{a})(g_1, ..., g_N) + \tilde{a}(\bar{g_1}, g_2, ..., g_N)$ has more than one coexistence state. This is a contradiction to the definition of λ_s. But, since (1) with $(1-\lambda_s)(g_1, ..., g_N) + \lambda_s(\bar{g_1}, g_2, ..., g_N)$ has a unique coexistence state and the Frechet derivative is invertible, Theorem 5 implies that λ_s can not be as defined. Therefore, for each $(g_1, .., g_N) \in \Gamma$, (1) with $(g_1, ..., g_N)$ has a unique coexistence state $(u_1, ..., u_N)$. Furthermore, by the assumption, for each $(g_1, ..., g_N) \in \Gamma$, the Frechet derivative of (1) with $(g_1, ..., g_N)$ at the unique solution $(u_1, ..., u_N)$ is invertible. Hence, Theorem 5 concluded that for each $(g_1, ..., g_N) \in \Gamma$, there is an open neighborhood $V_{(g_1,...,g_N)}$ of $(g_1, ..., g_N)$ in B such that if $(\bar{g_1}, ..., \bar{g_N}) \in V_{(g_1,...,g_N)}$, then (1) with $(\bar{g_1}, ..., \bar{g_N})$ has a unique coexistence state. Let $W = \bigcup_{(g_1,...,g_N) \in \Gamma} V_{(g_1,...,g_N)}$. Then W is an open set in B such that $\Gamma \subseteq W$ and for each $(\bar{g_1}, ..., \bar{g_N}) \in W$, (1) with $(\bar{g_1}, ..., \bar{g_N})$ has a unique coexistence state.

Apparently, Theorem 6 generalizes Theorem 5. □

8. Ecological applications

Within the academia of mathematical biology, extensive academic work has been devoted to investigation of the simple predator-prey model, commonly known as the Lotka-Volterra predator-prey model. This system describes the predator-prey interaction of two species residing in the same environment in the following manner:

Suppose N species of animals with predator-prey interaction, rabbits and tigers for instance, are residing in a bounded domain Ω. Let $u_i(x,t)$ be densities of prey($i=1$) and predators($i=2,...,N$) in the place x of Ω at time t, respectively. Then we have the dynamic predator-prey model

$$\begin{cases} (u_i)_t(x,t) = \Delta(u_i)(x,t) + u_i(x,t)(a_i + \sum_{j=1}^{N} b_{ij}u_j(x,t)) \quad \text{in } \Omega\times[0,\infty), \\ (u_i)(x,t) = 0, i=1,...,N \text{ for } x\in\partial\Omega, \end{cases}$$

where $a_i > 0, i=1,...,N$ are reproduction rates and b_{ij} are self-limitation and competition rates such that

$$b_{ij} \begin{cases} <0, i=1, j=1,...,N, \\ >0, i=2,...,N, j=1, \\ <0, i=2,...,N, j=2,...,N. \end{cases}$$

Here we are interested in the time independent, positive solutions, i.e. the positive solutions $u_i(x), i=1,...,N$ of the elliptic interacting system of N functions with homogeneous boundary conditions

$$\begin{cases} \Delta u_i + u_i(a_i + \sum_{j=1}^{N} b_{ij}u_j) = 0 \quad \text{in } \Omega, \\ u_i|_{\partial\Omega} = 0, i=1,...,N, \end{cases} \tag{8}$$

which are called the coexistence state or the steady state. The coexistence state is the positive density solution depending only on the spatial variable x, not on the time variable t, and so, its existence means that the two species of animals can live peacefully and forever.

We establish the following existence result from the Theorem 1:

Corollary 2. *If for each $i=2,...,N$,*

(A)
$$b_{11}b_{ii} + (N-1)b_{1i}b_{i1} > 0, i=2,...,N,$$

(B)
$$a_1 > \frac{b_{11}b_{ii}(\lambda_1 + \frac{(N-1)b_{1i}a_i}{b_{ii}})}{b_{11}b_{ii} + (N-1)b_{1i}b_{i1}}$$

and

(C)
$$a_i > \lambda_1 - (N-2)\inf_{j=2,...,N, j\neq i}\{\bar{u}_j b_{ij}\}, i=2,...,N,$$

where $\bar{u}_j$ is defined below, then (8) has a positive solution.

Biologically, the conditions in Corollary 2 implies that if the self-reproduction and self-limitation rates are relatively large, and the competition rates are relatively small, in other words, if members of each species interact strongly among themselves and weakly with members of the other species, then there is a positive steady state solution to (8), that is, the two species within the same domain will coexist indefinitely at population densities.

We also establish the following nonexistence results from the Theorem 2, which means that they can not peacefully coexist, in other words, they are extinct, if they don't have sufficiently large reproduction capacities.

Corollary 3. *Suppose $a_i \leq \lambda_1, i=1,...,N$. Then $u_i \equiv 0, i=1,...,N$ is the only nonnegative solution to (8).*

The Theorem 4 derives the following uniqueness result:

Corollary 4. *In addition to the Corollary 2, if*

(A)
$$a_1+\sum_{j=2,j\neq i}^{N} b_{1j}M_j+b_{1i}\left[\frac{a_1b_{i1}-a_ib_{11}}{b_{11}b_{ii}}\right]>\lambda_1,\ a_i+\sum_{j=2,j\neq i}^{N} b_{ij}M_j>\lambda_1, i=2,...,N,$$

and

(B)
$$2b_{11}+\sum_{i=2}^{N} b_{i1}\sup\frac{-\frac{1}{b_{ii}}\theta_{a_i-\frac{a_1b_{i1}}{b_{11}}}}{-\frac{1}{b_{11}}\theta_{a_1+\sum\limits_{j=2,j\neq i}^{N} b_{1j}M_j+b_{1i}\frac{a_1b_{i1}-a_ib_{11}}{b_{11}b_{ii}}}}<\sum_{j=2}^{N}\left(b_{1j}+b_{j1}\inf\frac{-\frac{1}{b_{jj}}\theta_{a_j+\sum\limits_{k=2,k\neq j}^{N} b_{jk}M_k}}{-\frac{1}{b_{11}}\theta_{a_1}}\right),$$

and

$$2b_{ii}+b_{i1}<\sum_{j=2,j\neq i}^{N}\left(b_{ij}+b_{ji}\sup\frac{-\frac{1}{b_{jj}}\theta_{a_j-\frac{a_1b_{j1}}{b_{11}}}}{-\frac{1}{b_{ii}}\theta_{a_i+\sum\limits_{k=2,k\neq i}^{N} b_{ik}M_k}}\right), \text{ for } i=2,...,N,$$

then (8) has a unique posotive solution.

The conditions imply that species interacts strongly among themselves and weakly with others. The following theorem is our main result about the perturbation of uniqueness derived from the Corollary 1:

Corollary 5. *If*

(A)
$$b_{11}b_{ii}+(N-1)b_{1i}b_{i1}>0, \text{ for } i=2,...,N,$$

(B)
$$\begin{aligned} a_1 >& \frac{b_{11}b_{ii}(\lambda_1+\frac{(N-1)b_{1i}a_i}{b_{ii}})}{b_{11}b_{ii}+(N-1)b_{1i}b_{i1}},\\ a_i >& \lambda_1-(N-2)\inf_{j=2,...,N,j\neq i}\{b_{ij}\bar{u}_j\},\end{aligned}$$

(C)
$$\begin{aligned} a_1+\sum_{j=2,j\neq i}^{N} b_{1j}M_j+b_{1i}\left(\frac{a_1b_{i1}-a_ib_{11}}{b_{11}b_{ii}}\right) >&\lambda_1,\\ a_1+\sum_{j=1,j\neq i}^{N} b_{1j}M_j >&\lambda_1,\\ a_i+\sum_{j=2,j\neq i}^{N} b_{ij}M_j >&\lambda_1, i=2,...,N,\end{aligned}$$

and

(D)
$$2b_{11}+\sum_{i=2}^{N} b_{i1}\sup\frac{-\frac{1}{b_{ii}}\theta_{a_i-\frac{a_1b_{i1}}{b_{11}}}}{-\frac{1}{b_{11}}\theta_{a_1+\sum\limits_{j=2,j\neq i}^{N} b_{1j}M_j+b_{1i}\frac{a_1b_{i1}-a_ib_{11}}{b_{11}b_{ii}}}}<\sum_{j=2}^{N}\left(b_{1j}+b_{j1}\inf\frac{-\frac{1}{b_{jj}}\theta_{a_j+\sum\limits_{k=2,k\neq j}^{N} b_{jk}M_k}}{-\frac{1}{b_{11}}\theta_{a_1}}\right),$$

and

$$2b_{ii} + b_{i1} < \sum_{j=2, j\neq i}^{N} \left(b_{ij} + b_{ji} \sup \frac{-\frac{1}{b_{jj}} \theta_{a_j - \frac{a_1 b_{j1}}{b_{11}}}}{-\frac{1}{b_{ii}} \theta_{a_i + \sum_{k=2, k\neq i}^{N} b_{ik} M_k}} \right) \textit{ for } i = 2, ..., N,$$

then there is a neighborhood V of $(a_1, ..., a_N)$ such that if $(\bar{a_1}, ..., \bar{a_N}) \in V$, then (8) with $(\bar{a_1}, ..., \bar{a_N})$ has a unique positive solution.

In biological terms, the result obtained in Corollary 5 confirms that under certain conditions, the species who relax ecologically can continue to coexist at fixed rates. The requirements given in (A) and (B) simply state that each species must interact strongly with itself and weakly with the other species.

These results provide insight into the predator-prey interactions of species operating under the conditions described in the Lotka-Volterra model.

9. Conclusions

In this paper, our investigation of the effects of perturbations on the general predator-prey model resulted in the development and proof of Theorem 5, Lemma 5, and Corollary 1 as detailed above. The three together assert that given the existence of a unique solution $(u_1, ..., u_N)$ to the system (5), perturbations of the birth rates $(g_1, ..., g_N)$, within a specified neighborhood, will maintain the existence and uniqueness of the positive steady state. Indeed, our results specifically outline conditions sufficient to maintain the positive, steady state solution when the general predator-prey model is perturbed within some region.

Applying this mathematical result to real world situations, our results establish that the species residing in the same environment can vary their interactions, within certain bounds, and continue to survive together indefinitely at unique densities. The conditions necessary for coexistence, as described in the theorem, simply require that members of each species interact strongly with themselves and weakly with members of the other species.

The research presented in this paper has a number of strengths, which confirm both the validity and the applicability of the project. First, the mathematical conditions required in Corollary 1 are identical to those required in Theorem 4. However, in the Theorem 4, we used these conditions to prove the existence and uniqueness of the positive steady state solution for the general predator-prey model. In contrast, the Corollary 1 employs the same conditions to establish that the existence and uniqueness of this solution is maintained when the model is perturbed within some neighborhood. Thus, our findings extend and improve established mathematical theory.

Secondly, perturbations of the general model render its implications more applicable both mathematically and biologically. Because our theorem extends the steady state to any value within some neighborhood of $(g_1, ..., g_N)$, results for the general model pertain to a far wider variety of values. Biologically, perturbations extend the model's description to species affected by factors that vary slightly yet erratically. Thus, the description of competitive interactions given by the model becomes a closer approximation of real-world population dynamics.

While our research therefore represents a progression in the field, the results obtained have an important limitation. Theorem 5, Lemma 5, and Corollary 1 establish that a region of perturbation exists within which the coexistence state is maintained for the general predator-prey model. However, the exact extent of that region remains unknown. Therefore, the results presented in this paper may serve as a platform for research of the question given above. Mathematicians should now attempt to establish the exact extent of the perturbation region in which coexistence is maintained for the general model. Such information would prove very useful not only mathematically but also biologically. Specifically, knowledge of the extent of the region would imply exactly how far the species can relax and yet continue to coexist. Thus, the results achieved through our research will enable the field to continue the development of theory on predator-prey interaction of populations.

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