Open Journal of
Mathematical Analysis

PSR Press

Article

Combinatorial identities of three complex parameters and their basic applications

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Received: 13 July 2025; Accepted: 25 August 2025; Published: 07 September 2025.

Abstract: Inspired by a problem proposal recently published in the journal The Fibonacci Quarterly we offer a generalization consisting of two combinatorial identities involving three complex parameters. These identities turn out to be immensely rich. We demonstrate this by providing basic applications to four different fields: polynomial identities, trigonometric identities, identities involving Horadam numbers, and combinatorial identities. Many of our findings will generalize existing results.

Keywords: combinatorial identity, complex parameter, binomial coefficient, trigonometric identity, Horadam number

MSC: 05A10, 05A19, 33B10.

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1. Introduction

In a recent issue of the journal The Fibonacci Quarterly [1] the second author asked the readers to prove the identity

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{a^k-b^k}{k}=\sum_{k=1}^{n}(-1)^{k-1}\frac{(a-1)^k-(b-1)^k}{k}, \tag{1}$$

valid for all complex numbers a and b. Although not hard to prove, such an identity provides an unusual but useful link between sums with and without binomial coefficients.

Our purpose in this paper is to derive a generalization of (1) involving an additional (that is third) complex parameter. Polynomial combinatorial identities are equations that express relations between polynomials and combinatorial quantities. These identities usually involve sums of polynomial terms weighted by quantities like binomial coefficients, falling or rising factorials or other counting numbers. Common examples are the binomial or multinomial theorem. Polynomial identities with a complex parameter in the binomial coefficient are not unusual and can be found in the literature. Examples for such identities were derived in the articles by Boyadzhiev [2], Wang and Wei [3] and Chen and Guo [4], for instance. Identities with two or even three complex parameters also exist but are rare. Two particular examples that come to mind are the Chu-Vandermonde identity and Hagen-Rothe identities [5–7]: For complex numbers x and y, and non-negative integers m and n the Chu-Vandermonde identity is

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y}{m-k}=\binom{x+y}{n}\binom{y-x}{m-n},$$

of which

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k}=\binom{x+y}{n},$$

is a special case. The Hagen-Rothe identities are similar. Still other sums related to ours were also studied by Egorychev [8] and Lyapin and Chandragiri [9].

Our three parameter generalization of (1) consists of two separate identities presented in a main lemma in the third section. These identities will turn out to be immensely rich and will allow us to deduce a big amount of important results as basic properties. These results will come from four different fields: polynomial identities, trigonometric sums, sums involving the Horadam sequence, and combinatorial identities. In the field of combinatorial identities we will focus on three different classes: Frisch-type identities, Klamkin-type identities and combinatorial sums involving powers of integers.

2. Preliminaries

Before presenting our main results, we first collect several preliminary identities that will be frequently used later.

Lemma 1. *If k and n are integers and x is a complex number, then*

$$\binom{x-k}{n-k} = (-1)^{n-k}\binom{n-x-1}{n-k}. \tag{2}$$

In particular,

$$\binom{-1-k}{n-k} = (-1)^{n-k}\binom{n}{k}, \tag{3}$$

$$\binom{-k}{n-k} = (-1)^{n-k}\binom{n-1}{k-1} = (-1)^{n-k}\frac{k}{n}\binom{n}{k}, \tag{4}$$

$$\binom{1-k}{n-k} = (-1)^{n-k}\binom{n-2}{k-2}. \tag{5}$$

Proof. Identity (2) follows directly from the -1 transformation. □

Lemma 2. *We have*

$$\binom{r+1/2}{s} = \binom{2r+1}{2s}\binom{2s}{s}\binom{r}{s}^{-1}2^{-2s}, \quad r,\, s \in \mathbb{C}\setminus\mathbb{Z}^-, \quad r-s \notin \mathbb{Z}^-, \quad s \neq -1/2, \tag{6}$$

$$\binom{1/2}{r} = (-1)^{r+1}\binom{2r}{r}\frac{2^{-2r}}{2r-1}, \quad r \in \mathbb{Z}, \tag{7}$$

$$\binom{r-1/2}{s} = \binom{2r}{r}\binom{r}{s}\binom{2(r-s)}{r-s}^{-1}2^{-2s}, \quad r,\, s \in \mathbb{C}\setminus\mathbb{Z}^-, \quad r-s \notin \mathbb{Z}^-, \tag{8}$$

$$\binom{-1/2}{r} = (-1)^r\binom{2r}{r}2^{-2r}, \quad r \in \mathbb{Z}, \tag{9}$$

$$\binom{-3/2}{r} = (-1)^r\binom{2r}{r}(2r+1)2^{-2r}, \quad r \in \mathbb{Z}, \tag{10}$$

$$\binom{-1/2-r}{s} = (-1)^s\binom{2\,(r+s)}{r+s}\binom{r+s}{r}\binom{2r}{r}^{-1}2^{-2s}, \quad r,\, s \in \mathbb{Z}. \tag{11}$$

Proof. These are consequences of the generalized binomial coefficients. They are easy to derive using the Gamma function. They can also be found in Gould's book [10]. □

Next, we recall some facts about Horadam sequences that will be needed later. The Horadam sequence $w_j = w_j(a, b; p, q)$ is defined, for all integers, by the recurrence relation [11]

$$w_0 = a,\ w_1 = b, \quad w_j = pw_{j-1} - qw_{j-2}, \quad j \geq 2,$$

with

$$w_{-j} = \frac{1}{q}\left(pw_{-j+1} - w_{-j+2}\right),$$

where a, b, p and q are arbitrary complex numbers with $p \neq 0$, $q \neq 0$, and $p^2 - 4q > 0$. The sequence w_j generalizes many important number and polynomial sequences, for instance, the Fibonacci sequence $F_j = w_j(0,1;1,-1)$, the Lucas sequence $L_j = w_j(2,1;1,-1)$, the Pell sequence $P_j = w_j(0,1;2,-1)$, the Chebyshev polynomials of the first and second kind given by $T_j(x) = w_j(1,x;2x,1)$ and $U_j(x) = w_j(1,2x;2x,1)$, and so on. The j-th term of a Horadam sequence is given by

$$w_j = w_j(p,q) = \frac{A\sigma^j(p,q) - B\tau^j(p,q)}{\sigma(p,q) - \tau(p,q)}, \tag{12}$$

where

$$A = w_1 - w_0\tau(p,q), \quad B = w_1 - w_0\sigma(p,q),$$

and $\sigma(p,q)$ and $\tau(p,q)$ are given by

$$\sigma = \sigma(p,q) = \frac{p+\delta}{2}, \quad \tau = \tau(p,q) = \frac{p-\delta}{2},$$

where $\delta = \sqrt{p^2 - 4q}$, so that $\sigma(p,q)\,\tau(p,q) = q$.

The sequences F_j and L_j are classical sequences and are indexed as sequences A000045 and A000032 in the On-Line Encyclopedia of Integer Sequences [12]. Koshy [13] and Vajda [14] have written excellent books on them. In addition, the sequences $u_j = w_j(0,1;p,q)$ and $v_j = w_j(2,p;p,q)$ are called the Lucas sequences of the first kind and the second kind, respectively. Their explicit forms equal

$$u_j = u_j(p,q) = \frac{\sigma^j(p,q) - \tau^j(p,q)}{\sigma(p,q) - \tau(p,q)} \qquad \text{and} \qquad v_j = v_j(p,q) = \sigma^j(p,q) + \tau^j(p,q).$$

Finally, we mention the gibonacci sequence (or generalized Fibonacci sequence) $G_j = G_j(a,b) = w_j(a,b;1,-1)$. This sequence was studied by Horadam [15] in 1961 under the notation H_j. Terms of the gibonacci sequence can be accessed directly through the Binet-like formula:

$$G_j = \frac{A\alpha^j - B\beta^j}{\alpha - \beta},$$

where $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, and $A = G_1 - G_0\beta$ and $B = G_1 - G_0\alpha$. It is readily established that

$$G_{-j} = (-1)^j(G_0 L_j - G_j).$$

Lemma 3. *For all $r, s \neq 0$ we have the relations*

$$\begin{aligned}
\sigma^r &= \sigma u_r - q u_{r-1}, \\
\tau^r &= \tau u_r - q u_{r-1}, \\
\sigma^r \delta &= \sigma v_r - q v_{r-1}, \\
\tau^r \delta &= -\tau v_r + q v_{r-1},
\end{aligned}$$

and more generally

$$\sigma^{rs} = \frac{u_{rs}}{u_s}\sigma^s - q^s \frac{u_{(r-1)s}}{u_s} \quad \text{and} \quad \tau^{rs} = \frac{u_{rs}}{u_s}\tau^s - q^s \frac{u_{(r-1)s}}{u_s}.$$

Proof. The statements can be verified directly by computation working with $u_s\sigma^{rs}$ (respectively $u_s\tau^{rs}$) and $q = \sigma\tau$. □

Lemma 4. *For all integers r, s and t we have*

$$\sigma^r u_{s-t} = \sigma^s u_{r-t} - q^{s-t}\sigma^t u_{r-s},$$
$$\tau^r u_{s-t} = \tau^s u_{r-t} - q^{s-t}\tau^t u_{r-s},$$
$$\sigma^r u_{s-t}\delta = \sigma^s v_{r-t} - q^{s-t}\sigma^t v_{r-s},$$
$$\tau^r u_{s-t}\delta = -\tau^s v_{r-t} + q^{s-t}\tau^t v_{r-s}.$$

3. The main lemma and its immediate consequences

Lemma 5. *If a, b and x are complex numbers and n is a non-negative integer, then*

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{x}{n-k}\frac{a^k-b^k}{k} = \sum_{k=1}^{n}\binom{x-k}{n-k}\frac{(1-b)^k-(1-a)^k}{k} \tag{13}$$

and

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{x}{n-k}\frac{a^k+b^k}{k} = 2\sum_{k=1}^{n}\binom{x-k}{n-k}\frac{1}{k} - \sum_{k=1}^{n}\binom{x-k}{n-k}\frac{(1-b)^k+(1-a)^k}{k}. \tag{14}$$

Proof. We have

$$\begin{aligned}(a-1)^k \pm (b-1)^k &= \sum_{j=0}^{k}\binom{k}{j}a^j(-1)^{k-j} \pm \sum_{j=0}^{k}\binom{k}{j}b^j(-1)^{k-j}\\ &= \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(a^j \pm b^j).\end{aligned}$$

Therefore,

$$\begin{aligned}(-1)^k\binom{x-k}{n-k}\frac{(a-1)^k \pm (b-1)^k}{k} &= \sum_{j=0}^{k}\binom{x-k}{n-k}\binom{k}{j}(-1)^j\frac{a^j \pm b^j}{k}\\ &= \sum_{j=1}^{k}\binom{x-k}{n-k}\binom{k}{j}(-1)^j\frac{a^j \pm b^j}{k} + (1\pm 1)\binom{x-k}{n-k}\frac{1}{k}\\ &= \sum_{j\geq 1}\binom{x-k}{n-k}\binom{k-1}{j-1}(-1)^j\frac{a^j \pm b^j}{j} + (1\pm 1)\binom{x-k}{n-k}\frac{1}{k},\end{aligned}$$

and thus

$$\begin{aligned}\sum_{k=1}^{n}(-1)^k\binom{x-k}{n-k}\frac{(a-1)^k \pm (b-1)^k}{k} &= \sum_{j\geq 1}(-1)^j\frac{a^j \pm b^j}{j}\sum_{k=1}^{n}\binom{x-k}{n-k}\binom{k-1}{j-1} + (1\pm 1)\sum_{k=1}^{n}\binom{x-k}{n-k}\frac{1}{k}\\ &= \sum_{j\geq 1}(-1)^j\frac{a^j \pm b^j}{j}\sum_{k=j}^{n}\binom{x-k}{n-k}\binom{k-1}{j-1} + (1\pm 1)\sum_{k=1}^{n}\binom{x-k}{n-k}\frac{1}{k}\\ &= \sum_{j\geq 1}(-1)^j\frac{a^j \pm b^j}{j}\binom{x}{n-j} + (1\pm 1)\sum_{k=1}^{n}\binom{x-k}{n-k}\frac{1}{k}\\ &= \sum_{j=1}^{n}(-1)^j\binom{x}{n-j}\frac{a^j \pm b^j}{j} + (1\pm 1)\sum_{k=1}^{n}\binom{x-k}{n-k}\frac{1}{k}\\ &= \sum_{k=1}^{n}(-1)^k\binom{x}{n-k}\frac{a^k \pm b^k}{k} + (1\pm 1)\sum_{k=1}^{n}\binom{x-k}{n-k}\frac{1}{k}.\end{aligned}$$

This completes the proof. □

When $x = n$ then identity (13) reduces to (1), which with $a = 1$ and $b = 0$ gives the classical identity [2]

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{1}{k} = H_n, \tag{15}$$

with $H_n = 1 + 1/2 + \cdots + 1/n$ being the nth harmonic number. Similarly, by setting $x = n + 1$, $a = 1$ and $b = 0$ in identity (13), we obtain

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n+1}{k+1}\frac{1}{k} = \sum_{k=1}^{n}\frac{n-k+1}{k} = (n+1)H_n - n.$$

It follows from $\binom{n+1}{k+1} = \frac{n+1}{k+1}\binom{n}{k}$ that

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{1}{k(k+1)} = H_n - 1 + \frac{1}{n+1}.$$

Note that the left-hand side of the equation above is

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{1}{k} - \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{1}{k+1},$$

hence we have

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\frac{1}{k+1} = \frac{1}{n+1}. \tag{16}$$

The last expression is also known. It is stated, for instance, in the book [16] as Exercise 27 in Chapter 2 (p.105).

The additional complex parameter x in the binomial coefficient provides a very rich source for various combinatorial identities. A first immediate consequence of the main Lemma 5 is the following result:

Theorem 1. *If n is a non-negative integer and a and x are complex numbers, then*

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{x}{n-k}\frac{1-a^k}{k} = \sum_{k=1}^{n}\binom{x-k}{n-k}\frac{(1-a)^k}{k},$$

and

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{x}{n-k}\frac{1+a^k}{k} = -\sum_{k=1}^{n}\binom{x-k}{n-k}\frac{(1-a)^k}{k} + 2\sum_{k=1}^{n}\binom{x-k}{n-k}\frac{1}{k}.$$

Proof. Set $b = 1$ in (13) and (14) and simplify. □

Another instant consequence are the following identities.

Theorem 2. *If n is a non-negative integer and a and x are complex numbers, then*

$$\sum_{k=1}^{n}\binom{x}{n-k}\frac{a^k - a^{-k}}{k} = \sum_{k=1}^{n}\binom{x-k}{n-k}\frac{(1+a)^k\left(1-a^{-k}\right)}{k}, \tag{17}$$

$$\sum_{k=1}^{n}\binom{x}{n-k}(a^k + a^{-k}) = \sum_{k=1}^{n}\binom{x-k}{n-k}(1+a)^k a\left(\frac{1-a^{-k}}{1+a} + \frac{1}{a^{k+1}}\right). \tag{18}$$

Proof. Identity (17) is obtained by setting $b = 1/a$ in (13) and writing $-a$ for a. Identity (18) follows from differentiating (17) with respect to a and multiplying through by a. □

We also get immediately the next known result.

Theorem 3. *If a and x are complex numbers and n is a non-negative integer, then*

$$\sum_{k=1}^{n}\binom{x-k}{n-k}(1+a)^{k-1}=\sum_{k=1}^{n}\binom{x}{n-k}a^{k-1}. \tag{19}$$

Proof. Differentiate (13) with respect to a and write $-a$ for a. $\square$

Remark 1. Identity (19) is not new and can be found in a different form in Gould's compendium [10] as equation (1.10). It is also recorded by Chu [17, Eq. (4)].

Another important consequence is the next theorem.

Theorem 4. *If n is a non-negative integer and x is a complex number and a is a complex variable, then*

$$\sum_{k=1}^{n}\binom{x}{n-k}\frac{a^k}{k}=\sum_{k=1}^{n}\binom{x-k}{n-k}\frac{(1+a)^k}{k}-\sum_{k=1}^{n}\binom{x-k}{n-k}\frac{1}{k}. \tag{20}$$

In particular, we have

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{x}{n-k}\frac{1}{k}=\sum_{k=1}^{n}\binom{x-k}{n-k}\frac{1}{k}. \tag{21}$$

Proof. Add (13) and (14) and write $-a$ for a. The particular case follows by substituting $a=-1$ in (20). $\square$

Proposition 1. *If n is a non-negative integer and a is a complex variable, then*

$$\sum_{k=1}^{n}\binom{n}{k}\frac{a^k}{k}=\sum_{k=1}^{n}\frac{(1+a)^k}{k}-H_n, \tag{22}$$

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{a^k}{k}=\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{(1+a)^k}{k}-H_n, \tag{23}$$

and

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(1+a)^k}{k+2}=\frac{1}{(1+a)^2}\left((-1)^n a^{n+1}\left(\frac{a}{n+2}+\frac{1}{n+1}\right)+\frac{1}{(n+1)(n+2)}\right). \tag{24}$$

In particular,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{k+2}=\frac{1}{(n+1)(n+2)}, \tag{25}$$

$$\sum_{k=0}^{n}\frac{\binom{n}{k}}{(1+n)^k(k+2)}=\frac{n+1}{n+2}. \tag{26}$$

Proof. Identity (22) is obtained by setting $x=n$ in (20). Identity (23) follows upon setting $x=-1$ in (20) and using Lemma 1, while (24) follows from $x=1$. Note that in deriving (23) we used the equation (15). Identity (26) comes from setting $a=-(n+2)/(n+1)$ in (24). $\square$

Remark 2. Identities (22) and (23) will be called dual identities. We also note that setting $x=0$ in (20) gives the binomial transform of the binomial theorem.

The next theorem generalizes (19).

Theorem 5. *If m is a positive integer, n is a non-negative integer, x is a complex number and a is a complex variable, then*

$$\sum_{k=m}^{n}\binom{x}{n-k}\binom{k}{m}\frac{a^{k-m}}{k}=\sum_{k=m}^{n}\binom{x-k}{n-k}\binom{k}{m}\frac{(1+a)^{k-m}}{k}. \tag{27}$$

Proof. The proof is by induction on m. The base case, $m = 1$, is valid because it is the derivative of (20). Assume the truth of the hypothesis (27). Differentiating (27) with respect to a shows that the identity is valid for $m+1$ whenever it is valid for m and the proof is complete. Note that

$$(k-m)\binom{k}{m} = (m+1)\binom{k}{m+1}.$$

□

By using the identity $\frac{1}{k}\binom{k}{m} = \frac{1}{m}\binom{k-1}{m-1}$ and shifting the summation index, equation (27) can be simplified into

$$\sum_{k=m}^{n}\binom{x+1}{n-k}\binom{k}{m}a^{k-m} = \sum_{k=m}^{n}\binom{x-k}{n-k}\binom{k}{m}(1+a)^{k-m}. \tag{28}$$

Remark 3. Any identity derived from (27) remains valid under the interchange $\binom{x-k}{n-k} \leftrightarrow (-1)^k\binom{x}{n-k}$. Similarly, the interchange $\binom{x-k}{n-k} \leftrightarrow (-1)^k\binom{x+1}{n-k}$ leaves any identity derived from (28) valid.

Substituting $a = -1$ and $a = 0$ into equation (28) yields the following results:

Proposition 2. *If m is a positive integer, n is a non-negative integer and x is a complex number, then*

$$\sum_{k=m}^{n}(-1)^k\binom{x+1}{n-k}\binom{k}{m} = (-1)^m\binom{x-m}{n-m},$$
$$\sum_{k=m}^{n}\binom{x-k}{n-k}\binom{k}{m} = \binom{x+1}{n-m}.$$

Evaluation at $x = -1/2$, on account of Lemma 2, yields the following combinatorial identities.

Proposition 3. *If m is a positive integer and n is a non-negative integer, then*

$$\sum_{k=m}^{n}\binom{2(n-k)}{n-k}\binom{k}{m}\frac{2^{2k}}{1-2(n-k)} = \binom{2n}{n}\binom{n}{m}\binom{2m}{m}^{-1}2^{2m}, \tag{29}$$

and

$$\sum_{k=m}^{n}(-1)^k\binom{n}{k}\binom{k}{m}\binom{2k}{k}^{-1}2^{2k} = (-1)^m\binom{2(n-m)}{n-m}\binom{2n}{n}^{-1}\frac{2^{2m}}{1-2(n-m)}. \tag{30}$$

Remark 4. By shifting the summation index it is not difficult to show that (29) also contains

$$\sum_{k=0}^{n}\binom{2k}{k}\frac{2^{-2k}}{1-2k} = \binom{2n}{n}2^{-2n}, \tag{31}$$

as a special case. The combinatorial sum (31) can be found in Riordan's book [18, p.130]. In addition we have from (30) its counterpart

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}^{-1}2^{2k} = \frac{1}{1-2n}. \tag{32}$$

Proposition 4. *If m and n are non-negative integers such that $n > m+1$, then*

$$\sum_{k=m}^{n}(-1)^k k\binom{n-m}{k-m} = 0.$$

In particular, for all $n > 1$,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0,$$

which is a well-known classical combinatorial identity that appears in numerous references; for instance, see Eq. (2.3.2) in [16].

Proof. Set $x = 0$ in (28), use Lemma 1, shift the summation index and set $a = 0$. □

Using the summation identity

$$\sum_{k=m}^{n} f(k) = \sum_{k=\lfloor (m+1)/2 \rfloor}^{\lfloor n/2 \rfloor} f(2k) + \sum_{k=\lfloor (m+2)/2 \rfloor}^{\lceil n/2 \rceil} f(2k-1),$$

together with (28), we obtain the following result:

Proposition 5. *Let m be a positive integer, n a non-negative integer, x a complex number and a a complex variable. Then*

$$\sum_{k=m}^{n} \binom{x-k}{n-k}\binom{k}{m}\frac{(1+a)^{k-m}+(1-a)^{k-m}}{2} = \begin{cases} \sum\limits_{k=\lfloor (m+1)/2 \rfloor}^{\lfloor n/2 \rfloor} \binom{x+1}{n-2k}\binom{2k}{m} a^{2k-m} & \text{if } m \text{ is even,} \\ \sum\limits_{k=\lfloor (m+2)/2 \rfloor}^{\lceil n/2 \rceil} \binom{x+1}{n-2k+1}\binom{2k-1}{m} a^{2k-m-1} & \text{if } m \text{ is odd,} \end{cases}$$

and

$$\sum_{k=m}^{n} \binom{x-k}{n-k}\binom{k}{m}\frac{(1+a)^{k-m}-(1-a)^{k-m}}{2} = \begin{cases} \sum\limits_{k=\lfloor (m+2)/2 \rfloor}^{\lceil n/2 \rceil} \binom{x+1}{n-2k+1}\binom{2k-1}{m} a^{2k-m-1} & \text{if } m \text{ is even,} \\ \sum\limits_{k=\lfloor (m+1)/2 \rfloor}^{\lfloor n/2 \rfloor} \binom{x+1}{n-2k}\binom{2k}{m} a^{2k-m} & \text{if } m \text{ is odd.} \end{cases}$$

Theorem 6. *If m and n are non-negative integers, x is a complex number and a is a complex variable, then*

$$\begin{aligned} \sum_{k=1}^{n} \binom{x}{n-k}\binom{k+m}{m}^{-1}\frac{a^{k+m}}{k} &= -a^m \sum_{k=1}^{n} \binom{x-k}{n-k}\frac{1}{k} + \sum_{k=1}^{n} \binom{x-k}{n-k}\binom{k+m}{m}^{-1}\frac{(1+a)^{k+m}}{k} \\ &\quad - \sum_{j=0}^{m-1} a^j \sum_{k=1}^{n} \binom{x-k}{n-k}\binom{k+m}{j}\binom{k+m}{m}^{-1}\frac{1}{k}. \end{aligned} \tag{33}$$

Proof. The proof is by induction on m. The base case, $m = 0$, is identity (20). Now assume the veracity of (33), the induction hypothesis, for a non-negative integer m. Replacing a with t and integrating with respect to t from 0 to a shows that (33) holds for $m + 1$ whenever it holds for m. Note that

$$(k+m+1)\binom{k+m}{m} = (m+1)\binom{k+m+1}{m+1}.$$

□

Remark 5. Identity (33) is also valid under the interchange stated in Remark 3. Thus, for example, we have

$$\begin{aligned} \sum_{k=1}^{n} (-1)^k \binom{x-k}{n-k}\binom{k+m}{m}^{-1}\frac{a^{k+m}}{k} &= -a^m \sum_{k=1}^{n} (-1)^k \binom{x}{n-k}\frac{1}{k} + \sum_{k=1}^{n} (-1)^k \binom{x}{n-k}\binom{k+m}{m}^{-1}\frac{(1+a)^{k+m}}{k} \\ &\quad - \sum_{j=0}^{m-1} a^j \sum_{k=1}^{n} (-1)^k \binom{x}{n-k}\binom{k+m}{j}\binom{k+m}{m}^{-1}\frac{1}{k}. \end{aligned} \tag{34}$$

Proposition 6. *If m and n are non-negative integers and a is a complex variable, then*

$$\sum_{k=1}^{n}\binom{n}{k}\binom{k+m}{m}^{-1}a^{k+m}=\binom{m+n}{m}^{-1}\left((1+a)^{m+n}-\sum_{k=0}^{m}\binom{m+n}{k}a^{k}\right). \tag{35}$$

Proof. Set $x=0$ in (34) and use Lemma 1. □

Remark 6. Identity (35) is equivalent to Gould [10, Identity (4.13), p.47].

Substituting $a=-1$ into (33) yields the following result.

Proposition 7. *If m and n are non-negative integers and x is a complex number, then*

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{x}{n-k}\binom{k+m}{m}^{-1}\frac{1}{k}=\sum_{k=1}^{n}\binom{x-k}{n-k}\frac{1}{k}\sum_{j=0}^{m-1}(-1)^{m-1-j}\sum_{k=1}^{n}\binom{x-k}{n-k}\binom{k+m}{j}\binom{k+m}{m}^{-1}\frac{1}{k}. \tag{36}$$

Proposition 8. *If m and n are non-negative integers, then*

$$\binom{2n}{n}^{-1}\sum_{k=1}^{n}\binom{2(n-k)}{n-k}\binom{k+m}{m}^{-1}\frac{2^{2k}}{k}=\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\binom{2k}{k}^{-1}\frac{2^{2k}}{k}$$
$$-(-1)^{m}\sum_{j=0}^{m-1}(-1)^{j}\sum_{k=1}^{n}(-1)^{k}\binom{n}{k}\binom{2k}{k}^{-1}\binom{k+m}{j}\binom{k+m}{m}^{-1}\frac{2^{2k}}{k}.$$

Proof. Set $x=-1/2$ in (36) and use Lemma 2. □

Substituting $x=0$, $x=-1$, and $x=1$ into (33), and applying Lemma 1, we obtain the following results.

Proposition 9. *If m and n are non-negative integers and a is a complex variable, then*

$$\sum_{k=1}^{n}(-1)^{k}\binom{n}{k}\binom{k+m}{m}^{-1}(1+a)^{k+m}-\sum_{j=0}^{m-1}a^{j}\sum_{k=1}^{n}(-1)^{k}\binom{n}{k}\binom{k+m}{j}\binom{k+m}{m}^{-1}$$
$$=a^{m}\left((-1)^{n}\binom{n+m}{m}^{-1}a^{n}-1\right),$$

$$\sum_{k=1}^{n}(-1)^{k}\binom{n}{k}\binom{k+m}{m}^{-1}\frac{(1+a)^{k+m}}{k}-\sum_{j=0}^{m-1}a^{j}\sum_{k=1}^{n}(-1)^{k}\binom{n}{k}\binom{k+m}{j}\binom{k+m}{m}^{-1}\frac{1}{k}$$
$$=a^{m}\left(\sum_{k=1}^{n}(-1)^{k}\binom{k+m}{m}^{-1}\frac{a^{k}}{k}-H_{n}\right)$$
$$=a^{m}\sum_{k=1}^{n}\left((-1)^{k}\binom{k+m}{m}^{-1}a^{k}-1\right)\frac{1}{k},$$

$$\sum_{k=1}^{n}(-1)^{k}\binom{n-2}{k-2}\binom{k+m}{m}^{-1}\frac{(1+a)^{k+m}}{k}-\sum_{j=0}^{m-1}a^{j}\sum_{k=1}^{n}(-1)^{k}\binom{n-2}{k-2}\binom{k+m}{j}\binom{k+m}{m}^{-1}\frac{1}{k}$$
$$=a^{m}\left(\left((-1)^{n}\binom{n+m}{m}^{-1}a^{n}-1\right)\frac{1}{n}-\left((-1)^{n-1}\binom{n+m-1}{m}^{-1}a^{n-1}-1\right)\frac{1}{n-1}\right).$$

Proposition 10. *If m and n are non-negative integers, x is a complex number and a and b are complex variables, then*

$$\begin{aligned}\sum_{k=1}^{n}\binom{x}{n-k}\binom{k+m}{m}^{-1}\frac{a^k-b^k}{k}&=\sum_{k=1}^{n}\binom{x-k}{n-k}\binom{k+m}{m}^{-1}\frac{1}{k}\left(\frac{(1+a)^{k+m}}{a^m}-\frac{(1+b)^{k+m}}{b^m}\right)\\&\quad-\sum_{j=0}^{m-1}\left(a^{j-m}-b^{j-m}\right)\sum_{k=1}^{n}\binom{x-k}{n-k}\binom{k+m}{j}\binom{k+m}{m}^{-1}\frac{1}{k}.\end{aligned}\tag{37}$$

Proof. Follows from (33). □

The next three sections are dedicated to illustrating important basic applications of the combinatorial identities derived in this section. We will explore three primary fields: identities involving trigonometric functions, identities related to Horadam sequence and finally, we will consider three certain types of combinatorial identities.

4. Some trigonometric identities

In this section we derive some possibly new trigonometric identities.

Proposition 11. *If n is a non-negative integer, x is a complex number and θ is a real number, then*

$$\sum_{k=1}^{n}\binom{x}{n-k}\frac{\cos(k\theta)}{k}=\sum_{k=1}^{n}\binom{x-k}{n-k}\frac{\cos(k\theta/2)}{k}\left(2\cos\left(\frac{\theta}{2}\right)\right)^k-\sum_{k=1}^{n}\binom{x-k}{n-k}\frac{1}{k},\tag{38}$$

and

$$\sum_{k=1}^{n}\binom{x}{n-k}\frac{\sin(k\theta)}{k}=\sum_{k=1}^{n}\binom{x-k}{n-k}\frac{\sin(k\theta/2)}{k}\left(2\cos\left(\frac{\theta}{2}\right)\right)^k.\tag{39}$$

Proof. Set $a=e^{i\theta}, i=\sqrt{-1}, \theta\in\mathbb{R}$ in the main identity (20) and use Euler's formula

$$e^{i\theta}=\cos(\theta)+i\sin(\theta),$$

together with

$$1+a=e^{i\theta/2}2\cos(\theta/2).$$

Compare the real and imaginary parts. This completes the proof. □

Corollary 1. *If n is a non-negative integer and θ is a real number, then*

$$\sum_{k=1}^{n}\binom{n}{k}\frac{\cos(k\theta)}{k}=\sum_{k=1}^{n}\frac{\cos(k\theta/2)}{k}\left(2\cos\left(\frac{\theta}{2}\right)\right)^k-H_n,$$

and

$$\sum_{k=1}^{n}\binom{n}{k}\frac{\sin(k\theta)}{k}=\sum_{k=1}^{n}\frac{\sin(k\theta/2)}{k}\left(2\cos\left(\frac{\theta}{2}\right)\right)^k.$$

Proof. Set $x=n$ in Proposition 11. □

Corollary 2. *If n is a non-negative integer and θ is a real number, then*

$$\sum_{k=1}^{n}(-1)^k\frac{\cos(k\theta)}{k}=\sum_{k=1}^{n}(-1)^k\binom{n}{k}\frac{\cos(k\theta/2)}{k}\left(2\cos\left(\frac{\theta}{2}\right)\right)^k+H_n,$$

and

$$\sum_{k=1}^{n}(-1)^k\frac{\sin(k\theta)}{k}=\sum_{k=1}^{n}(-1)^k\binom{n}{k}\frac{\sin(k\theta/2)}{k}\left(2\cos\left(\frac{\theta}{2}\right)\right)^k.$$

Proof. Set $x=-1$ in Proposition 11, use Lemma 1, and simplify. □

Corollary 3. *If n is a non-negative integer and θ is a real number, then*

$$\sum_{k=1}^{n}(-1)^k\binom{2(n-k)}{n-k}2^{2k}\frac{\cos(k\theta)}{k}=\binom{2n}{n}\sum_{k=1}^{n}(-1)^k2^{2k}\binom{n}{k}\binom{2k}{k}^{-1}\frac{1}{k}\left(\cos\left(\frac{k\theta}{2}\right)\left(2\cos\left(\frac{\theta}{2}\right)\right)^k-1\right),$$

and

$$\sum_{k=1}^{n}(-1)^k\binom{2(n-k)}{n-k}2^{2k}\frac{\sin(k\theta)}{k}=\binom{2n}{n}\sum_{k=1}^{n}(-1)^k2^{3k}\binom{n}{k}\binom{2k}{k}^{-1}\frac{1}{k}\sin\left(\frac{k\theta}{2}\right)\left(\cos\left(\frac{\theta}{2}\right)\right)^k.$$

Proof. Set $x=-1/2$ in Proposition 11, use Lemma 2, and simplify. □

Corollary 4. *If n is a non-negative integer and θ is a real number, then*

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{2(n-k)}{n-k}\frac{2^{2k}}{2(n-k)-1}\frac{\cos(k\theta)}{k}$$
$$=\binom{2(n-1)}{n-1}\sum_{k=1}^{n}(-1)^k2^{2k}\binom{n-1}{k-1}\binom{2(k-1)}{k-1}^{-1}\frac{1}{k}\left(\cos\left(\frac{k\theta}{2}\right)\left(2\cos\left(\frac{\theta}{2}\right)\right)^k-1\right),$$

and

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{2(n-k)}{n-k}\frac{2^{2k}}{2(n-k)-1}\frac{\sin(k\theta)}{k}$$
$$=\binom{2(n-1)}{n-1}\sum_{k=1}^{n}(-1)^k2^{3k}\binom{n-1}{k-1}\binom{2(k-1)}{k-1}^{-1}\frac{1}{k}\sin\left(\frac{k\theta}{2}\right)\left(\cos\left(\frac{\theta}{2}\right)\right)^k.$$

Proof. Set $x=1/2$ in Proposition 11, use Lemma 2, and simplify. □

Proposition 12. *Let n be a non-negative integer, x a complex number and θ a real number. If $\theta+\pi/2\notin\pi\mathbb{Z}$, then*

$$\sum_{k=1}^{n}\binom{x}{n-k}\frac{\tan^{2k}(\theta)}{k}=\sum_{k=1}^{n}\binom{x-k}{n-k}\frac{\cos^{-2k}(\theta)}{k}-\sum_{k=1}^{n}\binom{x-k}{n-k}\frac{1}{k}. \tag{40}$$

If $\theta\notin\pi\mathbb{Z}$, then

$$\sum_{k=1}^{n}\binom{x}{n-k}\frac{\tan^{-2k}(\theta)}{k}=\sum_{k=1}^{n}\binom{x-k}{n-k}\frac{\sin^{-2k}(\theta)}{k}-\sum_{k=1}^{n}\binom{x-k}{n-k}\frac{1}{k}. \tag{41}$$

Proof. Set $a=\tan^2(\theta)$ and $a=\tan^{-2}(\theta)$, in turn, in the main identity (20) and simplify. □

Proposition 13. *If n is a non-negative integer, x is a complex number and θ is a real number, then*

$$\sum_{k=1}^{n}\binom{x}{n-k}\frac{\cos^{k}(\theta/2)}{k}=\sum_{k=1}^{n}\binom{x-k}{n-k}2^k\frac{\cos^{2k}(\theta/4)}{k}-\sum_{k=1}^{n}\binom{x-k}{n-k}\frac{1}{k}, \tag{42}$$

and

$$\sum_{k=1}^{n}\binom{x}{n-k}\frac{\sin^{k}(\theta/2)}{k}=\sum_{k=1}^{n}\binom{x-k}{n-k}2^k\frac{\sin^{2k}((\theta+\pi)/4)}{k}-\sum_{k=1}^{n}\binom{x-k}{n-k}\frac{1}{k}. \tag{43}$$

Proof. Set $a=\cos(\theta/2)$ and $a=\sin(\theta/2)$, in turn, in the main identity (20) and simplify. □

To avoid repetitions we omit the special cases.

5. Identities involving Horadam sequences

In this section we state new identities involving Horadam sequences $w_n = w_n(w_0, w_1; p, q)$ introduced in Section 2.

Proposition 14. *If n is a non-negative integer, x is a complex number, t is an integer, and r, s are positive integers, then*

$$\sum_{k=1}^{n} \binom{x}{n-k} \frac{(-1)^k}{k} \left(\frac{u_{rs}}{q^s u_{(r-1)s}} \right)^k w_{sk+t} = \sum_{k=1}^{n} \binom{x-k}{n-k} \frac{(-1)^k}{k} \left(\frac{u_s}{q^s u_{(r-1)s}} \right)^k w_{rsk+t} - w_t \sum_{k=1}^{n} \binom{x-k}{n-k} \frac{1}{k}.$$

In particular,

$$\sum_{k=1}^{n} \binom{x}{n-k} \frac{(-1)^k}{k} \left(\frac{u_r}{q u_{r-1}} \right)^k w_{k+t} = \sum_{k=1}^{n} \binom{x-k}{n-k} \frac{(-1)^k}{k} \left(\frac{1}{q u_{r-1}} \right)^k w_{rk+t} - w_t \sum_{k=1}^{n} \binom{x-k}{n-k} \frac{1}{k}.$$

Proof. Set $a = -u_{rs}\sigma^s/(q^s u_{(r-1)s})$ in (20). Then Lemma 3 yields

$$\sum_{k=1}^{n} \binom{x}{n-k} \frac{(-1)^k}{k} \left(\frac{u_{rs}}{q^s u_{(r-1)s}} \right)^k \sigma^{sk} = \sum_{k=1}^{n} \binom{x-k}{n-k} \frac{(-1)^k}{k} \left(\frac{u_s}{q^s u_{(r-1)s}} \right)^k \sigma^{rsk} - \sum_{k=1}^{n} \binom{x-k}{n-k} \frac{1}{k}. \tag{44}$$

Similarly, with $a = -u_{rs}\tau^s/(q^s u_{(r-1)s})$ in (20) in conjunction with Lemma 3 we obtain

$$\sum_{k=1}^{n} \binom{x}{n-k} \frac{(-1)^k}{k} \left(\frac{u_{rs}}{q^s u_{(r-1)s}} \right)^k \tau^{sk} = \sum_{k=1}^{n} \binom{x-k}{n-k} \frac{(-1)^k}{k} \left(\frac{u_s}{q^s u_{(r-1)s}} \right)^k \tau^{rsk} - \sum_{k=1}^{n} \binom{x-k}{n-k} \frac{1}{k}. \tag{45}$$

The identity follows upon multiplying (44) and (45) by σ^t, respectively τ^t, and combining according to the Binet form (12). The special case is obtained by setting $s = 1$. □

Proposition 15. *If m and n are non-negative integers, r is an integer and x is a complex number, then*

$$\begin{aligned} \sum_{k=1}^{n} (-1)^{k-1} \binom{x}{n-k} \binom{k+m}{m}^{-1} \frac{u_{rk}}{v_r^k k} &= \sum_{j=0}^{m-1} (-1)^{j-m} \frac{u_{r(j-m)}}{v_r^{j-m}} \sum_{k=1}^{n} \binom{x-k}{n-k} \binom{k+m}{j} \binom{k+m}{m}^{-1} \frac{1}{k} \\ &\quad + \frac{(-1)^m}{q^{rm}} \sum_{k=1}^{n} \binom{x-k}{n-k} \binom{k+m}{m}^{-1} \frac{u_{2rm+rk}}{v_r^k k}. \end{aligned} \tag{46}$$

In particular,

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{x}{n-k} \frac{u_{rk}}{v_r^k k} = \sum_{k=1}^{n} \binom{x-k}{n-k} \frac{u_{rk}}{v_r^k k};$$

with the special value

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{u_{rk}}{v_r^k} = \frac{u_{rn}}{v_r^n},$$

which has the interesting property that it is its own binomial transform.

Proof. Set $a = -\sigma^r/v_r$ and $b = -\tau^r/v_r$ in (37). □

Remark 7. In view of Remark 3, identity (46) also implies

$$\begin{aligned} \sum_{k=1}^{n} \binom{x-k}{n-k} \binom{k+m}{m}^{-1} \frac{u_{rk}}{v_r^k k} &= \sum_{j=0}^{m-1} (-1)^{j-m} \frac{u_{r(j-m)}}{v_r^{j-m}} \sum_{k=1}^{n} (-1)^{k-1} \binom{x}{n-k} \binom{k+m}{j} \binom{k+m}{m}^{-1} \frac{1}{k} \\ &\quad + \frac{(-1)^m}{q^{rm}} \sum_{k=1}^{n} (-1)^{k-1} \binom{x}{n-k} \binom{k+m}{m}^{-1} \frac{u_{2rm+rk}}{v_r^k k}. \end{aligned}$$

In particular, at $x = 0$ we obtain

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\binom{k+m}{m}^{-1}\frac{u_{rk}}{v_r^k} = (-1)^m\binom{n+m}{m}^{-1}\left(\sum_{j=0}^{m-1}(-1)^j\frac{u_{r(j-m)}}{v_r^{j-m}}\binom{n+m}{j} + \frac{u_{2rm+rn}}{q^{rm}v_r^n}\right).$$

Remark 8. These results should be regarded as basic. To keep the paper readable we do not state the spacial cases. We can obtain more general results by utilizing Lemma 4.

6. Combinatorial identities

Lemma 6. *If r, k and s are complex numbers and x is a complex variable, then*

$$\int_0^1 y^{r+k-s}(1-y)^{s-1}\,dy = \frac{1}{s}\binom{k+r}{s}^{-1}, \quad \Re(r+k-s+1) > 0 \text{ and } 0 \neq s \notin \mathbb{Z}^-; \tag{47}$$

$$\int_0^1 y^{r-s}(1-y)^{k+s-1}\,dy = \frac{1}{k+s}\binom{k+r}{k+s}^{-1}, \quad \Re(r-s+1) > 0 \text{ and } \Re(k+s) > 0; \tag{48}$$

$$\int_0^1 y^{k+s}(1-y)^{r-k-s}\,dy = \frac{1}{r+1}\binom{r}{k+s}^{-1}, \quad \Re(k+s+1) > 0 \text{ and } \Re(r-k-s+1) > 0, \tag{49}$$

and

$$\int_0^1 y^{n-k+s}(1-y)^{r-n-s} = \frac{1}{r-k+1}\binom{r-k}{r-s-n}, \quad \Re(n-k+s+1) > 0 \text{ and } \Re(r-n-s+1) > 0. \tag{50}$$

Proof. The integrals in (47)– (50) are immediate consequences of the Beta function, $B(r,s)$, defined, as usual, for complex numbers r and s such that $\Re(r) > 0$ and $\Re(s) > 0$, by

$$B(r,s) = B(s,r) = \int_0^1 y^{r-1}(1-y)^{s-1}.$$

With the help of the Gamma function, the integral is evaluated as

$$B(r,s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)} = \frac{1}{s}\binom{r+s-1}{s}^{-1} = \frac{1}{r}\binom{r+s-1}{r}^{-1}.$$

Note that in obtaining (49) and (50), we also used

$$\binom{u+1}{v+1} = \frac{u+1}{v+1}\binom{u}{v},$$

an identity which we will often use without comment in this paper. □

6.1. Frisch-type identities

The following combinatorial identity is attributed to Frisch [19–22]:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{b+k}{c}^{-1} = \frac{c}{n+c}\binom{n+b}{b-c}^{-1}, \quad b, c, b-c \in \mathbb{C}\setminus\mathbb{Z}^-. \tag{51}$$

Here, we derive generalizations and variants of this identity.

Theorem 7. *If m is a positive integer, n is a non-negative integer, r and s are complex numbers such that $\Re(r-s+1) > 0$ and s is not a non-positive integer and x is a complex number, then*

$$\sum_{k=m}^{n}\frac{(-1)^{k-m}}{k}\binom{x}{n-k}\binom{k}{m}\binom{k+r}{s}^{-1} = \sum_{k=m}^{n}\frac{s}{k(k-m+s)}\binom{x-k}{n-k}\binom{k}{m}\binom{k+r}{k-m+s}^{-1}, \tag{52}$$

and

$$\sum_{k=m}^{n}\frac{1}{k}\binom{x-k}{n-k}\binom{k}{m}\binom{k+r}{s}^{-1}=\sum_{k=m}^{n}\frac{(-1)^{k-m}s}{k\,(k-m+s)}\binom{x}{n-k}\binom{k}{m}\binom{k+r}{k-m+s}^{-1}. \tag{53}$$

Proof. Write $-a$ for a in (27) and multiply through by $a^{r-s+m}(1-a)^{s-1}$ to obtain

$$\sum_{k=m}^{n}(-1)^{k-m}\frac{1}{k}\binom{x}{n-k}\binom{k}{m}a^{k+r-s}\,(1-a)^{s-1}=\sum_{k=m}^{n}\frac{1}{k}\binom{x-k}{n-k}\binom{k}{m}a^{r-s+m}\,(1-a)^{k-m+s-1},$$

and hence (52) after term-wise integration from 0 to 1 with respect to a by (47) and (48). Identity (53) follows from (52) by the

$$(-1)^{k-m}\binom{x}{n-k}\leftrightarrow\binom{x-k}{n-k},$$

symmetry of (27). □

Corollary 5. *If m and n are non-negative integers and r and s are complex numbers such that $\Re(r-s+1)>0$ and s is not a non-positive integer, then*

$$\sum_{k=m}^{n}\frac{(-1)^k}{k-m+s}\binom{n-m}{k-m}\binom{k+r}{k-m+s}^{-1}=\frac{(-1)^m}{s}\binom{n+r}{s}^{-1}, \tag{54}$$

and

$$\sum_{k=m}^{n}(-1)^k\binom{n-m}{k-m}\binom{k+r}{s}^{-1}=\frac{(-1)^m s}{n-m+s}\binom{n+r}{n-m+s}^{-1}. \tag{55}$$

Proof. Set $x=0$ in Theorem 7 and use (4). Note that $x=0$ in Theorem 7 removes the singularity at $m=0$ on account of (4). □

Remark 9. Identity (55) generalizes Frisch's identity (51) to which it reduces at $m=0$. In addition, new combinatorial identities can be derived by setting $s=\pm 1/2$ in Corollary 5. We leave this little exercise to the interested reader.

Corollary 6. *If m is a positive integer, n is a non-negative integer, r and s are complex numbers such that $\Re(r-s+1)>0$ and s is not a non-positive integer, then*

$$\begin{aligned}&\sum_{k=m}^{n}(-1)^k\binom{n-2}{k-2}\binom{k}{m}\binom{k+r}{k-m+s}^{-1}\frac{s}{k(k-m+s)}\\&=(-1)^m\left(-\binom{n-1}{m}\binom{n+r-1}{s}^{-1}\frac{1}{n-1}+\binom{n}{m}\binom{n+r}{s}^{-1}\frac{1}{n}\right),\end{aligned} \tag{56}$$

$$\begin{aligned}&\sum_{k=m}^{n}(-1)^k\binom{n-2}{k-2}\binom{k}{m}\binom{k+r}{s}^{-1}\frac{1}{k}\\&=(-1)^m\Big(-\binom{n-1}{m}\binom{n+r-1}{n-m+s-1}^{-1}\frac{1}{(n-1)(n-m+s-1)}\\&\quad+\binom{n}{m}\binom{n+r}{n-m+s}^{-1}\frac{1}{n(n-m+s)}\Big)s.\end{aligned} \tag{57}$$

Proof. Set $x=1$ in Theorem 7. □

Corollary 7. *If m is a positive integer, n is a non-negative integer, r and s are complex numbers such that $\Re(r-s+1)>0$ and s is not a non-positive integer, then*

$$\sum_{k=m}^{n}(-1)^k\binom{n}{k}\binom{k}{m}\binom{k+r}{k-m+s}^{-1}\frac{s}{k(k-m+s)}=(-1)^m\sum_{k=m}^{n}\binom{k}{m}\binom{k+r}{s}^{-1}\frac{1}{k}, \tag{58}$$

$$\sum_{k=m}^{n}(-1)^k\binom{n}{k}\binom{k}{m}\binom{k+r}{s}^{-1}\frac{1}{k}=(-1)^m\sum_{k=m}^{n}\binom{k}{m}\binom{k+r}{k-m+s}^{-1}\frac{s}{k(k-m+s)}. \tag{59}$$

Proof. Set $x=-1$ in Theorem 7. □

Proposition 16. *If m and n are non-negative integers and s is a complex number such that $\Re(\frac{1}{2}-s)>0$ and s is not a non-positive integer, then*

$$\sum_{k=m}^{n}(-1)^k\binom{n-m}{k-m}\binom{2k}{k}^{-1}\binom{k}{m-s}^{-1}\frac{2^{2(k-m)}}{k-m+s}=(-1)^m\binom{2(n-s)}{n-s}\binom{2(m-s)}{m-s}^{-1}\binom{2n}{n}^{-1}\binom{n}{s}^{-1}\frac{1}{s}, \tag{60}$$

$$\sum_{k=m}^{n}(-1)^k\binom{n-m}{k-m}\binom{2(k-s)}{k-s}\binom{2k}{k}^{-1}\binom{k}{s}^{-1}=(-1)^m\binom{2(m-s)}{m-s}\binom{2n}{n}^{-1}\binom{n}{m-s}^{-1}\frac{2^{2(n-m)}s}{n-m+s}. \tag{61}$$

Proof. Set $r=-1/2$ in Corollary 5. □

Proposition 17. *If m and n are non-negative integers and s is a complex number such that $\Re(\frac{3}{2}-s)>0$ and s is not a non-positive integer, then*

$$\begin{aligned}&\sum_{k=m}^{n}(-1)^k\binom{n-m}{k-m}\binom{k}{k-m+s}\binom{2k+1}{2(k-m+s)}^{-1}\binom{2(k-m+s)}{k-m+s}^{-1}\frac{2^{2(k-m)}}{k-m+s}\\&=(-1)^m\binom{n}{s}\binom{2n+1}{2s}^{-1}\binom{2s}{s}^{-1}\frac{1}{s},\end{aligned} \tag{62}$$

$$\begin{aligned}&\sum_{k=m}^{n}(-1)^k\binom{n-m}{k-m}\binom{k}{s}\binom{2k+1}{2s}^{-1}\\&=(-1)^m\binom{n}{n-m+s}\binom{2s}{s}\binom{2n+1}{2(n-m+s)}^{-1}\binom{2(n-m+s)}{n-m+s}^{-1}\frac{2^{2(n-m)}s}{n-m+s}.\end{aligned} \tag{63}$$

Proof. Set $r=1/2$ in Corollary 5. □

Proposition 18. *If m is a positive integer, n is a non-negative integer, s is a complex number such that $\Re(\frac{1}{2}-s)>0$ and s is not a non-positive integer, then*

$$\sum_{k=m}^{n}(-1)^k\frac{\binom{n-2}{k-2}\binom{k}{m}2^{2k}}{\binom{2k}{k}\binom{k}{m-s}k(k-m+s)}=\frac{(-1)^m2^{2m}}{\binom{2(m-s)}{m-s}s}\left(\frac{\binom{n}{m}\binom{2(n-s)}{n-s}}{\binom{2n}{n}\binom{n}{s}n}-\frac{\binom{n-1}{m}\binom{2(n-s-1)}{n-s-1}}{\binom{2(n-1)}{n-1}\binom{n-1}{s}(n-1)}\right), \tag{64}$$

$$\begin{aligned}&\sum_{k=m}^{n}(-1)^k\frac{\binom{n-2}{k-2}\binom{k}{m}\binom{2(k-s)}{k-s}}{\binom{2k}{k}\binom{k}{s}k}\\&=\frac{(-1)^m\binom{2(m-s)}{m-s}s}{2^{2m}}\left(\frac{\binom{n}{m}2^{2n}}{\binom{2n}{n}\binom{n}{m-s}n(n-m+s)}-\frac{\binom{n-1}{m}2^{2(n-1)}}{\binom{2(n-1)}{n-1}\binom{n-1}{m-s}(n-1)(n-m+s-1)}\right).\end{aligned} \tag{65}$$

Proof. Set $r=-1/2$ in Corollary 6. □

Proposition 19. *If m is a positive integer, n is a non-negative integer, s is a complex number such that $\Re(\frac{3}{2}-s)>0$ and s is not a non-positive integer, then*

$$\sum_{k=m}^{n}(-1)^k\frac{\binom{n-2}{k-2}\binom{k}{m}\binom{k}{m-s}2^{2k}}{\binom{2k+1}{2(k-m+s)}\binom{2(k-m+s)}{k-m+s}k(k-m+s)}=\frac{(-1)^m2^{2m}}{s}\left(\frac{\binom{n}{m}\binom{n}{s}}{\binom{2n+1}{2s}\binom{2s}{s}n}-\frac{\binom{n-1}{m}\binom{2(n-s)}{n-s}}{\binom{2n}{n}\binom{n}{s}(n-1)}\right), \tag{66}$$

$$\begin{aligned}\sum_{k=m}^{n}(-1)^k\frac{\binom{n-2}{k-2}\binom{k}{m}\binom{k}{s}}{\binom{2k+1}{2s}k}=\frac{(-1)^m\binom{2s}{s}s}{2^{2m}}\Bigg(&\frac{\binom{n}{m}\binom{n}{m-s}2^{2n}}{\binom{2n+1}{2(n-m+s)}\binom{2(n-m+s)}{n-m+s}n(n-m+s)}\\&-\frac{\binom{n-1}{m}\binom{2(m-s+1)}{m-s+1}2^{2(n-1)}}{\binom{2n}{n}\binom{n}{m-s+1}(n-1)(n-m+s-1)}\Bigg).\end{aligned} \tag{67}$$

Proof. Set $r=1/2$ in Corollary 6. □

Proposition 20. *If m is a positive integer, n is a non-negative integer, s is a complex number such that $\Re(\frac{1}{2}-s)>0$ and s is not a non-positive integer, then*

$$\sum_{k=m}^{n}(-1)^k\frac{\binom{n}{k}\binom{k}{m}2^{2k}}{\binom{2k}{k}\binom{k}{m-s}k(k-m+s)}=\frac{(-1)^m2^{2m}}{\binom{2(m-s)}{m-s}s}\sum_{k=m}^{n}\frac{\binom{k}{m}\binom{2(k-s)}{k-s}}{\binom{2k}{k}\binom{k}{s}k}, \tag{68}$$

$$\sum_{k=m}^{n}(-1)^k\frac{\binom{n}{k}\binom{k}{m}\binom{2(k-s)}{k-s}}{\binom{2k}{k}\binom{k}{s}k}=\frac{(-1)^m\binom{2(m-s)}{m-s}s}{2^{2m}}\sum_{k=m}^{n}\frac{\binom{k}{m}2^{2k}}{\binom{2k}{k}\binom{k}{m-s}k(k-m+s)}. \tag{69}$$

Proof. Set $r=-1/2$ in Corollary 7. □

Proposition 21. *If m is a positive integer, n is a non-negative integer, s is a complex number such that $\Re(\frac{3}{2}-s)>0$ and s is not a non-positive integer, then*

$$\sum_{k=m}^{n}(-1)^k\frac{\binom{n}{k}\binom{k}{m}\binom{k}{m-s}2^{2k}}{\binom{2k+1}{2(k-m+s)}\binom{2(k-m+s)}{k-m+s}k(k-m+s)}=\frac{(-1)^m2^{2m}}{\binom{2s}{s}s}\sum_{k=m}^{n}\frac{\binom{k}{m}\binom{k}{s}}{\binom{2k+1}{2s}k}, \tag{70}$$

$$\sum_{k=m}^{n}(-1)^k\frac{\binom{n}{k}\binom{k}{m}\binom{k}{s}}{\binom{2k+1}{2s}k}=\frac{(-1)^m\binom{2s}{s}s}{2^{2m}}\sum_{k=m}^{n}\frac{\binom{k}{m}\binom{k}{m-s}2^{2k}}{\binom{2k+1}{2(k-m+s)}\binom{2(k-m+s)}{k-m+s}k(k-m+s)}. \tag{71}$$

Proof. Set $r=1/2$ in Corollary 7. □

Remark 10. Again, four additional interesting special cases will come from setting $s=\pm1/2$ in Corollaries 6 and 7.

6.2. Klamkin-type identities

The identity

$$\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k+b}^{-1}=\frac{x+1}{x-n+1}\binom{x-n}{b}^{-1}, \tag{72}$$

is attributed to Klamkin [19,21,22]. Here, we derive generalizations and variants of this identity.

Theorem 8. *Let m be a positive integer, n a non-negative integer, r and s complex numbers such that $\Re(r-n-s+1)>0$ and s is not a negative integer. Let x be a complex number. Then*

$$\sum_{k=m}^{n}\frac{1}{k}\binom{x}{n-k}\binom{k}{m}\binom{r}{k+s}^{-1}=\sum_{k=m}^{n}\frac{r+1}{k\,(m+r-k+1)}\binom{x-k}{n-k}\binom{k}{m}\binom{m+r-k}{m+s}^{-1} \tag{73}$$

and

$$\sum_{k=m}^{n}\frac{(-1)^k}{k}\binom{x-k}{n-k}\binom{k}{m}\binom{r}{k+s}^{-1}=\sum_{k=m}^{n}\frac{(r+1)\,(-1)^k}{k\,(m+r-k+1)}\binom{x}{n-k}\binom{k}{m}\binom{m+r-k}{m+s}^{-1}. \tag{74}$$

Proof. Multiply through (27) by $a^{s+m}(1-a)^{r-m-s}$ and integrate term-wise from 0 to 1 using (49) and (50). □

Corollary 8. *Let m be a non-negative integer, n a non-negative integer, r and s complex numbers such that $\Re(r-n-s+1)>0$ and s is not a negative integer. Then*

$$\sum_{k=m}^{n}\frac{(-1)^k}{m+r-k+1}\binom{n-m}{k-m}\binom{m+r-k}{m+s}^{-1}=\frac{(-1)^n}{r+1}\binom{r}{n+s}^{-1}, \tag{75}$$

and

$$\sum_{k=m}^{n}\binom{n-m}{k-m}\binom{r}{k+s}^{-1}=\frac{r+1}{m+r-n+1}\binom{m+r-n}{m+s}^{-1}. \tag{76}$$

Proof. Set $x=0$ in Theorem 8. Again, note that the singularity at $m=0$ was removed by virtue of (4). □

Remark 11. Identity (76) reduces to Klamkin's identity (72) at $m=0$. The choices $s=\pm 1/2$ in Corollary 8 will yield two additional sums.

Corollary 9. *Let m be a positive integer, n a non-negative integer, r and s complex numbers such that $\Re(r-n-s+1)>0$ and s is not a negative integer. Then*

$$\begin{aligned}&\sum_{k=m}^{n}(-1)^k\binom{n-2}{k-2}\binom{k}{m}\binom{m+r-k}{m+s}^{-1}\frac{r+1}{k(m+r-k+1)}\\&=(-1)^n\left(\binom{n-1}{m}\binom{r}{n+s-1}^{-1}\frac{1}{n-1}+\binom{n}{m}\binom{r}{n+s}^{-1}\frac{1}{n}\right),\end{aligned} \tag{77}$$

$$\begin{aligned}&\sum_{k=m}^{n}\binom{n-2}{k-2}\binom{k}{m}\binom{r}{k+s}^{-1}\frac{1}{k}\\&=(r+1)\left(-\binom{n-1}{m}\binom{m+r-n+1}{m+s}^{-1}\frac{1}{(n-1)(m+r-n+2)}\right.\\&\quad\left.+\binom{n}{m}\binom{m+r-n}{m+s}^{-1}\frac{1}{n(m+r-n+1)}\right).\end{aligned} \tag{78}$$

Proof. Set $x=1$ in Theorem 8. □

Corollary 10. *Let m be a positive integer, n a non-negative integer, r and s complex numbers such that $\Re(r-n-s+1)>0$ and s is not a negative integer. Then*

$$\sum_{k=m}^{n}(-1)^k\binom{k}{m}\binom{r}{k+s}^{-1}\frac{1}{k}=\sum_{k=m}^{n}(-1)^k\binom{n}{k}\binom{k}{m}\binom{m+r-k}{m+s}^{-1}\frac{r+1}{k(m+r-k+1)}, \tag{79}$$

$$\sum_{k=m}^{n}\binom{n}{k}\binom{k}{m}\binom{r}{k+s}^{-1}\frac{1}{k}=(r+1)\sum_{k=m}^{n}\binom{k}{m}\binom{m+r-k}{m+s}^{-1}\frac{1}{k(m+r-k+1)}. \tag{80}$$

Proof. Set $x=-1$ in Theorem 8. □

Proposition 22. *Let m be a non-negative integer, n a non-negative integer, s a complex number such that $\Re(\frac{1}{2}-n-s)>0$ and s is not a negative integer. Then*

$$\sum_{k=m}^{n}(-1)^k\binom{n-m}{k-m}\binom{2(k-m)}{k-m}\binom{2(k+s)}{k+s}^{-1}\binom{k+s}{m+s}^{-1}\frac{1}{2m-2k+1}=(-1)^m\binom{2(n+s)}{n+s}^{-1}2^{2(n-m)}, \tag{81}$$

$$\sum_{k=m}^{n}(-1)^k\binom{n-m}{k-m}\binom{2(k+s)}{k+s}^{-1}2^{2k}=(-1)^m\binom{2(n-m)}{n-m}\binom{2(n+s)}{n+s}^{-1}\binom{n+s}{m+s}^{-1}\frac{2^{2m}}{2m-2n+1}. \tag{82}$$

Proof. Set $r=-1/2$ in Corollary 8. □

Proposition 23. *Let m be a non-negative integer, n a non-negative integer, s is a complex number such that $\Re(\frac{3}{2}-n-s)>0$ and s is not a negative integer. Then*

$$\begin{aligned}\sum_{k=m}^{n}(-1)^k\binom{n-m}{k-m}\binom{m-k}{m+s}\binom{2(m-k)+1}{2(m+s)}^{-1}\frac{1}{2m-2k+3}\\=(-1)^{s+1}\binom{2(m+s)}{m+s}\binom{2(n+s)}{n+s}^{-1}\frac{2^{2(n-m)}(2(n+s)-1)}{3},\end{aligned} \tag{83}$$

$$\begin{aligned}\sum_{k=m}^{n}(-1)^k\binom{n-m}{k-m}\binom{2(k+s)}{k+s}^{-1}2^{2k}\,(2(k+s)-1)\\=(-1)^{s+1}\binom{m-n}{m+s}\binom{2(m-n)+1}{2(m+s)}^{-1}\binom{2(m+s)}{m+s}^{-1}\frac{2^{2m}3}{2m-2n+3}.\end{aligned} \tag{84}$$

Proof. Set $r=1/2$ in Corollary 8. □

Proposition 24. *Let m be a positive integer, n a non-negative integer, s is a complex number such that $\Re(\frac{1}{2}-n-s)>0$ and s is not a negative integer. Then*

$$\sum_{k=m}^{n}(-1)^k\frac{\binom{n-2}{k-2}\binom{k}{m}\binom{2(k-m)}{k-m}}{\binom{2(k+s)}{k+s}\binom{k+s}{k-m}k(2m-2k+1)}=\frac{(-1)^m}{2^{2m}}\left(\frac{\binom{n}{m}2^{2n}}{\binom{2(n+s)}{n+s}n}-\frac{\binom{n-1}{m}2^{2(n-1)}}{\binom{2(n+s-1)}{n+s-1}(n-1)}\right), \tag{85}$$

$$\begin{aligned}\sum_{k=m}^{n}(-1)^k\frac{\binom{n-2}{k-2}\binom{k}{m}2^{2k}}{\binom{2(k+s)}{k+s}k}=2^{2m}\Bigg((-1)^m\frac{\binom{n}{m}\binom{2(n-m)}{n-m}}{\binom{2(n+s)}{n+s}\binom{n+s}{n-m}n(2m-2n+1)}\\-(-1)^s\frac{\binom{n-1}{m}\binom{m-n}{m+s}}{\binom{2m-2n+1}{2(m+s)}\binom{2(m+s)}{m+s}(n-1)(2m-2n+3)}\Bigg).\end{aligned} \tag{86}$$

Proof. Set $r=-1/2$ in Corollary 9. □

Proposition 25. *Let m be a positive integer, n a non-negative integer, s is a complex number such that $\Re(\frac{3}{2}-n-s)>0$ and s is not a negative integer. Then*

$$\begin{aligned}\sum_{k=m}^{n}(-1)^k\frac{\binom{n-2}{k-2}\binom{k}{m}\binom{m-k}{m+s}}{\binom{2m-2k+1}{2(m+s)}k(2m-2k+3)}\\=\frac{(-1)^s\binom{2(m+s)}{m+s}}{2^{2m}3}\left(\frac{\binom{n-1}{m}(2n+2s-3)2^{2(n-1)}}{\binom{2(n+s-1)}{n+s-1}(n-1)}-\frac{\binom{n}{m}(2n+2s-1)2^{2n}}{\binom{2(n+s)}{n+s}n}\right),\end{aligned} \tag{87}$$

$$\sum_{k=m}^{n}(-1)^k\frac{\binom{n-2}{k-2}\binom{k}{m}(2k+2s-1)2^{2k}}{\binom{2(k+s)}{k+s}k} = \frac{(-1)^s 2^{2m}3}{\binom{2(m+s)}{m+s}}\left(\frac{\binom{n-1}{m}\binom{m-n+1}{m+s}}{\binom{2m-2n+3}{2(m+s)}(n-1)(2m-2n+5)} - \frac{\binom{n}{m}\binom{m-n}{m+s}}{\binom{2m-2n+1}{2(m+s)}n(2m-2n+3)}\right). \tag{88}$$

Proof. Set $r=1/2$ in Corollary 9. □

Proposition 26. *Let m be a positive integer, n a non-negative integer, s is a complex number such that $\Re(\frac{1}{2}-n-s)>0$ and s is not a negative integer. Then*

$$\sum_{k=m}^{n}(-1)^k\frac{\binom{k}{m}2^{2k}}{\binom{2(k+s)}{k+s}k} = (-1)^m 2^{2m}\sum_{k=m}^{n}\frac{\binom{n}{k}\binom{k}{m}\binom{2(k-m)}{k-m}}{\binom{2(k+s)}{k+s}\binom{k+s}{k-m}k(2m-2k+1)}, \tag{89}$$

$$\sum_{k=m}^{n}(-1)^k\frac{\binom{n}{k}\binom{k}{m}2^{2k}}{\binom{2(k+s)}{k+s}k} = (-1)^m 2^{2m}\sum_{k=m}^{n}\frac{\binom{k}{m}\binom{2(k-m)}{k-m}}{\binom{2(k+s)}{k+s}\binom{k+s}{k-m}k(2m-2k+1)}. \tag{90}$$

Proof. Set $r=-1/2$ in Corollary 10. □

Proposition 27. *Let m be a positive integer, n a non-negative integer, s is a complex number such that $\Re(\frac{3}{2}-n-s)>0$ and s is not a negative integer. Then*

$$\sum_{k=m}^{n}(-1)^k\frac{\binom{k}{m}(2k+2s-1)2^{2k}}{\binom{2(k+s)}{k+s}k} = \frac{(-1)^{s+1}2^{2m}3}{\binom{2(m+s)}{m+s}}\sum_{k=m}^{n}\frac{\binom{n}{k}\binom{k}{m}\binom{m-k}{m+s}}{\binom{2m-2k+1}{2(m+s)}k(2m-2k+3)}, \tag{91}$$

$$\sum_{k=m}^{n}(-1)^k\frac{\binom{n}{k}\binom{k}{m}(2k+2s-1)2^{2k}}{\binom{2(k+s)}{k+s}k} = \frac{(-1)^{s+1}2^{2m}3}{\binom{2(m+s)}{m+s}}\sum_{k=m}^{n}\frac{\binom{k}{m}\binom{m-k}{m+s}}{\binom{2m-2k+1}{2(m+s)}k(2m-2k+3)}. \tag{92}$$

Proof. Set $r=1/2$ in Corollary 10. □

Remark 12. Again, four additional interesting special cases will come from setting $s=\pm 1/2$ in Corollaries 9 and 10.

6.3. Combinatorial identities involving powers of integers

Lemma 7. *If r and k are non-negative integers, then*

$$\frac{d^r}{dy^r}\left((1-e^y)^k\right)\Big|_{y=0} = \sum_{i=0}^{k}(-1)^i\binom{k}{i}i^r. \tag{93}$$

Proof. Since

$$(1-e^y)^k = \sum_{i=0}^{k}(-1)^i\binom{k}{i}e^{iy};$$

we have

$$\frac{d^r}{dy^r}(1-e^y)^k = \sum_{i=0}^{k}(-1)^i\binom{k}{i}i^r e^{iy};$$

and hence (93).
□

Remark 13. The evaluated derivatives in (93) can also be expressed as

$$\frac{d^r}{dy^r}(1-e^y)^k\Big|_{y=0} = (-1)^k k!\left\{ {r \atop k} \right\},$$

where $\left\{ {r \atop k} \right\}$ are the Stirling numbers of the second kind, defined by

$$\left\{ {r \atop k} \right\} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^r,$$

and having the useful property

$$\left\{ {r \atop k} \right\} = 0 \text{ if } r < k. \tag{94}$$

Theorem 9. *If x is a complex number, m is a positive integer and n and r are non-negative integers, then*

$$\sum_{k=0}^{n} \frac{(-1)^k}{k+m} \binom{x}{n-k} \binom{k+m}{m} k^r = \sum_{k=0}^{r} \frac{(-1)^k k!}{k+m} \binom{x-k-m}{n-k} \binom{k+m}{m} \left\{ {r \atop k} \right\} \tag{95}$$

and

$$\sum_{k=0}^{n} \frac{1}{k+m} \binom{x-k-m}{n-k} \binom{k+m}{m} k^r = \sum_{k=0}^{r} \frac{k!}{k+m} \binom{x}{n-k} \binom{k+m}{m} \left\{ {r \atop k} \right\}. \tag{96}$$

Proof. Write $-\exp a$ for a in (27), differentiate r times with respect to a and evaluate at $a = 0$ to obtain

$$\sum_{k=m}^{n} \frac{(-1)^{k-m}}{k} \binom{x}{n-k} \binom{k}{m} (k-m)^r = \sum_{k=0}^{r+m} \frac{(-1)^{k-m} (k-m)!}{k} \binom{x-k}{n-k} \binom{k}{m} \left\{ {r \atop k-m} \right\},$$

which can be written as (95) after shifting indices. Identity (96) follows from (95) by symmetry. □

Corollary 11. *If m is a positive integer and n and r are non-negative integers, then*

$$\sum_{k=0}^{n} \frac{(-1)^k}{k+m} \binom{n+m}{k+m} \binom{k+m}{m} k^r = \sum_{k=0}^{r} \frac{(-1)^k k!}{k+m} \binom{k+m}{m} \left\{ {r \atop k} \right\}, \tag{97}$$

and

$$\sum_{k=0}^{n} \frac{1}{k+m} \binom{k+m}{m} k^r = \sum_{k=0}^{r} \frac{k!}{k+m} \binom{n+m}{k+m} \binom{k+m}{m} \left\{ {r \atop k} \right\}. \tag{98}$$

Proof. Set $x = n + m$ in (95) and (96) and use (4). □

Remark 14. Identity (98) generalizes the known identity (consult for instance [23,24])

$$\sum_{k=0}^{n} k^r = \sum_{k=0}^{r} k! \binom{n+1}{k+1} \left\{ {r \atop k} \right\},$$

to which it reduces at $m = 1$ and which expresses the sum of powers of integers in terms of Stirling numbers of the second kind.

Remark 15. Setting $x = \pm 1/2 + m$ in Theorem 9 will yield two other interesting sums.

7. Conclusion

The motivation for writing this paper was Problem B-1358 in the 2024 issue of the Fibonacci Quarterly [1]. What we considered initially more or less a note, turned out to be a very powerful result. Our generalized identities presented in Lemma 5 enabled us to provide a range of applications to four different fields: polynomial identities, trigonometric identities, identities involving Horadam numbers, and combinatorial identities. In each field we have found generalizations of existing results. Our findings dealing with Frisch-type identities, Klamkin-type identities and power sums are important examples of such generalizations. It is worth mentioning, however, that there are gaps remaining. As indicated in Remarks 8, 9-12, and 15 more appealing results are still to be discovered. This is left as a potential future work.

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