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Comparison principles for anisotropic conformable fractional elliptic equations via Picone identities

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Abstract: Fractional differential equations is a rapidly growing field of mathematical analysis with a wide and robust applicability in several areas of physics and geometry. Picone identity is a powerful tool which has been applied extensively in the study of second order elliptic equations. In this paper we prove some nonlinear anisotropic Picone type identities and give its applications to deriving Sturmian comparison principle and Liouville type results for anisotropic conformable fractional elliptic differential equations and systems.

Keywords: fractional derivatives, anisotropic differential operator, elliptic equations, picone identities, sturmian comparison principle, Liouville theorems

MSC: 26D10, 46E30, 47J10, 65Mxx.

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1. Introduction

In recent time, differential equations and inequalities of fractional order have attracted serious interests of mathematicians and scientists owing to their applications in several situations involving nonlocal structure and memory effects. These situations arise from science, engineering, medicine, economics, finance and so on. The study of fractional order differential equations emanated from the field of fractional calculus, which is dated back to the conversation about half-order derivative of a function between L'Hopital and Leibniz in 1965 [1,2]. The main objects of concern in fractional calculus are integrals and derivatives of non-integer orders. Over time, various definitions of these objects, such as those by Riemann-Liouville, Riez, Weyl, Hadamard, Grünwald-Letnikov, and Caputo, have appeared and been improved. Fractional derivatives of Riemann-Liouville and Caputo have been seriously engaged in this regard [3–7].

The Riemann-Liouville (left) fractional integral of order $\alpha > 0$ is defined for a continuous function f with some singular kernel as follows:

$$ {}_a\mathbb{I}_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(x)}{(t-x)^{1-\alpha}} dx, \tag{1} $$

where $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$ is the usual Gamma function.

Riemann-Liouville fractional derivatives of order $\alpha \in (n-1, n]$ for f is defined by

$$ {}^{RL}{}_aD_t^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx = \frac{d^n}{dt^n}\, {}_a\mathbb{I}_t^{n-\alpha} f(t). \tag{2} $$

Caputo fractional derivatives of order $\alpha \in (n-1, n]$ for f is defined by

$$ {}^{C}{}_aD_t^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{1}{(t-x)^{\alpha-n+1}} \frac{d^n f(x)}{dt^n} dx = {}_a\mathbb{I}_t^{n-\alpha} f^{(n)}(t). \tag{3} $$

By Definitions (2) and (3), it is seen that both Riemann-Liouville and Caputo fractional derivatives are given with respect to fractional integrals with singular kernel. Hence, they inherit some nonlocal structures and historic memory. However, there are shortcomings in the definitions of Riemann-Liouville and Caputo fractional derivatives in the sense that they lack certain vital properties associated with the classical derivative. They do not satisfy such properties as the product rule, quotient rule, chain rule, power rule, Rolle's theorem, mean value theorem, Green's theorem and the property that the derivative of constant should be zero, (Caputo fractional derivative of a constant is zero, though), and so on. These shortcomings limit the applicability of these derivatives to real life phenomena. In order to circumvent these challenges, Khalil et al. [8] introduced a new definition of fractional derivative known as conformable fractional derivative of a function, which involves a limit instead of integral with singular kernel.

Given a function $f : [0, \infty) \to \mathbb{R}$, then the conformable fractional derivative of order α is defined in [8] by

$$\left(T_t^\alpha f\right)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \tag{4}$$

for all $t > 0$ and $\alpha \in (0, 1]$. If f is α-differentiable in some interval $(0, a)$, $a > 0$, and if $\lim_{t\to 0} \left(T_t^\alpha f\right)(t)$ exists, then it is known that $(T_t^\alpha f)(0) = \lim_{t\to 0} \left(T_t^\alpha f\right)(t)$. It can be seen that definition (4) thus extends the limit definition of the classical derivative. This local structure thereby makes it more flexible for conformable fractional derivative to accommodate many classical theorems of calculus, which in turn allows for the extension of classical results to the fractional order set up. Further properties of conformable derivative are examined by authors in [9–11], some of its applications to science and engineering can be found in [12–14]. One can see [15,16] for geometric and physical interpretations of the conformable fractional derivative. Basic concepts of conformable fractional calculus as related to the present work are highlighted in §2 of this paper.

The anisotropic conformable fractional differential operator is defined for continuous α-differentiable function u by

$$\sum_{k=1}^{n} \frac{\partial^\alpha}{\partial x_k^\alpha} \left(\left| \frac{\partial^\alpha u}{\partial x_k^\alpha} \right|^{p_k - 2} \frac{\partial^\alpha u}{\partial x_k^\alpha} \right), \tag{5}$$

with $\alpha \in (0, 1]$, $p_k > 1$, $k = 1, \cdots, n$ and $| \cdot |$ denotes Euclidean norm. (See §2 for the definition of $\frac{\partial^\alpha}{\partial x_k^\alpha}$). Setting $p_k = 2$ and $p_k = p$ for all k in (5), this operator reduces to conformable Laplacian and the conformable pseudo-p-Laplacian, respectively. The anisotropic Laplacian has numerous applications in several areas of mathematical analysis and across various fields in engineering and sciences. In particulars, they are used in modeling fluid dynamics in the anisotropic media having different conductivities in different direction [17] and also to model anisotropic characteristics of some reinforced materials [18] as well as to model the propagation of epidemic diseases in nonhomogeneous domain. Models involving anisotropic Laplacian arise also in image segmentation and computer vision [19,20].

Anisotropic Picone identities for classical gradient operator is proved in [21] for differentiable functions $u \geq 0$, $v > 0$ in a domain of $\mathbb{R}^n$ and exponents $p_k > 1$ as follows:

$$\sum_{k=1}^{n} \left| \frac{\partial u}{\partial x_k} \right|^{p_k} - \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left(\frac{u^{p_k}}{v^{p_k - 1}} \right) \left| \frac{\partial v}{\partial x_k} \right|^{p_k - 2} \frac{\partial v}{\partial x_k} \geq 0, \tag{6}$$

with equality if and only if $u = cv$ for some constant $c > 0$. Picone type identities have proved to an effective tool in the study of existence and nonexistence of positive solutions to differential equations, Sturmian comparison principle, domain monotonicity, Hardy's inequality, Caccioppoli inequality, e. t. c. (see [21–26] and the references cited therein). Motivated by (6), we [27] recently prove the following conformable Picone type inequalities for anisotropic fractional gradient operator ($D_{x_k}^\alpha := \frac{\partial^\alpha}{\partial x_k^\alpha}$) on a compatible domain

$$\sum_{k=1}^{n} |D_{x_k}^\alpha u|^{p_k} - \sum_{k=1}^{n} D_{x_k}^\alpha \left(\frac{u^{p_k}}{v^{p_k - 1}} \right) |D_{x_k}^\alpha v|^{p_k - 2} D_{x_k}^\alpha v \geq 0, \tag{7}$$

where several Hardy type inequalities and Heisenberg Pauli-Weyl uncertainty principles are derived as its consequences. Similar Picone identities have been proved for single valued functions by authors in [28,29].

The aim of this paper therefore is to prove some nonlinear anisotropic Picone type identities and give its applications to deriving Sturmian comparison principle and Liouville type results for anisotropic fractional conformable elliptic differential equations and systems. Elliptic partial differential equations and systems occur as models of various phenomena in physics, such as electrostatics, heat and mass diffusion and hydrodynamics, quantum wave etc., (see [30,31] for instance). Their practical applications are also abound in all branches of mathematics, including harmonic analysis, geometry, Lie theory, and spectral analysis (see [32,33] for instance). The prototypes of elliptic PDEs are the Laplacian and Poisson equations.

The remaining part of this paper is as follows: §2 collects basic definitions and fundamental results relating to the concept of conformable fractional calculus as will be relevant to our results. In §3, Picone identities are proved and their applications to Sturmian comparison and Liouville principles are presented.

2. Preliminaries

This section collects basic definitions and fundamental results relating to the concept of conformable fractional calculus, functional spaces and conformable fractional eigenvalue problem as will be used in the rest of this paper.

2.1. Conformable fractional derivative

Consider the conformable fractional derivative of a function $f : [0, \infty) \to \mathbb{R}$ with order α as defined by (4). It has been well established that a function $f : [0, \infty) \to \mathbb{R}$ is continuous at $x_0 > 0$ if f is α-differentiable for $\alpha \in (0, 1]$. The next theorem presents some basic properties which conformable fractional derivative inherits from the classical derivative.

Theorem 1. [8,9] *Let $\alpha \in (0, 1]$, and let f, g be α-differentiable at a point $x > 0$. Then*

1. *Linearity:* $T_x^\alpha(af + bg)(x) = a(T_x^\alpha f)(x) + b(T_x^\alpha g)(x), \ \ a, b \in \mathbb{R}$.
2. *Product rule:* $T_x^\alpha(fg)(x) = g(T_x^\alpha f)(x) + f(T_x^\alpha g)(x)$.
3. *Quotient rule:* $T_x^\alpha\left(\frac{f}{g}\right)(x) = \frac{g(T_x^\alpha f)(x) - f(T_x^\alpha g)(x)}{g^2(x)}$, $g \neq 0$.
4. *Vanishing derivative of a constant function:* $T_x^\alpha(\lambda) = 0$ *for all constant function* $f(x) = \lambda$.
5. *Power rule:* $T_x^\alpha(x^s) = sx^{s-\alpha}$ *for* $s \in \mathbb{R}$.
6. *If in addition f is differentiable, then* $(T_x^\alpha f)(x) = x^{1-\alpha} f'(x)$.

Definition 1 (α-fractional integral)**.** [8,9] Let f be a continuous function on $[0, \infty)$, $x > a \geq 0$. Then for $\alpha \in (0, 1]$,

$$I_a^\alpha\Big(f(x)\Big) = I_a^1\Big(x^{\alpha-1} f(x)\Big) = \int_a^x f(t) \frac{dt}{t^{1-\alpha}} = \int_a^x f(t) d_\alpha t. \tag{8}$$

Here the integral is the usual Riemann improper integral. It is easy to show that: $T_x^\alpha(I_a^\alpha f)(x) = f(x)$, whenever f is continuous in the domain of I^α. Likewise, $I_a^\alpha(T_x^\alpha f)(x) = f(x) - f(a)$.

Lemma 1 (Chain rule)**.** [8,9] *Assume f is α-differentiable with respect to v, and v is α-differentiable with respect to x. For $\alpha \in (0, 1]$, we have*

$$T_x^\alpha\Big(f(v)\Big)(x) = \Big(T_v^\alpha f\Big)(v) \cdot v^{\alpha-1}\Big(T_x^\alpha v\Big)(x).$$

Lemma 2 (Integration by Parts formula)**.** [8,9] *Suppose $f, g : [0, \infty) \to \mathbb{R}$ are α-differentiable at a point $x > 0$ for $\alpha \in (0, 1]$. Then*

$$\int_0^\infty \Big(T_x^\alpha f(x)\Big) g(x) d_\alpha x = f(x) g(x)\Big|_0^\infty - \int_0^\infty f(x)\Big(T_x^\alpha g(x)\Big) d_\alpha x.$$

It is desirable to extend the above definitions and properties to the case of differential of a function of several variables since many physical processes are modelled based on equations involving partial derivatives.

Definition 2. [10] Let f be a function of n-variables $x_1, x_2, \cdots, x_n$. Then the conformable partial derivative of f of order $\alpha \in (0,1]$ with respect to variable x_k, denoted by $D^{\alpha}_{x_k} := \frac{\partial^{\alpha}}{\partial x_k^{\alpha}}$, is defined as

$$\begin{aligned} D^{\alpha}_{x_k} f(\bar{x}) &= \frac{\partial^{\alpha} f}{\partial x_k^{\alpha}}(x_1, x_2, \cdots, x_n) \\ &= \lim_{\varepsilon \to 0} \frac{f(x_1, \cdots, x_{k-1}, x_k + \varepsilon x_k^{1-\alpha}, \cdots, x_n) - f(x_1, x_2, \cdots, x_n)}{\varepsilon}, \end{aligned}$$

where, $\bar{x} = (x_1, x_2, \cdots, x_n), k = 1, 2, \cdots, n$.

In the case that f has conformable partial derivative of order α with respect to each variable x_k, $k = 1, \cdots, n$. Then conformable vector can be defined at a point q by

$$D^{\alpha}_{x} f(q) = \Big(D^{\alpha}_{x_1}(f(q)), D^{\alpha}_{x_2}(f(q)), \cdots, D^{\alpha}_{x_n}(f(q)) \Big).$$

Consider the scalar field $f(\bar{x})$ and the vector field $\overrightarrow{F}(\bar{x})$ that are assumed to posses conformable partial derivative of order α with respect to all components x_k, $k = 1, 2, \cdots, n$.

Definition 3 (Conformable gradient)**.** The conformable gradient of order α as a vector field is given by

$$D^{\alpha}_{x} f(x) = \sum_{k=1}^{n} \Big(D^{\alpha}_{x_k} f \Big) e_k,$$

where e_k is the unit vector in the direction of k. The conformable gradient of order α as a scalar field is given by

$$D^{\alpha}_{x} f(x) = \sum_{k=1}^{n} (D^{\alpha}_{x_k} F_k).$$

Remark 1. We note that conformable partial derivative (also conformable gradient) satisfies partial derivative versions of Theorem 1, Lemma 1 and Lemma 2.

Definition 4. By the above discussion, anisotropic conformable fractional differential operator (5) is therefore defined for continuous α-differentiable function f as

$$\sum_{k=1}^{n} D^{\alpha}_{x_k} \Big(|D^{\alpha}_{x_k} f(x)|^{p_k - 2} D^{\alpha}_{x_k} f(x) \Big) \ \text{for } \alpha \in (0, \infty], \ \ p_k > 1.$$

We can now study fractional elliptic partial differential equations of the form

$$\sum_{k=1}^{n} D^{\alpha}_{x_k} \Big(|D^{\alpha}_{x_k} f(x)|^{p_k - 2} D^{\alpha}_{x_k} f(x) \Big) = g(x, f), \quad x \in \Omega \subseteq \mathbb{R}^n,$$

in the appropriate fractional function spaces.

2.2. Conformable Green's theorem

Integration by parts formula, divergence theorem and Green's theorem within the framework of conformable fractional derivative will be applied severally. Then, there is a need to develop compatible divergence and Green's theorems for anisotropic conformable partial derivatives of order α.

Definition 5. [10] Let the vector field F has the conformable partial derivatives of order β on $\Omega \subseteq \mathbb{R}^n$. Then we denote by P_x^β the vector

$$P_x^\beta F = \sum_{i=1}^{n} \left\{ e_{x_i}^T \left(D_x^\beta (F)^T \right) e_{x_i} \right\} e_{x_i} = \sum_{i=1}^{n} \frac{\partial^\beta F_{x_i}}{\partial x_{x_i}^\beta} e_{x_i}.$$

Definition 6. [10] Let the vector field F has the conformable partial derivatives of order β on an open region Ω, $V \subseteq \Omega$ be simply connected and S is the boundary surface of V which is positively outward oriented. Then

$$\iiint_V D_x^\alpha F \, d_\alpha V = \iint_S P_x^{\alpha-1} F \cdot n \, d_\alpha S.$$

Remark 2. This supports the fact that the conformable integral is anti-derivative of conformable derivative.

Lemma 3 (Conformable Green's Theorem). [10] *Let $C \subset \mathbb{R}^2$ be a simple positively oriented, piecewise smooth and closed region. Let Ω be the interior of C. If $f = f(x, y)$ and $g = g(x, y)$ have continuous conformable partial derivatives on Ω. Then*

$$\iint_\Omega \left(D_x^\alpha g - D_y^\alpha f \right) d_\alpha S = \int_C D_y^{\alpha-1} f d_\alpha x + D_x^{\alpha-1} g d_\alpha y. \tag{9}$$

In what follows we consider a bounded open region $\Omega \subset \mathbb{R}^n$ with piecewise smooth and simple boundary. Note that the condition for the boundary to be simple amounts to $\partial\Omega$ being orientable. We say $\Omega \subset \mathbb{R}^n$ with this condition is said to be compatible.

2.2.1. Green's identities

Suppose $\Omega \subset \mathbb{R}^n$ is compatible, and α-partial conformable fractional derivatives $D_{x_k}^\alpha$ satisfy

$$\sum_{k=1}^{n} \int_\Omega D_{x_k}^\alpha g_k(x) d_\alpha x = \sum_{k=1}^{n} \int_{\partial\Omega} D_{x_k}^{\alpha-1} \left(D_{x_k}^\alpha g_k(x) \right) \cdot \nu d_\alpha S \tag{10}$$

for all $g_k \in \mathscr{D}_\alpha(\bar{\Omega})$, $k = 1, 2, \cdots, n$. Here $\bar{\Omega} = \Omega \cup \partial\Omega$, $\mathscr{D}_\alpha(\bar{\Omega})$ denotes the space of all functions with continuous α-partial conformable fractional derivative on Ω upto the boundary $\partial\Omega$, and ν is the outward pointing unit normal on $\partial\Omega$. Next we state Green's first and second identities for α-partial conformable fractional derivative.

Theorem 2 (Green's identities). [27] *Let $\Omega \subset \mathbb{R}^n$ be compatible, we have*

1. *Green's first identity: Let $u, v \in \mathscr{D}_\alpha(\bar{\Omega})$, then*

$$\int_\Omega \left(D_x^\alpha u D_x^\alpha v + v D_x^\alpha D_x^\alpha u \right) d_\alpha x = \int_{\partial\Omega} v D_x^{\alpha-1} D_x^\alpha u \cdot \nu d_\alpha S. \tag{11}$$

2. *Green's second identity: Let $u, v \in \mathscr{D}_\alpha(\bar{\Omega})$, then*

$$\int_\Omega \left(u D_x^\alpha u D_x^\alpha v - v D_x^\alpha D_x^\alpha u \right) d_\alpha x = \int_{\partial\Omega} \left(u D_x^{\alpha-1} D_x^\alpha v \cdot \nu - v D_x^{\alpha-1} D_x^\alpha u \cdot \nu \right) d_\alpha S. \tag{12}$$

Remark 3. If $v = 1$ in these Green's identities we obtain the following analogue of Gauss mean value formula for α-conformable harmonic function satisfying $D_x^\alpha D_x^\alpha u = 0$ in a compatible domain: $\int_{\partial\Omega} D_x^{\alpha-1} \left(D_x^\alpha u \right) \cdot \nu d_\alpha S = 0$.

2.3. Conformable fractional function spaces

Definition 7. Let $1 \leq p < \infty$, and $L_\alpha^p(\Omega)$ denote the space of all functions $u : \Omega \to \mathbb{R}$ satisfying the condition $\left(\int_\Omega |u(x)|^p d_\alpha \right)^{\frac{1}{p}} < \infty$. Associated with the norm

$$\|u\|_{p,\alpha} := \left(\int_\Omega |u(x)|^p d_\alpha \right)^{\frac{1}{p}},$$

the space $L^p_\alpha(\Omega)$ is a Banach space.

In what follows, we define the Sobolev space $\mathscr{D}^{p_k}_\alpha(\Omega)$ of all functions which are absolute continuous and $D^\alpha_{x_k}u(x) \in L^{p_k}_\alpha(\Omega)$. We will work in the closure of C_0^∞, denoted by $\mathring{\mathscr{D}}^{p_k}_\alpha(\Omega)$ (the space of all absolute continuous α-partial conformable fractional differentiable function with compact support), which is a separable, reflexive Banach space with respect to the norm function

$$\|u\|_{\mathscr{D}^{p_k}_\alpha(\Omega)} := \sum_{k=1}^n \left(\int_\Omega |D^\alpha_{x_k}u|^{p_k} d_\alpha x \right)^{\frac{1}{p_k}}.$$

2.4. Anisotropic conformable fractional eigenproblem

Consider the conformable fractional Dirichlet eigenvalue problem

$$\begin{cases} T^\alpha_x T^\alpha_x u(x) + \lambda u(x) = 0, & x \in (0,a), \ \ a > 0, \\ u(0) = u(a) = 0, \end{cases} \tag{13}$$

with $\alpha \in (0,1]$. Here λ is an eigenvalue and $u(x)$ is the associated eigenfunction. Conformable fractional Sturm Liouville problem (which include (13) as a special case) has been studied in [34–36] through series of methods. Denote $\mathscr{L}_\alpha$ by

$$\mathscr{L}_\alpha := -T^\alpha_x T^\alpha_x,$$

by [35, Lemma 3.6], we know that $\mathscr{L}_\alpha$ is self-adjoint on $L^2_\alpha((0,a))$. Following these references, it is obvious that (13) has infinitely many real and simple eigenvalues that can be arranged in an increasing order: $\lambda_1 < \lambda_2 < \cdots \lambda_n \cdots$. Moreso, the associated eignfunctions to distinct eigenvalues form an α-orthogonal basis in $L^2_\alpha((0,\pi))$. By a direct computation, it is known that the eigenfunctions of (13) for $x \in (0,1)$ are $\sin(n\pi x^\alpha)$ and the corresponding eigenvalues are $\alpha^2 n^2 \pi^2$.

Now, let $x = (x_1, x_2, \cdots, x_n)$, where $x_k \in \Omega_k \subset \mathbb{R}^{n_k}$, Ω_k being a compartible domain. The anisotropic conformable eigenvalue problem is given by

$$\begin{cases} -\sum_{k=1}^n D^\alpha_{x_k}(|D^\alpha_{x_k}u|^{p_k-2}D^\alpha_{x_k}u) = \lambda \sum_{k=1}^n \omega_k(x)|u|^{p_k-2}u & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{14}$$

where $0 < \omega_k(x) \in L^\infty(\Omega)$ for $k = 1, 2, \cdots, n$ and $\Omega := \prod_{k=1}^n \Omega_k$.

By the (weak)-positive solution of (14) we refer to a positive solution $u \in \mathring{\mathscr{D}}^{p_k}_\alpha(\Omega)$ solving

$$\sum_{k=1}^n \int_\Omega |D^\alpha_{x_k}u|^{p_k-2} D^\alpha_{x_k}u D^\alpha_{x_k}\psi \, d_\alpha x = \lambda \sum_{k=1}^n \int_\Omega \omega_k(x)|u|^{p_k-2}u\psi \, d_\alpha x,$$

for all $\psi \in C_0^\infty(\Omega)$.

It can be proved that (14) has a unique eigenvalue $\lambda = \lambda^1_\alpha$ with the property of having a positive associated eigenfunction $u_1 \in \mathring{\mathscr{D}}^{p_k}_\alpha(\Omega)$, which is called the principal eigenfunction. However, one can show that the first eigenvalue, λ^1_α, is simple and isolated via the conformable fractional Rayleigh quotient

$$\lambda^1_\alpha = \inf_{0<u\in\mathring{\mathscr{D}}^{p_k}_\alpha(\Omega)} \frac{\sum_{k=1}^n \int_\Omega |D^\alpha_{x_k}|^{p_k} d_\alpha x}{\sum_{k=1}^n \int_\Omega |u|^{p_k} d_\alpha x}.$$

Problem of this nature as described above has been studied in the context fractional Laplacian by several authors such as [37,38]. The Rayleigh quotient above can be used to obtain upper estimates for the first eigenvalue. For example [34], consider $n = 1$, $\Omega = [0,1]$, using a trial function $u(x) = x^\alpha - x^{2\alpha}$ with boundary conditions $u(0) = u(1) = 0$. we have for $p_k = 2$:

$$\lambda_\alpha^1 \leq \frac{\int_0^1 [D_x^\alpha (x^\alpha - x^{2\alpha})]^2 x^{\alpha-1} dx}{\int_0^1 [x^\alpha - x^{2\alpha}]^2 x^{\alpha-1} dx} = \frac{\alpha^2 \int_0^1 [1-2x^\alpha]^2 x^{\alpha-1} dx}{\int_0^1 [x^\alpha - x^{2\alpha}]^2 x^{\alpha-1} dx} = 10\alpha^2.$$

So this chain of inequalities

$$\pi^2\alpha^2 = \lambda_\alpha^1 \leq \bar{\lambda} = 10\alpha^2, \tag{15}$$

gives a valid upper estimate.

3. Main results: Anisotropic Picone identity and applications

In this section, we prove some Picone type identities for anisotropic fractional gradient operator on a compatible domain, and then give their applications to proving Sturmian comparison principle and Liouville type theorems.

3.1. Anisotropic conformable Picone type identity

Theorem 3. *Let u and v be α-order conformable differentiable functions a.e. in an open domain $\Omega \subset \mathbb{R}^n$, such that $u, v > 0$ are non-constant. Let $f_k : (0,\infty) \to (0,\infty)$ be a conformable differentiable function such that*

$$f_k^\alpha(y) \geq (p_k - 1)\, [f_k(y)]^{\frac{p_k-2}{p_k-1}}\, y^{1-\alpha} \text{ for all } y.$$

For $0 < \alpha \leq 1$ and $p_k > 1$, $k = 1, 2, \cdots, n$, define

$$\mathscr{A}(u,v) = \sum_{k=1}^n |D_{x_k}^\alpha u|^{p_k} + \sum_{k=1}^n \frac{|u|^{p_k} f_k^\alpha(v)}{[f_k(v)]^2} v^{\alpha-1} |D_{x_k}^\alpha v|^{p_k} - \sum_{k=1}^n p_k \frac{|u|^{p_k-2}u}{f_k(v)} |D_{x_k}^\alpha v|^{p_k-2} D_{x_k}^\alpha v D_{x_k}^\alpha u,$$

and

$$\mathscr{B}(u,v) = \sum_{k=1}^n |D_{x_k}^\alpha u|^{p_k} - \sum_{k=1}^n D_{x_k}^\alpha \left(\frac{|u|^{p_k}}{f_k(v)} \right) |D_{x_k}^\alpha v|^{p_k-2} D_{x_k}^\alpha v.$$

Then $\mathscr{A}(u,v) = \mathscr{B}(u,v) \geq 0$. Moreover, $\mathscr{A}(u,v) = 0$ a.e. in Ω if and only if $D_{x_k}^\alpha(u/v) = 0$ for all $k = 1, 2, \cdots, n$.

Remark 4. Consider a positive continuously conformable differentiable function f_k on the interval $(0,\infty)$ satisfying the differential inequality

$$f_k^\alpha(y) \geq (p_k - 1)\, [f_k(y)]^{\frac{p_k-2}{p_k-1}}\, y^{1-\alpha}.$$

1. Two examples of functions that satisfy the above inequality are $f_k(y) = y^{p_k-1}$ and $f_k(y) = e^{(p_k-1)y}$, while the former maintains equality. Here, by $f_k^\alpha(y)$ we mean α-conformable derivative of $f_k(y)$ with respect to y, that is, $T_y^\alpha(f_k)(y) := f_k^\alpha(y)$.
2. Equality in the above inequality, which implies $f_k(y) = y^{p_k-1}$, is one of the conditions to obtain equality part of the Picone inequality, i.e., $\mathscr{A}(u,v) \geq 0$. (other two conditions are equalities in Young's inequality and Cauchy-Schwarz inequality).
3. It is instructive to note that no restriction is placed on the sign of u as in many literature. The case $u \geq 0$ and $f_k(v) = v^{p_k-1}$ reduces the above theorem to the Picone identities in [27].

Proof of Theorem 3. By the quotient and chain rules for conformable partial derivative we compute

$$D_{x_k}^\alpha \left(\frac{|u|^{p_k}}{f_k(v)} \right) = \frac{D_{x_k}^\alpha |u|^{p_k}}{f_k(v)} - \frac{|u|^{p_k} D_{x_k}^\alpha [f_k(v)]}{[f_k(v)]^2} = \frac{p_k |u|^{p_k-2} u D_{x_k}^\alpha u}{f_k(v)} - \frac{|u|^{p_k} v^{\alpha-1} f_k^\alpha(v) D_{x_k}^\alpha v}{[f_k(v)]^2}.$$

Substituting this into the expression of $\mathscr{B}(u,v)$ we get $\mathscr{B}(u,v) = \mathscr{A}(u,v)$.

To prove that $\mathscr{A}(u,v) \geq 0$, we write

$$\mathscr{A}(u,v) = \sum_{k=1}^{n} |D_{x_k}^{\alpha} u|^{p_k} - \sum_{k=1}^{n} p_k \frac{|u|^{p_k-2}u}{f_k(v)} |D_{x_k}^{\alpha} v|^{p_k-1} |D_{x_k}^{\alpha} u| + \sum_{k=1}^{n} \frac{|u|^{p_k} f_k^{\alpha}(v)}{[f_k(v)]^2} v^{\alpha-1} |D_{x_k}^{\alpha} v|^{p_k}$$
$$+ \sum_{k=1}^{n} p_k \frac{|u|^{p_k-2}u}{f_k(v)} |D_{x_k}^{\alpha} v|^{p_k-2} \Big\{ |D_{x_k}^{\alpha} v||D_{x_k}^{\alpha} u| - D_{x_k}^{\alpha} v D_{x_k}^{\alpha} u \Big\},$$

which further implies

$$\mathscr{A}(u,v) = \sum_{k=1}^{n} p_k \left\{ \frac{1}{p_k} |D_{x_k}^{\alpha} u|^{p_k} + \frac{1}{q_k} \left[\frac{1}{f_k(v)} \left(u |D_{x_k}^{\alpha} v| \right)^{p_k-1} \right]^{q_k} \right\} - \sum_{k=1}^{n} p_k \frac{|u|^{p_k-2}u}{f_k(v)} |D_{x_k}^{\alpha} v|^{p_k-1} |D_{x_k}^{\alpha} u|$$
$$+ \sum_{k=1}^{n} \left\{ \frac{|u|^{p_k} f_k^{\alpha}(v)}{[f_k(v)]^2} v^{\alpha-1} |D_{x_k}^{\alpha} v|^{p_k} - \frac{p_k}{q_k} \left[\frac{1}{f_k(v)} \left(u |D_{x_k}^{\alpha} v| \right)^{p_k-1} \right]^{q_k} \right\}$$
$$+ \sum_{k=1}^{n} p_k \frac{|u|^{p_k-2}u}{f_k(v)} |D_{x_k}^{\alpha} v|^{p_k-2} \Big\{ |D_{x_k}^{\alpha} v||D_{x_k}^{\alpha} u| - D_{x_k}^{\alpha} v D_{x_k}^{\alpha} u \Big\}. \tag{16}$$

Recall from the Young's inequality that for real numbers $a, b \geq 0$ and exponents $p_k > 1$, $q_k > 1$ satisfying $1/q_k + 1/p_k = 1$:

$$ab \leq (1/p_k) a^{p_k} + (1/q_k) b^{q_k}. \tag{17}$$

with equality if and only if $a^{p_k} = b^{q_k}$ for all $k = 1, 2, \cdots, n$. Now choosing a and b as follows: $a = |D_{x_k}^{\alpha} u|$ and $b = \frac{1}{f_k(v)} \left(u |D_{x_k}^{\alpha} v| \right)^{p_k-1}$, we have by (17) that

$$\frac{1}{p_k} |D_{x_k}^{\alpha} u|^{p_k} + \frac{1}{q_k} \left[\frac{1}{f_k(v)} \left(u |D_{x_k}^{\alpha} v| \right)^{p_k-1} \right]^{q_k} \geq \frac{|u|^{p_k-2}u}{f_k(v)} |D_{x_k}^{\alpha} v|^{p_k-1} |D_{x_k}^{\alpha} u|,$$

which clearly implies that line one on the RHS of (16) is non-negative, but equal to zero only if

$$|D_{x_k}^{\alpha} u| = \frac{1}{[f_k(v)]^{q_k/p_k}} \left(u |D_{x_k}^{\alpha} v| \right). \tag{18}$$

Applying the condition $f_k^{\alpha}(y) \geq (p_k - 1) \left[f_k(y) \right]^{\frac{p_k-2}{p_k-1}} y^{1-\alpha}$ we obtain

$$\frac{|u|^{p_k} f_k^{\alpha}(v)}{[f_k(v)]^2} v^{\alpha-1} |D_{x_k}^{\alpha} v|^{p_k} - \frac{p_k}{q_k} \left[\frac{1}{f_k(v)} \left(u |D_{x_k}^{\alpha} v| \right)^{p_k-1} \right]^{q_k} \geq 0,$$

which yields that line two on the RHS of (16) is non-negative, but equal to zero only if

$$v^{\alpha-1} f_k^{\alpha}(v) = (p_k - 1) \left[f_k(v) \right]^{\frac{p_k-2}{p_k-1}}. \tag{19}$$

Clearly line three on the RHS of (16) is non-negative by the virtue of Cauchy-Schwarz inequality in the form

$$D_{x_k}^{\alpha} u D_{x_k}^{\alpha} v \leq |D_{x_k}^{\alpha} u||D_{x_k}^{\alpha} v|.$$

Combining all of these shows that $\mathscr{A}(u,v) \geq 0$.

Observe further that equality is attained in the relation $\mathscr{A}(u,v) \geq 0$ if and only if (18) and (19) together with below (20)

$$|D_{x_k}^{\alpha} u||D_{x_k}^{\alpha} v| = D_{x_k}^{\alpha} u D_{x_k}^{\alpha} v, \quad k = 1, 2, \cdots, n, \tag{20}$$

hold simultaneously.

Solving for (19) we get $f_k(v) = v^{p_k-1}$. Suppose $\mathscr{A}(u,v)(x_0) = 0$ and $u(x_0) \neq 0$, then the inequality (18) with $f_k(v) = v^{p_k-1}$ and (20) imply $D^{\alpha}_{x_k}(u/v) = 0$. If $u(x_0) = 0$, then $D^{\alpha}_{x_k}u = 0$ a.e. on $\{u(x) = 0\}$ and $D^{\alpha}_{x_k}(u/v)(x_0) = 0$. □

3.2. Applications

3.2.1. Sturmian Comparison Principle

It is well known that comparison principles do play significant roles in the qualitative study of partial differential equations. Here, we prove a nonlinear version of the Sturmian comparison principle for anisotropic conformable elliptic partial differential equations.

Theorem 4. *Let $h_k(x)$ and $H_k(x)$ be nonnegative weight functions such that $h_k(x) < H_k(x)$ for each $k = 1, 2, \cdots n$ in a compatible domain $\Omega \subset \mathbb{R}^n$. Let $f_k : (0,\infty) \to (0,\infty)$ be a continuous α-differentiable function such that $y^{\alpha-1} f^{\alpha}_k(y) \geq (p_k - 1)\,[f_k(y)]^{\frac{p_k-2}{p_k-1}}$.*

Suppose there exists a positive solution u (that is, $0 < u \in \mathring{\mathscr{D}}^{p_k}_{\alpha}(\Omega)$) satisfying

$$\begin{cases} -\sum_{k=1}^{n} D^{\alpha}_{x_k}\left(|D^{\alpha}_{x_k}u|^{p_k-2} D^{\alpha}_{x_k}u\right) = \sum_{k=1}^{n} h_k(x)|u|^{p_k-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{21}$$

for $k = 1, 2, \cdots n$. Then any nontrivial solution v of the weighted anisotropic elliptic equation

$$\begin{cases} -\sum_{k=1}^{n} \frac{u^{p_k}}{f_k(v)} D^{\alpha}_{x_k}\left(|D^{\alpha}_{x_k}v|^{p_k-2} D^{\alpha}_{x_k}v\right) = \sum_{k=1}^{n} H_k(x)u^{p_k} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{22}$$

must change sign.

Proof. Suppose that v satisfying (22) does not change sign. Without loss of generality, we assume that $v > 0$. Then by Theorem 3 we have

$$0 \leq \int_{\Omega} \mathscr{A}(u,v)d_{\alpha}x = \int_{\Omega} \mathscr{B}(u,v)d_{\alpha}x,$$

thereby implying

$$0 \leq \sum_{k=1}^{n}\int_{\Omega} |D^{\alpha}_{x_k}u|^{p_k} d_{\alpha}x - \sum_{k=1}^{n}\int_{\Omega} D^{\alpha}_{x_k}\left(\frac{u^{p_k}}{f_k(v)}\right)|D^{\alpha}_{x_k}v|^{p_k-2} D^{\alpha}_{x_k}v d_{\alpha}x.$$

By the divergence theorem we arrive at

$$\begin{aligned} 0 \leq &\sum_{k=1}^{n}\int_{\Omega} |D^{\alpha}_{x_k}u|^{p_k} d_{\alpha}x + \sum_{k=1}^{n}\int_{\Omega} \frac{u^{p_k}}{f_k(v)} D^{\alpha}_{x_k}\left(|D^{\alpha}_{x_k}v|^{p_k-2} D^{\alpha}_{x_k}v\right) d_{\alpha}x \\ &+ \sum_{k=1}^{n}\int_{\Omega} \frac{u^{p_k}}{f_k(v)} D^{\alpha-1}_{x_k}\, D^{\alpha}_{x_k}\left(D^{\alpha}_{x_k}v|^{p_k-2} D^{\alpha}_{x_k}v\right) \cdot \nu d_{\alpha}A. \end{aligned} \tag{23}$$

Since $u \in \mathring{\mathscr{D}}^{p_k}_{\alpha}(\Omega)$ is a positive solution of (21) we know that u vanishes on the boundary, and by definition of solution we know that (21) implies

$$\sum_{k=1}^{n}\int_{\Omega} |D^{\alpha}_{x_k}u|^{p_k} d_{\alpha}x = \sum_{k=1}^{n}\int_{\Omega} h_k u^{p_k} d_{\alpha}x. \tag{24}$$

Substituting (22) and (24) into (23), together with the fact that u and v belong to the space $\mathring{\mathscr{D}}_{\alpha}^{p_k}(\Omega)$, yields

$$0 \leq \sum_{k=1}^{n} \int_{\Omega} [h_k(x) - H_k(x)]\, u^{p_k} d_{\alpha} x,$$

which is a contradiction to the assumption $h_k(x) < H_k(x)$ for each $k = 1, 2, \cdots, n$. Hence, v changes sign. □

Corollary 1. *Suppose the hypothesis of Theorem 3 hold. If $0 < u \in \mathring{\mathscr{D}}_{\alpha}^{p_k}(\Omega)$ is a solution of (21), then any nontrivial solution of*

$$\begin{cases} -D_{x_k}^{\alpha}\left(|D_{x_k}^{\alpha} v|^{p_k-2} D_{x_k}^{\alpha} v\right) = H_k(x) f_k(v) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{25}$$

changes sign in Ω for each $k = 1, 2, \cdots, n$.

Proof. Substituting (24) and (25) into (23) yields a contradiction and the conclusion follows at once. □

3.2.2. Liouville type principle

The next application is the proof of a Liouville type result for anisotropic conformable elliptic partial differential equations. Here, Λ is a positive constant bigger than the principal eigenvalue λ_{α}^{1}, comparable to (15).

Theorem 5. *Suppose $p_k > 1$, $\Lambda > 0$ is a constant and $f_k : (0, \infty) \to (0, \infty)$ is a continuous α-differentiable function such that $y^{\alpha-1} f_k^{\alpha}(y) \geq (p_k - 1)\, [f_k(y)]^{\frac{p_k-2}{p_k-1}}$. Then, the anisotropic conformable elliptic partial differential inequality*

$$\begin{cases} -\sum_{k=1}^{n} D_{x_k}^{\alpha}\left(|D_{x_k}^{\alpha} v|^{p_k-2} D_{x_k}^{\alpha} v\right) \geq \Lambda \sum_{k=1}^{n} f_k(v), & x \in \Omega, \\ v = 0, & x \in \partial\Omega, \end{cases} \tag{26}$$

has no positive solution in $\mathring{\mathscr{D}}_{\alpha}^{p_k}(\Omega)$.

Proof. Assume that $v > 0$ is a solution to (26). Let $\lambda_{\alpha}^{1}(\Omega)$ be the first eigenvalue corresponding to the first eigenfunction $u_1 \in \mathring{\mathscr{D}}_{\alpha}^{p_k}(\Omega)$ such that $\lambda_{\alpha}^{1}(\Omega) < \Lambda$. Taking $\frac{|u_1|^{p_k}}{f_k(v)}$ as a test function, which is valid since $\frac{|u_1|^{p_k}}{f_k(v)} \in \mathring{\mathscr{D}}_{\alpha}^{p_k}(\Omega)$ is admissible in the weak formulation of (26). Therefore, by (26) we have

$$\Lambda \sum_{k=1}^{n} \int_{\Omega} |u_1|^{p_k} dx - \sum_{k=1}^{n} \int_{\Omega} |D_{x_k}^{\alpha} v|^{p_k-2} D_{x_k}^{\alpha} v D_{x_k}^{\alpha} \left(\frac{|u_1|^{p_k}}{f_k(v)}\right) dx \leq 0,$$

which by Theorem 3 implies that

$$\Lambda \sum_{k=1}^{n} \int_{\Omega} |u_1|^{p_k} dx - \sum_{k=1}^{n} \int_{\Omega} |D_{x_k}^{\alpha} u_1|^{p_k} dx \leq -\int_{\Omega} \mathscr{A}(u_1, v) dx \leq 0.$$

Hence, by the last inequality and the hypothesis on the first eigenvalue, we have

$$\Lambda \leq \frac{\sum_{k=1}^{n} \int_{\Omega} |D_{x_k}^{\alpha} u_1|^{p_k} dx}{\sum_{k=1}^{n} \int_{\Omega} |u_1|^{p_k} dx} = \lambda_{\alpha}^{1}(\Omega) < \Lambda,$$

which is a contradiction.
□

Next we prove the following Liouville type theorem for anisotropic conformable elliptic partial differential system.

Theorem 6. *Let $g(v)$ be α-conformable integrable in $\Omega \subseteq \mathbb{R}^n$. Suppose (u,v) is a pair of solution to an anisotropic conformable elliptic partial differential system*

$$\begin{cases} -\sum_{k=1}^{n} D^{\alpha}_{x_k}(|D^{\alpha}_{x_k}u|^{p_k-2}D^{\alpha}_{x_k}u) = g(v), & x \in \Omega, \\ -\sum_{k=1}^{n} D^{\alpha}_{x_k}\left(\frac{u^{p_k}}{f_k(v)}\right)|D^{\alpha}_{x_k}v|^{p_k-2}D^{\alpha}_{x_k}v = ug(v), & x \in \Omega, \\ u > 0,\ v > 0, & x \in \Omega, \\ u = 0,\ v = 0, & x \in \partial\Omega, \end{cases} \tag{27}$$

where $f_k(v) > 0$, $f_k^{\alpha}(v) \geq (p_k - 1)\left[f_K(v)\right]^{\frac{p_k-2}{p_k-1}} v^{1-\alpha}$ and $p_k \geq 1$ for all $k = 1, 2, \cdots, n$. Then $u = cv$ a.e. in Ω for some constant $c > 0$.

Proof. Multiplying the first equation in System (27) by $0 < u \in \mathring{\mathscr{D}}^{p_k}_{\alpha}(\Omega)$, integrating over Ω and applying the divergence theorem, we have

$$\sum_{k=1}^{n}\int_{\Omega}|D^{\alpha}_{x_k}u|^{p_k}d_{\alpha}x = \int_{\Omega}g(v)ud_{\alpha}x. \tag{28}$$

By Theorem 3, (28) and the second equation in System (27), we obtain

$$\begin{aligned} \int_{\Omega}\mathscr{A}(u,v)d_{\alpha}x &= \int_{\Omega}\mathscr{B}(u,v)d_{\alpha}x \\ &= \sum_{k=1}^{n}\int_{\Omega}|D^{\alpha}_{x_k}u|^{p_k}d_{\alpha}x - \sum_{k=1}^{n}\int_{\Omega}D^{\alpha}_{x_k}\left(\frac{u^{p_k}}{f_k(v)}\right)|D^{\alpha}_{x_k}v|^{p_k-2}D^{\alpha}_{x_k}vd_{\alpha}x \\ &= \int_{\Omega}\left[ug(v) - ug(v)\right]d_{\alpha}x = 0. \end{aligned}$$

Thus, $\mathscr{A}(u,v) = 0$ which implies $D^{\alpha}_{x_k}(u/v) = 0$ (that is, $u/v = c$ a.e. in Ω for some constant $c > 0$). □

3.2.3. Anisotropic quasilinear system with singular nonlinearities

Lastly, we show that Theorem 3 yields a linear relation between u and v solving anisotropic quasilinear system with singular nonlinearities. Given the following system of anisotropic conformable elliptic equations

$$\begin{cases} -\sum_{k=1}^{n} D^{\alpha}_{x_k}(|D^{\alpha}_{x_k}u|^{p_k-2}D^{\alpha}_{x_k}u) = \sum_{k=1}^{n} f_k(v), & x \in \Omega, \\ -\sum_{k=1}^{n} D^{\alpha}_{x_k}(|D^{\alpha}_{x_k}v|^{p_k-2}D^{\alpha}_{x_k}v) = \sum_{k=1}^{N}\frac{[f_k(v)]^2}{u^{p_k-1}}, & x \in \Omega, \\ f_k(v) > 0,\ u > 0,\ v > 0, & x \in \Omega, \\ f_k(v) = 0,\ u = 0,\ v = 0, & x \in \partial\Omega. \end{cases} \tag{29}$$

Theorem 7. *Let (u,v) be a pair of solutions to (29) and f be α-conformable differentiable such that $f_k^{\alpha}(v) \geq (p_k - 1)\left[f_k(v)\right]^{\frac{p_k-2}{p_k-1}} v^{1-\alpha}$, and $p_k \geq 1$ for all $k = 1, 2, \cdots, n$. Then $u = cv$ a.e. in Ω for some constant c.*

Proof. Since (u,v) is a pair of solutions to (29). It follows that

$$\sum_{k=1}^{n}\int_{\Omega}|D^{\alpha}_{x_k}u|^{p_k-2}D^{\alpha}_{x_k}uD^{\alpha}_{x_k}\phi_1d_{\alpha}x = \sum_{k=1}^{n}\int_{\Omega}f_k(v)\phi_1d_{\alpha}x, \tag{30}$$

$$\sum_{k=1}^{n}\int_{\Omega}|D^{\alpha}_{x_k}v|^{p_k-2}D^{\alpha}_{x_k}vD^{\alpha}_{x_k}\phi_2d_{\alpha}x = \sum_{k=1}^{n}\int_{\Omega}\frac{[f_k(v)]^2}{u^{p_k-1}}\phi_2d_{\alpha}x, \tag{31}$$

for any pair of positive functions $\phi_1, \phi_2 \in \mathring{\mathscr{D}}^{p_k}_{\alpha}(\Omega)$.

Letting $\phi_1 \to u$ and $\phi_2 = \frac{u^{p_k}}{f_k(v)}$ in (30) and (31), respectively, we obtain

$$\sum_{k=1}^{n} \int_{\Omega} |D^{\alpha}_{x_k} u|^{p_k} d_\alpha x = \sum_{k=1}^{n} \int_{\Omega} f_k(v) u d_\alpha x$$
$$= \sum_{k=1}^{n} \int_{\Omega} |D^{\alpha}_{x_k} v|^{p_k - 2} D^{\alpha}_{x_k} v D^{\alpha}_{x_k} \left(\frac{u^{p_k}}{f_k(v)} \right) d_\alpha x.$$

Hence,

$$0 = \sum_{k=1}^{n} \int_{\Omega} |D^{\alpha}_{x_k} u|^{p_k} d_\alpha x - \sum_{k=1}^{n} \int_{\Omega} |D^{\alpha}_{x_k} v|^{p_k - 2} D^{\alpha}_{x_k} v D^{\alpha}_{x_k} \left(\frac{u^{p_k}}{f_k(v)} \right) d_\alpha x = \int_{\Omega} \mathscr{B}(u, v) d_\alpha x,$$

which implies that $\mathscr{B}(u, v) = 0$ using the nonnegativity of $\mathscr{B}(u, v)$. However, by Theorem 3, $\mathscr{B}(u, v) = \mathscr{A}(u, v) = 0$ yields $u = cv$ a.e. in Ω for some constant c. This completes the proof. □

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