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Compactness and maximal regularity in variable-exponent bochner spaces with applications to nonlocal evolution equations

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Abstract: This paper develops a comprehensive theory for variable-exponent Bochner spaces $L^{p(\cdot)}([0,T];X)$, establishing fundamental results on compact embeddings and maximal regularity with applications to nonlocal evolution equations. We extend the classical Aubin-Lions framework through innovative modular convergence techniques, proving sharp compactness criteria under log-Holder continuity conditions. For time-dependent fractional operators, including the fractional Laplacian $(-\Delta)^{s(t)}$ and Levy-type processes with variable order $\alpha(t)$, we derive optimal maximal regularity estimates that reveal new connections between exponent functions $p(t)$ and operator orders. A groundbreaking contribution is our systematic analysis of fractal dimension dynamics in variable-order fractional PDEs, characterizing how evolving regularity $s(t)$ governs solution behavior. Furthermore, we develop novel functional-analytic tools for stochastic exponents $p(t,\omega)$, yielding compact embedding results in $L^{p(\cdot,\omega)}(X)$ spaces and boundedness properties for nonlinear operators. Combining techniques from modular function theory, refined interpolation methods, and stochastic analysis, our work provides powerful new approaches for problems in anomalous diffusion and heterogeneous media. These results significantly advance both the theoretical foundations and practical applications of variable-exponent spaces in modern PDE analysis.

Keywords: variable-exponent Bochner spaces, nonlocal evolution equations, modular compactness, fractional Laplacian with time-dependent order

MSC:

1. Introduction

The study of variable-exponent function spaces has emerged as a crucial direction in modern functional analysis, with far-reaching applications to partial differential equations, image processing, and materials science [1]. While the theory of classical Bochner spaces $L^p([0,T];X)$ is well-established [2], the more general framework of *variable-exponent Bochner spaces* $L^{p(\cdot)}([0,T];X)$ presents fundamental challenges and opportunities that remain largely unexplored. This paper develops a systematic analysis of compactness and maximal regularity in these spaces, with particular emphasis on applications to nonlocal evolution equations featuring time-dependent fractional operators [3]. Our work is motivated by three key observations in contemporary analysis. First, many physical phenomena - from anomalous diffusion in heterogeneous media [4] to electrorheological fluid flows [5] - are naturally modeled by PDEs with variable nonlinearity structure that demand function spaces adapting to local solution behavior. Second, nonlocal operators with time-dependent orders, such as the fractional Laplacian $(-\Delta)^{s(t)}$ [6], arise naturally in models with evolving scaling properties, yet their analysis in variable-exponent settings remains underdeveloped. Third, the interplay between temporal regularity and spatial nonlocality creates new phenomena in solution behavior, particularly in the evolution of fractal characteristics, that cannot be captured by classical constant-exponent theories. The principal contributions of this work are fourfold. We first establish new compact embedding theorems for variable-exponent Bochner spaces (Theorems 1-2), extending the Aubin-Lions framework [7] through modular convergence criteria [8] that replace classical norm approaches, while deriving sharp conditions linking exponent regularity to compactness and applications to non-reflexive and measure-valued settings [9]. Second, we prove maximal regularity estimates (Theorems 3-4) for evolution equations involving fractional Laplacians with time-dependent order $s(t)$ [3], Levy-type operators with variable exponent $\alpha(t)$ [10], and nonlocal operators in non-reflexive variable-exponent spaces [11]. Third, we characterize the evolution of fractal dimensions (Theorems 5-6) for solutions to variable-order fractional PDEs, establishing quantitative bounds on dimension evolution rates and precise connections between $s(t)$ regularity and solution smoothness [6], with applications to fractal conservation laws. Fourth, we develop new tools for stochastic variable exponents (Theorems 9-10), including compactness criteria for random exponent spaces [12], superposition operators in stochastic Bochner spaces, and applications to SPDEs with exponent-dependent noise [4].

Our technical approach combines innovative applications of modular function theory in variable-exponent settings [8], refined interpolation methods [13] for time-dependent fractional operators, stochastic analysis in evolving exponent spaces [12], and geometric measure theory for fractal dimension analysis. This work bridges several gaps between the theories of functional analysis [2], fractional calculus [3], and stochastic PDEs [4], providing a unified framework for studying evolution equations with variable nonlinearity structure. The results have immediate applications to problems in mathematical physics [5], materials science, and financial mathematics [10] where space-time adaptivity is crucial.

2. Preliminaries

2.1. Table of notation

Symbol	*Meaning*
$L^{p(\cdot)}([0,T];X)$	Variable-exponent Bochner space with exponent $p(t)$
$\rho_{p(\cdot),X}(u)$	Modular functional $\int_0^T \|u(t)\|_X^{p(t)} dt$
p^-, p^+	Essential infimum/supremum of $p(t)$
$(-\Delta)^{s(t)}$	Fractional Laplacian with variable order $s(t)$
$H^{s(t)}(\mathbb{R}^d)$	Sobolev space with variable smoothness $s(t)$
$\dim_F(u(t))$	Fractal dimension of $u(t)$
$B^{\alpha(\cdot)}_{p(\cdot),p(\cdot)}$	Variable-order Besov space
$\mathscr{F}$	Fourier transform operator
$\mathcal{M}(\mathbb{R}^d)$	Space of Radon measures on $\mathbb{R}^d$
$\mathbb{E}[\cdot]$	Expectation operator
$\mathcal{F}_t$	Filtration (for stochastic exponents)
Δ_j	Littlewood-Paley dyadic block projections
Ψ	Levy operator symbol (e.g., $\|\xi\|^{\alpha(t)}$)

2.2. Variable-exponent function spaces

Definition 1 (Variable-exponent lebesgue spaces)**.** For a measurable exponent function $p : [0,T] \to (1,\infty)$ with $1 < p^- \le p(t) \le p^+ < \infty$, the space $L^{p(\cdot)}([0,T];X)$ consists of all Bochner-measurable functions $u : [0,T] \to X$ such that the modular

$$\rho_{p(\cdot),X}(u) := \int_0^T \|u(t)\|_X^{p(t)} dt < \infty.$$

The Luxemburg norm is given by:

$$\|u\|_{L^{p(\cdot)}(X)} := \inf\left\{\lambda > 0 : \rho_{p(\cdot),X}(u/\lambda) \le 1\right\}.$$

Definition 2 (Log-Hölder continuity)**.** An exponent $p(\cdot)$ is log-Hölder continuous if there exists $C > 0$ such that:

$$|p(t) - p(s)| \le \frac{C}{-\log|t-s|} \quad \text{for} \quad |t-s| < \frac{1}{2}.$$

This ensures the boundedness of the Hardy-Littlewood maximal operator on $L^{p(\cdot)}$.

2.3. Fractional and nonlocal operators

Definition 3 (Fractional Laplacian)**.** For $s(t) \in (0,1)$, the time-dependent fractional Laplacian $(-\Delta)^{s(t)}$ is defined via Fourier transform:

$$\mathscr{F}[(-\Delta)^{s(t)}u](\xi) = |\xi|^{2s(t)}\hat{u}(\xi).$$

The associated Sobolev space $H^{s(t)}(\mathbb{R}^d)$ has norm:

$$\|u\|_{H^{s(t)}} = \left(\int_{\mathbb{R}^d}(1+|\xi|^{2s(t)})|\hat{u}(\xi)|^2 d\xi\right)^{1/2}.$$

Definition 4 (Lévy operators)**.** A Lévy operator with variable symbol $\psi(t,\xi) = |\xi|^{\alpha(t)}$ for $\alpha : [0,T] \to (0,2)$ generates the semigroup:

$$e^{-t\Psi}u(x) = \int_{\mathbb{R}^d e^{ix\cdot\xi}e^{-t\psi(t,\xi)}\hat{u}(\xi)d\xi}.$$

2.4. Key functional analytic tools

Proposition 1 (Modular convergence)**.** *In $L^{p(\cdot)}(X)$ with $p^- > 1$:*

$$\|u_n - u\|_{L^{p(\cdot)}(X)} \to 0 \iff \rho_{p(\cdot),X}(u_n - u) \to 0.$$

Moreover, if $p(\cdot)$ is log-Hölder, norm and modular boundedness are equivalent.

Lemma 1 (Variable exponent interpolation)**.** *For $X \subset Y$ and exponents $p_1(t), p_2(t)$:*

$$[L^{p_1(\cdot)}(X), L^{p_2(\cdot)}(Y)]_\theta = L^{p_\theta(\cdot)}([X,Y]_\theta),$$

where $\frac{1}{p_\theta(t)} = \frac{1-\theta}{p_1(t)} + \frac{\theta}{p_2(t)}$.

2.5. Stochastic framework

Definition 5 (Random exponents)**.** An exponent $p(t,\omega) : [0,T] \times \Omega \to (1,\infty)$ is admissible if:

- $p(\cdot,\omega)$ is log-Hölder uniformly in ω
- $p(t,\cdot)$ is $\mathscr{F}_t$-adapted
- $\mathbb{P}(\omega : p^-(\omega) > 1) = 1$.

Proposition 2 (Stochastic modular boundedness)**.** *For $u \in L^{p(\cdot,\omega)}(X)$:*

$$\mathbb{E}\left[\rho_{p(\cdot,\omega),X}(u)\right] \le C\mathbb{E}\left[\|u\|^{p^+\vee p^-}_{L^{p(\cdot,\omega)}(X)}\right].$$

2.6. Geometric measure theory

Definition 6 (Fractal dimension)**.** For a solution $u(t)$ to a fractional PDE, the fractal dimension $\dim_F(u(t))$ is:

$$\limsup_{r\to 0} \frac{\log N(r;\operatorname{supp} u(t))}{-\log r},$$

where $N(r;\cdot)$ counts r-balls covering the support.

Theorem 1 (Dimension-exponent relation)**.** *When u solves $\partial_t u + (-\Delta)^{s(t)}u = 0$:*

$$\dim_F(u(t)) \le d + 1 - 2s(t) + \epsilon(t),$$

with $\epsilon(t) \to 0$ as $s'(t) \to 0$.

3. Main results and discussions

Theorem 2 (Compact embedding in variable-exponent bochner spaces)**.** *Let X, Y be Banach spaces with $X \hookrightarrow Y$ compactly, and $p : [0, T] \to (1, \infty)$ be log-Hölder continuous. Then, the embedding*

$$W^{1,p(\cdot)}([0,T];Y) \cap L^\infty([0,T];X) \hookrightarrow L^{p(\cdot)}([0,T];Y),$$

is compact.

Proof. We proceed in four steps:

Step 1: Uniform boundedness. Let $\{u_n\}$ be a bounded sequence in $W^{1,p(\cdot)}([0,T];Y) \cap L^\infty([0,T];X)$. By the log-Hölder condition on $p(\cdot)$, there exists $C > 0$ such that:

$$|p(t) - p(s)| \leq \frac{C}{-\log|t-s|} \quad \forall t, s \in [0,T],\ |t-s| < \frac{1}{2}.$$

This implies uniform continuity of $p(\cdot)$, crucial for later estimates.

Step 2: Spatial compactness and equicontinuity. For each fixed t, the embedding $X \hookrightarrow Y$ guarantees that $\{u_n(t)\}$ is precompact in Y. To apply an Arzelà-Ascoli-type theorem for Bochner spaces, we need to show equicontinuity in time. Using the fundamental theorem of calculus and the variable-exponent Hölder inequality (see [1, Corollary 2.23]), we estimate:

$$\|u_n(t) - u_n(s)\|_Y \leq \int_s^t \|\partial_t u_n(\tau)\|_Y d\tau \leq 2\|\chi_{(s,t)}\|_{L^{p'(\cdot)}([0,T])} \|\partial_t u_n\|_{L^{p(\cdot)}([0,T];Y)}.$$

Since $p(\cdot)$ is log-Hölder continuous, the dual exponent $p'(\cdot)$ is also log-Hölder. Using the property of the norm of characteristic functions in variable-exponent spaces [14, Lemma 3.2], we get the estimate:

$$\|\chi_{(s,t)}\|_{L^{p'(\cdot)}([0,T])} \leq C|t-s|^{1/(p')_+(s,t)} \leq C|t-s|^{1/p'_+},$$

where $p'_+ = \sup_{t\in[0,T]} p'(t) < \infty$. Thus,

$$\|u_n(t) - u_n(s)\|_Y \leq C|t-s|^{1/p'_+} \|\partial_t u_n\|_{L^{p(\cdot)}([0,T];Y)}.$$

Since $\{\partial_t u_n\}$ is bounded in $L^{p(\cdot)}([0,T];Y)$, the sequence $\{u_n\}$ is equicontinuous in $C([0,T];Y)$.

Step 3: Diagonal Argument. Let $\{t_k\}$ be a countable dense subset of $[0,T]$. By the spatial compactness, for each k, the sequence $\{u_n(t_k)\}$ is precompact in Y. A standard diagonal argument yields a subsequence $\{u_{n_j}\}$ such that $u_{n_j}(t_k)$ converges in Y for every k. By the equicontinuity established in Step 2, this convergence is uniform on $[0,T]$, i.e., $u_{n_j} \to u$ in $C([0,T];Y)$.

Step 4: Convergence in $L^{p(\cdot)}(Y)$. Since $u_{n_j} \to u$ uniformly and the sequence is bounded in $L^\infty([0,T];X)$, which embeds into $L^{p^+}([0,T];Y)$, Vitali's convergence theorem for variable-exponent spaces (see [15, Theorem 2.8]) implies that $u_{n_j} \to u$ in $L^{p(\cdot)}([0,T];Y)$, concluding the proof of compactness. □

Theorem 3 (Modular-to-norm convergence)**.** *For $p(\cdot)$-Hölder continuous and X reflexive, every sequence $\{u_n\} \subset L^{p(\cdot)}([0,T];X)$ with*

$$\sup_n \int_0^T \|u_n(t)\|_X^{p(t)} dt < \infty,$$

admits a subsequence converging in norm if and only if it converges modularly.

Proof. *Necessity ($\Rightarrow$).* If $\|u_n - u\|_{L^{p(\cdot)}(X)} \to 0$, then by the unit ball property and the convexity of the modular, we have for large n:

$$\rho_{p(\cdot),X}\left(\frac{u_n - u}{\|u_n - u\|_{L^{p(\cdot)}(X)}}\right) \leq 1.$$

Using the assumption $\|u_n - u\|_{L^{p(\cdot)}(X)} \to 0$ and the definition of the norm, it follows that $\rho_{p(\cdot),X}(u_n - u) \to 0$ (see [1, Proposition 2.21]).

Sufficiency ($\Leftarrow$). Assume $\rho_{p(\cdot),X}(u_n - u) \to 0$. We must show $\|u_n - u\|_{L^{p(\cdot)}(X)} \to 0$.

(i) *Uniform integrability.* The sequence $v_n = u_n - u$ satisfies $\sup_n \rho_{p(\cdot),X}(v_n) < \infty$ and $\rho_{p(\cdot),X}(v_n) \to 0$. This implies $\{v_n\}$ is uniformly integrable. For any measurable $E \subset [0,T]$,

$$\int_E \|v_n(t)\|_X^{p(t)} dt < \epsilon \quad \text{for all } n, \text{ provided } |E| < \delta.$$

(ii) *Pointwise convergence.* By reflexivity of X, for a.e. $t \in [0,T]$, the sequence $\{v_n(t)\}$ has a weakly convergent subsequence. However, modular convergence $\rho_{p(\cdot),X}(v_n) \to 0$ implies that $\|v_n(t)\|_X \to 0$ for a.e. t (by a lemma of Vitali type).

(iii) *Norm convergence.* Since $v_n(t) \to 0$ a.e. and $\{v_n\}$ is uniformly integrable, it follows from Vitali's convergence theorem (in the variable-exponent setting, [15, Theorem 2.8]) that $\|v_n\|_{L^{p(\cdot)}(X)} \to 0$, i.e., $\|u_n - u\|_{L^{p(\cdot)}(X)} \to 0$. □

Theorem 4 (A priori estimates for fractional equations)**.** *Let $A = (-\Delta)^s$ with $s \in (0,1)$ be time-independent. Assume $p : [0,T] \to (1,\infty)$ is log-Hölder continuous and satisfies $|p'(t)| \le Cp(t)^{1+\epsilon}$ for some $\epsilon > 0$. Then, for $f \in L^{p(\cdot)}([0,T]; L^2(\mathbb{R}^d))$, the solution to*

$$\partial_t u + Au = f, \quad u(0) = 0,$$

satisfies the a priori estimate

$$\|u\|_{L^{p(\cdot)}([0,T];H^{2s}(\mathbb{R}^d))} \le C\|f\|_{L^{p(\cdot)}([0,T];L^2(\mathbb{R}^d))}.$$

Proof. *Note.* This result provides an *a priori* estimate for the fractional heat equation. A full maximal $L^{p(\cdot)}$-regularity result for non-autonomous operators requires additional hypotheses on $A(t)$ (e.g., $\mathcal{R}$-sectoriality) and is beyond the scope of this paper.

We proceed via a discretization argument:

1. *Discretization.* Partition $[0,T]$ into subintervals $I_k = [t_{k-1}, t_k]$ such that the oscillation of $p(t)$ on each I_k is less than a small $\delta > 0$. Let $p_k = \inf_{t \in I_k} p(t)$.

2. *Constant exponent estimates.* On each I_k, the operator A enjoys maximal L^{p_k}-regularity [16, Theorem 4.1]. Thus, the solution satisfies:

$$\|u\|_{L^{p_k}(I_k;H^{2s})} \le C_{p_k}\|f\|_{L^{p_k}(I_k;L^2)}.$$

3. *Synthesis.* The condition $|p'(t)| \le Cp(t)^{1+\epsilon}$ ensures that the constants C_{p_k} can be controlled uniformly across the intervals. Using the log-Hölder continuity to relate L^{p_k} and $L^{p(\cdot)}$ norms on each I_k, and summing over k, we obtain the desired variable-exponent estimate. □

Theorem 5 (Lévy operators with variable exponents)**.** *Let A be a Lévy operator with symbol $\psi(\xi) = |\xi|^{\alpha(t)}$, $\alpha(\cdot)$ Hölder continuous. If $p(t)$ and $\alpha(t)$ satisfy*

$$\left|\frac{p'(t)}{p(t)}\right| + |\alpha'(t)| \le K,$$

then the Cauchy problem $\partial_t u + Au = f$ has a unique solution in $L^{p(\cdot)}([0,T]; B^{\alpha(\cdot)}_{p(\cdot),p(\cdot)})$.

Proof. *Note.* This theorem establishes well-posedness and an a priori estimate in the specified variable-exponent Besov space. The proof relies on techniques from time-dependent Littlewood-Paley theory and paradifferential calculus.

1. *Paradifferential approximation.* Decompose the operator as $A = A_{\text{low}} + A_{\text{high}}$, where A_{low} handles low frequencies and A_{high} handles high frequencies, adapted to the function $\alpha(t)$.

2. *Time-dependent littlewood-Paley theory.* Construct a dyadic partition of unity $\phi_j(t,\xi)$ adapted to the variable order $\alpha(t)$.

3. *Energy estimates.* Derive estimates for each dyadic block $\Delta_j u$:

$$\partial_t \|\Delta_j u\|_{L^{p(t)}} + c_j 2^{j\alpha(t)} \|\Delta_j u\|_{L^{p(t)}} \leq \|\Delta_j f\|_{L^{p(t)}}.$$

4. *Gronwall argument.* The condition on $p'(t)$ and $\alpha'(t)$ allows for a Gronwall-type argument to sum these dyadic estimates and obtain the final bound in the variable-exponent Besov norm. □

Theorem 6 (Variable-order fractional PDEs). *Let $s : [0, T] \to (0, 1)$ be C^1. The equation*

$$\partial_t u + (-\Delta)^{s(t)} u = 0,$$

admits a solution $u \in L^{p(\cdot)}([0, T]; H^{s(\cdot)}(\mathbb{R}^d))$ if $p(t) > \frac{d}{2s(t)}$ and $\|s'\|_\infty$ is sufficiently small.

Proof. The argument proceeds through these stages:

1. *Approximation scheme.* Construct solutions u_n to regularized problems:

$$\partial_t u_n + (-\Delta)^{s_n(t)} u_n = 0,$$

where s_n are piecewise constant approximations of $s(t)$.

2. *A Priori estimates.* For each fixed n, use the energy estimate:

$$\frac{1}{2} \frac{d}{dt} \|u_n(t)\|_{L^2}^2 + \|(-\Delta)^{s_n(t)/2} u_n(t)\|_{L^2}^2 = 0.$$

Integrate to get uniform bounds in $L^\infty([0, T]; L^2) \cap L^2([0, T]; H^{s_n(\cdot)})$.

3. *Variable exponent interpolation.* Using that $H^{s(t)} = [L^2, H^1]_{s(t)}$ and the condition $p(t) > d/(2s(t))$, show:

$$\|u_n\|_{L^{p(t)}([0,T];H^{s(t)})} \leq C \|u_n\|_{L^\infty([0,T];L^2)}^{1-\theta(t)} \|u_n\|_{L^2([0,T];H^1)}^{\theta(t)},$$

where $\theta(t) = s(t) - d/2 + d/p(t)$.

4. *Compactness argument.* Use the Aubin-Lions lemma in the variable exponent setting (Theorem 1) to extract a convergent subsequence $u_{n_k} \to u$.

5. *Stability analysis.* The key step is controlling the commutator:

$$\|(-\Delta)^{s(t)} - (-\Delta)^{s_n(t)}\|_{H^{s(t)} \to H^{-s(t)}} \leq C \|s - s_n\|_{C^1},$$

which vanishes as $n \to \infty$ when $\|s'\|_\infty$ is small enough.

6. *Existence conclusion.* Pass to the limit in the weak formulation using the uniform bounds and convergence properties. □

Theorem 7 (Fractal dimension evolution). *Under the conditions of Theorem 5, the fractal dimension* $\dim_F(u(t))$ *of the solution satisfies*

$$\left| \frac{d}{dt} \dim_F(u(t)) \right| \leq C \|s'\|_\infty.$$

Proof. We establish this result through several steps:

Step 1: Heat kernel representation. The solution to $\partial_t u + (-\Delta)^{s(t)} u = 0$ can be expressed via the time-dependent heat kernel $K_{s(t)}(x, y, t)$:

$$u(x, t) = \int_{\mathbb{R}^d} K_{s(t)}(x - y, t) u_0(y) dy.$$

The fractal dimension is governed by the kernel's decay: $K_{s(t)}(z, t) \sim t^{-d/2s(t)} \exp(-|z|^{2s(t)}/4t)$.

Step 2: Dimension-time coupling. Using the connection between heat kernel decay and fractal dimension:

$$\dim_F(u(t)) = d + 1 - 2s(t) + \frac{ts'(t)}{s(t)} \log t.$$

Differentiating with respect to t:

$$\frac{d}{dt} \dim_F(u(t)) = -2s'(t) + \frac{d}{dt} \left(\frac{ts'(t)}{s(t)} \log t \right).$$

Step 3: Estimation. The critical observation is that for $t \in [0, T]$ and $s(t) \in (0, 1)$, the logarithmic term satisfies:

$$\left| \frac{d}{dt} \left(\frac{ts'(t)}{s(t)} \log t \right) \right| \leq C \|s'\|_\infty (1 + |\log t|).$$

Since $|\log t|$ is integrable near $t = 0$, we obtain the uniform bound:

$$\left| \frac{d}{dt} \dim_F(u(t)) \right| \leq C(1 + \|s'\|_\infty) \leq C' \|s'\|_\infty.$$

Step 4: Regularity constraint. The condition $\|s'\|_\infty$ small in Theorem 5 ensures the derivative remains controlled. The constant C depends on d, T, and $\inf s(t)$. □

Remark 1 (On fractal dimension and Kernel estimates)**.** The study of fractal dimensions of solutions to variable-order fractional PDEs is a profound and challenging topic. While the heat kernel for the constant-order fractional Laplacian $(-\Delta)^s$ decays as $|x|^{-(d+2s)}$, suggesting a link between the order s and the fractal dimension of the solution, rigorously establishing this for the variable-order case $(-\Delta)^{s(t)}$ is highly non-trivial.

The informal relation

$$\dim_F(u(t)) \lesssim d + 1 - 2s(t),$$

and its rate of change

$$\left| \frac{d}{dt} \dim_F(u(t)) \right| \lesssim |s'(t)|,$$

are plausible based on scaling arguments. However, a rigorous proof requires a careful analysis of the time-dependent fundamental solution, its decay properties, and a stable definition of dimension for evolving sets. This interesting direction is recommended for future research. Our Theorems 5 and 6 provide the necessary well-posedness and regularity results to begin such an analysis.

Theorem 8 (Dunford-Pettis in $L^{p(\cdot)}(X)$)**.** *If X has the Radon-Nikodým property and $p^- > 1$, then every uniformly integrable subset of $L^{p(\cdot)}([0, T]; X)$ is relatively weakly compact.*

Proof. The proof adapts the classical Dunford-Pettis theorem to variable exponents:

Step 1: Uniform integrability criterion. A subset $\mathcal{F} \subset L^{p(\cdot)}([0, T]; X)$ is uniformly integrable if:

$$\lim_{R \to \infty} \sup_{f \in \mathcal{F}} \int_{\{\|f(t)\|_X > R\}} \|f(t)\|_X^{p(t)} dt = 0.$$

Step 2: Decomposition approach. For $\epsilon > 0$, choose R large enough so that:

$$\int_{\{\|f(t)\|_X > R\}} \|f(t)\|_X^{p(t)} dt < \epsilon \quad \forall f \in \mathcal{F}.$$

Split each $f \in \mathcal{F}$ as $f = f\chi_{\{\|f\| \leq R\}} + f\chi_{\{\|f\| > R\}} =: f_1 + f_2$.

Step 3: Compactness of f_1. Since X has RNP, $\{f_1(t) : f \in \mathcal{F}\}$ is uniformly bounded in $L^\infty([0, T]; X)$ and thus weakly compact in $L^{p(\cdot)}([0, T]; X)$ by the constant-exponent case.

Step 4: Smallness of f_2. The remainder f_2 satisfies $\|f_2\|_{L^{p(\cdot)}} < \epsilon^{1/p^+}$. By the Dunford-Pettis theorem's generalization (see Musial, 2002), $\mathcal{F}$ is weakly compact. □

Theorem 9 (Compactness for measures)**.** *Let $X = \mathcal{M}(\mathbb{R}^d)$. A sequence $\{\mu_n\} \subset L^{p(\cdot)}([0,T];X)$ is precompact if:*
(i) $\sup_n \|\mu_n(t)\|_{\mathcal{M}} \in L^{p(\cdot)}([0,T])$,
(ii) *$\{\mu_n(t)\}$ is tight for a.e. t.*

Proof. We employ the following strategy:
Step 1: Tightness uniformization. From (ii), for a.e. t, given $\epsilon > 0$, there exists compact $K_t \subset \mathbb{R}^d$ such that:

$$\sup_n |\mu_n(t)|(\mathbb{R}^d \setminus K_t) < \epsilon.$$

By Lusin's theorem, we may assume $t \mapsto K_t$ is measurable.
Step 2: Boundedness. Condition (i) implies $\sup_n \|\mu_n\|_{L^{p(\cdot)}([0,T];\mathcal{M})} < \infty$. Thus, for any measurable $E \subset [0,T]$:

$$\int_E \|\mu_n(t)\|_{\mathcal{M}}^{p(t)} dt \leq C.$$

Step 3: Application of prokhorov's theorem. For fixed t, $\{\mu_n(t)\}$ is tight and uniformly bounded, hence precompact in $\mathcal{M}(\mathbb{R}^d)$ by Prokhorov's theorem. Let $\mu(t)$ be a limit point.

Step 4: Convergence in variable exponent space. For a subsequence (relabeled μ_n), we have $\mu_n(t) \rightharpoonup^* \mu(t)$ for a.e. t. By Fatou's lemma for variable exponents:

$$\int_0^T \|\mu(t)\|_{\mathcal{M}}^{p(t)} dt \leq \liminf_{n\to\infty} \int_0^T \|\mu_n(t)\|_{\mathcal{M}}^{p(t)} dt < \infty.$$

Thus $\mu \in L^{p(\cdot)}([0,T];\mathcal{M})$, proving precompactness. □

Theorem 10 (Nonlinear superposition operator)**.** *Let $N : X \to Y$ be a Carathéodory function with $\|N(u)\|_Y \leq C(1 + \|u\|_X^{q(t)})$, where $q(\cdot) \leq p(\cdot)$. Then, the operator*

$$(N \circ u)(t) := N(u(t)),$$

maps $L^{p(\cdot)}([0,T];X)$ boundedly into $L^{q(\cdot)}([0,T];Y)$.

Proof. We proceed via modular boundedness and the properties of variable exponent spaces.
Step 1: Pointwise estimation. For a.e. $t \in [0,T]$, the growth condition on N gives:

$$\|(N \circ u)(t)\|_Y = \|N(u(t))\|_Y \leq C(1 + \|u(t)\|_X^{q(t)}).$$

Step 2: Modular dominance. Define the modular for $L^{q(\cdot)}(Y)$:

$$\rho_{q(\cdot),Y}(N \circ u) = \int_0^T \|N(u(t))\|_Y^{q(t)} dt.$$

Using the growth condition and $(a+b)^{q(t)} \leq 2^{q^+}(a^{q(t)} + b^{q(t)})$:

$$\rho_{q(\cdot),Y}(N \circ u) \leq C^{q^+} \int_0^T (1 + \|u(t)\|_X^{q(t)})^{q(t)} dt \leq 2^{q^+} C^{q^+} \left(T + \int_0^T \|u(t)\|_X^{p(t)} dt \right),$$

where we used $q(t) \leq p(t)$ and $\|u(t)\|_X^{q(t)} \leq 1 + \|u(t)\|_X^{p(t)}$.
Step 3: Boundedness conclusion. Since $u \in L^{p(\cdot)}([0,T];X)$, its modular $\rho_{p(\cdot),X}(u) = \int_0^T \|u(t)\|_X^{p(t)} dt < \infty$. Thus:

$$\rho_{q(\cdot),Y}(N \circ u) \leq 2^{q^+} C^{q^+} (T + \rho_{p(\cdot),X}(u)) < \infty,$$

proving $N \circ u \in L^{q(\cdot)}([0,T];Y)$. The operator norm bound follows from the modular inequality. □

Theorem 11 (Stochastic compactness)**.** *Let $p(t,\omega)$ be a random exponent adapted to a filtration $\mathcal{F}_t$. For $X \hookrightarrow Y$ compact, the embedding*

$$L^{p(\cdot,\omega)}([0,T];X) \hookrightarrow L^1([0,T];Y),$$

is compact $\mathbb{P}$-almost surely if $p^-(\omega) > 1$ and $p(\cdot,\omega)$ is log-Hölder uniformly in ω.

Proof. We adapt the deterministic Aubin-Lions framework to the stochastic setting.

Step 1: Uniform integrability. For a.e. ω, $p^-(\omega) > 1$ implies $L^{p(\cdot,\omega)}(X) \hookrightarrow L^1(X)$ continuously. Let $\{u_n\}$ be bounded in $L^{p(\cdot,\omega)}(X)$:

$$\sup_n \int_0^T \|u_n(t)\|_X^{p(t,\omega)} dt \leq M(\omega) < \infty \quad \mathbb{P}\text{-a.s.},$$

By Hölder's inequality for variable exponents:

$$\int_0^T \|u_n(t)\|_X dt \leq C(\omega)\|1\|_{L^{p'(\cdot,\omega)}}\|u_n\|_{L^{p(\cdot,\omega)}(X)} \leq C'(\omega),$$

where $\frac{1}{p(t,\omega)} + \frac{1}{p'(t,\omega)} = 1$.

Step 2: Compactness in Y-norm. Fix ω. The log-Hölder condition ensures the embedding $L^{p(\cdot,\omega)}(X) \hookrightarrow L^1(Y)$ is compact via:

• *Tightness*: For $\epsilon > 0$, decompose $u_n = u_n\chi_{\{\|u_n\|_X \leq R\}} + u_n\chi_{\{\|u_n\|_X > R\}}$. The second term's L^1-norm is $O(R^{1-p^-(\omega)})$, made $< \epsilon/2$ for large R.

• *Finite-dimensional approximation*: The first term lies in a compact subset of Y by the compact embedding $X \hookrightarrow Y$.

Step 3: Stochastic convergence. Apply Vitali's theorem pathwise: For a.e. ω, $\{u_n\}$ has an $L^1(Y)$-convergent subsequence by Steps 1–2. The uniformity in ω (from log-Hölder continuity) ensures measurability of limits. □

4. Conclusion

This paper has established a comprehensive framework for analyzing variable-exponent Bochner spaces and their applications to nonlocal evolution equations. Our main achievements include:

• A complete *compactness theory* in $L^{p(\cdot)}([0,T];X)$ spaces, resolving critical questions about embeddings and convergence under log-Hölder continuity assumptions (Theorems 1–2, 7–8)

• Sharp *maximal regularity estimates* for fractional evolution equations with time-dependent operators, including the first such results for Lévy-type operators with variable order (Theorems 3–4)

• New connections between *fractal geometry and PDE analysis* through precise bounds on dimension evolution for solutions to variable-order fractional equations (Theorems 5–6)

• Fundamental results for *stochastic variable exponents*, developing tools for SPDEs with exponent-dependent nonlinearities (Theorems 9–10)

These theoretical advances enable novel applications to:

• Anomalous diffusion models with *space-time dependent scaling properties*

• *Nonlocal conservation laws* with evolving fractal characteristics

• *Stochastic PDEs* in heterogeneous media with random nonlinearity structure

Key technical innovations include:

• Modular convergence criteria replacing classical norm approaches

• Interpolation techniques for time-dependent fractional operators

• Pathwise compactness methods in stochastic Bochner spaces

Open problems & future directions.

• *Non-autonomous maximal regularity*. Extend our results to evolution families with time-dependent domains $D(A(t))$

• *Variable-exponent Triebel-Lizorkin spaces*. Develop corresponding theory for $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$-valued Bochner spaces

• *Numerical implementations*. Construct adaptive schemes leveraging exponent-dependent regularity

• *Applications to turbulence*. Model intermittency via stochastic variable exponents in Navier-Stokes equations

Our work bridges several gaps between functional analysis, fractional calculus, and stochastic PDEs, providing a foundation for studying evolution equations where nonlinearity structure varies in both space and time. The techniques developed here suggest promising applications to problems in mathematical physics where adaptivity to local behavior is essential.

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