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Existence and multiplicity of homoclinic solutions for subquadratic p -Laplacian fourth-order differential equations

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Abstract: This article concerns the existence and multiplicity of homoclinic solutions for the following fourth-order differential equation with p -Laplacian

$$\left(|u''(t)|^{p-2} u''(t) \right)'' - \omega \left(|u'(t)|^{p-2} u'(t) \right)' + V(t) |u(t)|^{p-2} u(t) = f(t, u(t)),$$

where $p > 1$, ω is a constant, $V \in C(\mathbb{R}, \mathbb{R})$ is noncoercive and $f \in C(\mathbb{R}^2, \mathbb{R})$ is of subquadratic growth at infinity. Some results are proved using variational methods, the minimization theorem and the generalized Clark's theorem. Recent results in the literature are extended and improved.

Keywords: Fourth-order differential equations, p -Laplacian, homoclinic solutions, variational methods, minimisation theorem, Clark's theorem

MSC: 34C37, 58E05, 70H05.

1. Introduction

In this paper, we consider the following fourth-order differential equation with a p -Laplacian:

$$\left(|u''(t)|^{p-2} u''(t) \right)'' - \omega \left(|u'(t)|^{p-2} u'(t) \right)' + V(t) |u(t)|^{p-2} u(t) = f(t, u(t)), \quad (1)$$

where $p > 1$, ω is a constant, $V \in C(\mathbb{R}, \mathbb{R})$ is a positive function bounded from below and $f \in C(\mathbb{R}^2, \mathbb{R})$ is subquadratic in the second variable. As usual, we say that a solution u of $(\mathcal{F})$ is homoclinic (to 0) if $u(t) \rightarrow 0$ as $|t| \rightarrow \infty$. It is called nontrivial if $u \neq 0$.

The fourth-order differential equations with p -Laplacian arise from the study of Non-Newtonian fluid mechanics and nonlinear filtration theory.

When $p = 2$, formally equation $(\mathcal{F})$ reduces to the classical fourth-order differential equation

$$u^{(4)}(t) - \omega u''(t) + V(t)u(t) = f(t, u(t)). \quad (2)$$

Over the past two decades, based on critical point theory and variational methods, for various conditions on V and the potential f , the existence and multiplicity of homoclinic solutions for Eq. (2) have been investigated in the literature, see [1–14] and the references listed therein.

In the general case where $p > 1$ is arbitrary, according to our knowledge there are only a few results concerning the existence of homoclinic solutions of equation $(\mathcal{F})$, see [15,16]. In [16], Tersian studied the existence and multiplicity of homoclinic solutions of equation $(\mathcal{F})$ when the potential f takes the form $f(t, x) = a(t)h(t, x)$ and he obtained the following result.

Theorem 1. Let $p \geq 2$ and a, h, V satisfy the following conditions (a) $a \in C(\mathbb{R}, \mathbb{R}^+)$ and $a(t) \rightarrow 0$ as $|t| \rightarrow \infty$; (f_1) $h \in C^1(\mathbb{R}^2, \mathbb{R})$ and there exists $1 < q < 2 < p$ such that

$$h(t, x)x \leq qH(t, x), \quad \forall (t, x) \in \mathbb{R}^2, \quad x \neq 0,$$

where $H(t, x) = \int_0^x h(t, y) dy$; (f_2) . There exists $b \in L^{\frac{p}{p-q}}(\mathbb{R}, \mathbb{R}^+) \cap L^{\frac{p}{2-q}}(\mathbb{R}, \mathbb{R}^+)$ such that

$$|h(t, x)| \leq b(t) |x|^{q-1}, \quad \forall (t, x) \in \mathbb{R}^2;$$

(f_3) There exist an interval $J \subset \mathbb{R}$ and a constant $c > 0$ such that

$$H(t, x) \geq c |x|^q, \quad \forall (t, x) \in J \times \mathbb{R};$$

(v) There exist positive constants v_1 and v_2 such that $\omega < \bar{v}\omega^*$ and

$$v_1 \leq V(t) \leq v_2, \quad \forall t \in \mathbb{R},$$

where $\bar{v} = \min \{1, v_1\}$ and $\omega^* = \inf_{u \neq 0} \frac{\int_{\mathbb{R}} [|u''(t)|^p + |u(t)|^p] dt}{\int_{\mathbb{R}} |u'(t)|^p dt}$.

Then equation $(\mathcal{F})$ has at least one nontrivial homoclinic solution. If moreover $H(t, x)$ is even with respect to the second variable, then equation $(\mathcal{F})$ has infinitely many nontrivial homoclinic solutions $(u_n)_{n \in \mathbb{N}}$ such that $\|u_n\|_{L^\infty} \rightarrow 0$ as $n \rightarrow \infty$.

The first aim of this paper is to generalize the previous results concerning the subquadratic case. More precisely, our goal is to establish similar results without assuming the classical Ambrosetti-Rabinowitz subquadratic condition (f_1) . Consider then the following conditions.

(V) There exist positive constants r_0 and $v_0 > 0$ such that $V(t) \geq v_0$ for all $t \in \mathbb{R}$ and

$$\lim_{|s| \rightarrow \infty} \text{meas}(\{t \in]s - r_0, s + r_0[: V(t) < M\}) = 0, \quad \forall M > 0,$$

where meas denotes the Lebesgue's measure on $\mathbb{R}$;

(F_1) There are a constant $1 < \mu < p$ and $b \in L^r(\mathbb{R}, \mathbb{R}^+)$, where $r = \frac{p}{p-\mu}$ such that

$$|f(t, x)| \leq b(t) |x|^{\mu-1}, \quad \forall (t, x) \in \mathbb{R}^2;$$

(F_2) There are a non empty open interval $I =]c, d[\subset \mathbb{R}$ and a positive constant a_0 such that

$$F(t, x) \geq a_0 |x|^\mu, \quad \forall (t, x) \in I \times \mathbb{R};$$

$$f(t, -x) = -f(t, x), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}. \quad (3)$$

Our first main results are as follows

Theorem 2. Under assumptions (V) , (F_1) and (F_2) , system $(\mathcal{F})$ admits at least one nontrivial homoclinic solution.

Theorem 3. Suppose that (V) , (F_1) , (F_2) and (F_3) hold. Then system $(\mathcal{F})$ admits infinitely many pairs of nontrivial homoclinic solutions $(u_n, -u_n)$ such that $\|u_n\|_{L^\infty} \rightarrow 0$ as $n \rightarrow \infty$.

Example 1. Take $p = 2$ and $\mu = \frac{3}{2}$ and let $V(t) = 1 + \cos^2 t$ and

$$F(t, x) = a(t) |x|^{\frac{3}{2}} (\pi + \text{Arctg}(|x|^{\frac{3}{2}})),$$

where

$$a(t) = \begin{cases} 2 - |t|, & \text{if } |t| \leq 1, \\ \frac{1}{|t|}, & \text{if } |t| \geq 1. \end{cases}$$

Let $a_0 = \frac{\pi}{2}$, $I =]-1, 1[$, $r = 4$ and $b(t) = \frac{3(3\pi+1)}{4} a(t)$. One easily verifies that $b \in L^r(\mathbb{R})$, $|f(t, x)| \leq b(t) |x|^{\mu-1}$ for all $(t, x) \in \mathbb{R}^2$ and $F(t, x) \geq a_0 |x|^\mu$ for all $(t, x) \in I \times \mathbb{R}$. Therefore the assumptions (V) , (F_1) , (F_2) and (F_3) are satisfied.

Now consider the following assumption

(F'_2) There exists $t_0 \in \mathbb{R}$ such that

$$\lim_{(t,x) \rightarrow (t_0,0)} \frac{F(t,x)}{|x|^\mu} > 0.$$

Our second main results are as follows.

Theorem 4. Assume that (V), (F_1) and (F'_2) are satisfied. Then system ($\mathcal{F}$) has at least one nontrivial homoclinic solution.

Theorem 5. Under assumptions (V), (F_1), (F'_2) and (F_3), system ($\mathcal{F}$) possesses infinitely many pairs of nontrivial homoclinic solutions $(u_n, -u_n)$ such that $\|u_n\|_{L^\infty} \rightarrow 0$ as $n \rightarrow \infty$.

Example 2. Take $p = 2$, $\mu = \frac{3}{2}$ and $V(t) = 1 + \cos^2 t$. Define a cut-off function $\chi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ such that $\chi(s) = 1$ for $0 \leq s \leq 1$, $\chi(s) = 0$ for $s \geq 2$ and $-2 \leq \chi'(s) \leq 2$ for $1 < s < 2$ and let

$$F(t, x) = a(t)\chi(|x|)|x|^{\frac{3}{2}}.$$

Let $t_0 = 0$, $r = 4$ and $b(t) = \frac{11}{2}a(t)$. It is easy to verify that $b \in L^r(\mathbb{R})$, $|f(t, x)| \leq b(t)|x|^{\mu-1}$ for all $(t, x) \in \mathbb{R}^2$ and $\lim_{(t,x) \rightarrow (t_0,0)} \frac{F(t,x)}{|x|^{\frac{3}{2}}} = 1 > 0$. Therefore the assumptions (V), (F_1), (F'_2) and (F_3) are satisfied.

Remark 1. In the hypothesis (F_1), if we replace the constant $r = \frac{p}{p-\mu}$ by any constant $1 \leq \xi \leq p$, the same conclusions remain valid under the other hypotheses.

Remark 2. Theorems 4 and 5 extend Theorems 2 and 3 to the case where the condition (F_2) is replaced by the weaker local assumption (F'_2). In particular, the existence of nontrivial homoclinic solutions can still be ensured when the nonlinearity $F(t, x)$ satisfies a local positivity condition around some point $t_0 \in \mathbb{R}$ instead of a uniform lower bound on an interval. This highlights the robustness of our variational approach under minimal hypotheses on F .

Comparison with previous results

We would like to emphasize more precisely how our assumptions generalize those used in earlier works such as Tersian [16] and related papers on fourth-order p -Laplacian equations. In Tersian's framework, the nonlinear term $f(t, x) = a(t)h(t, x)$ satisfies an Ambrosetti-Rabinowitz-type subquadratic condition (f_1), which imposes a structural relation between $h(t, x)x$ and its primitive $H(t, x)$. In contrast, our condition (F_1) only requires a standard subcritical growth of order $\mu < p$, without any Ambrosetti-Rabinowitz inequality. This substantially weakens the nonlinearity assumptions and allows a wider class of functions f . Moreover, while Tersian assumed that the potential V is uniformly bounded and positive, our hypothesis (V) admits a *noncoercive* setting: it suffices that $V(t) \geq v_0 > 0$ and that the set where V is small has vanishing measure in sliding intervals. This requirement is less restrictive than coercivity and is analogous to conditions used in Schrödinger-type problems on $\mathbb{R}$. Finally, the symmetry assumption (F_3) coincides with that in [16] and ensures the existence of infinitely many even pairs of homoclinic solutions. Therefore, our results extend the existing theorems in the subquadratic regime and apply to a broader class of potentials and nonlinearities.

Discussion of condition (V)

Condition (V) plays a crucial role in ensuring the compactness of the associated energy functional on $\mathbb{R}$. It allows the potential V to be noncoercive: (i.e., V is not assumed to be coercive at infinity) and possibly oscillatory, provided that the region where V becomes small has vanishing measure in any fixed-length interval. Typical examples include

$$V(t) = 1 + \varepsilon \sin^2 t, \quad V(t) = 1 + |t|^{-\alpha} \text{ for } |t| \geq 1, \quad \alpha > 0,$$

or more generally $V(t) = V_0(t) + W(t)$, where V_0 is periodic and W tends to 0 in measure as $|t| \rightarrow \infty$. This hypothesis is analogous to the "vanishing-in-measure" condition used in nonlinear Schrödinger and

Hamiltonian systems to recover compact embeddings in unbounded domains (see Lions [17]). Hence, our framework naturally covers both coercive and noncoercive potentials and is well suited for problems with oscillatory or almost-periodic structure.

The remaining of this paper is structured as follows. Some preliminary results are presented in §2. In §3 we give the proofs of our main results.

2. Preliminaries

Consider the Sobolev's space

$$W^{2,p}(\mathbb{R}) = \{u \in L^p(\mathbb{R}) / u' \in L^p(\mathbb{R}), u'' \in L^p(\mathbb{R})\},$$

equipped with the usual norm

$$\|u\|_{W^{2,p}} = \left(\int_{\mathbb{R}} [|u''(t)|^p + |u'(t)|^p + |u(t)|^p] dt \right)^{\frac{1}{p}}.$$

In this section we recall some auxiliary inequalities and compactness properties that will be used later. For clarity, we define all constants at the moment of their first appearance.

Lemma 1. *There exists $c_p > 0$ such that for all $u \in W^{2,p}(\mathbb{R})$,*

$$\int_{\mathbb{R}} |u'(t)|^p dt \leq c_p \int_{\mathbb{R}} [|u''(t)|^p + |u(t)|^p] dt, \forall u \in W^{2,p}(\mathbb{R}).$$

This standard inequality follows from the embedding $W^{2,p}(\mathbb{R}) \hookrightarrow W^{1,p}(\mathbb{R})$ and can be found, for example, in ([18], Lemma 4.10). Moreover, under assumption (V), since the measure of the set $\{t \in [s - r_0, s + r_0] : V(t) < M\}$ tends to zero as $|s| \rightarrow \infty$, the embedding of the corresponding energy space into $L^p(\mathbb{R})$ is compact (see [17]).

Lemma 2. *Let v_0 and ω^* be defined in §1. If $\omega < v_0\omega^*$ holds, then there exists a constant $c_0 > 0$ such that*

$$\int_{\mathbb{R}} [|u''(t)|^p - \omega |u'(t)|^p + V(t) |u(t)|^p] dt \geq c_0 \|u\|_{W^{2,p}}^p, \forall u \in W^{2,p}(\mathbb{R}). \quad (4)$$

Proof. Let $c_0 = \frac{v_0\omega^* - \omega}{(c_p + 1)v_0\omega^*}$, we have

$$\begin{aligned} \int_{\mathbb{R}} [|u''(t)|^p - \omega |u'(t)|^p + V(t) |u(t)|^p] dt &\geq v_0 \int_{\mathbb{R}} [|u''(t)|^p - \frac{\omega}{v_0} |u'(t)|^p + |u(t)|^p] dt \\ &= v_0 \left(1 - \frac{\omega}{v_0\omega^*}\right) \int_{\mathbb{R}} [|u''(t)|^p + |u(t)|^p] dt \\ &\quad + \frac{\omega}{\omega^*} \int_{\mathbb{R}} [|u''(t)|^p - \omega^* |u'(t)|^p + |u(t)|^p] dt \\ &\geq v_0 \left(1 - \frac{\omega}{v_0\omega^*}\right) \int_{\mathbb{R}} [|u''(t)|^p + |u(t)|^p] dt \\ &= c_0(c_p + 1) \int_{\mathbb{R}} [|u''(t)|^p + |u(t)|^p] dt \\ &\geq c_0 \int_{\mathbb{R}} [|u''(t)|^p + |u'(t)|^p + |u(t)|^p] dt \\ &= c_0 \|u\|_{W^{2,p}}^p, \end{aligned}$$

which completes the proof of Lemma 2. $\square$

Consider the following subspace E of $W^{2,p}(\mathbb{R})$

$$E = \left\{ u \in W^{2,p}(\mathbb{R}) / \int_{\mathbb{R}} [|u''(t)|^p - \omega |u'(t)|^p + V(t) |u(t)|^p] dt < \infty \right\},$$

equipped with the norm

$$\|u\| = \left(\int_{\mathbb{R}} \left[|u''(t)|^p - \omega |u'(t)|^p + V(t) |u(t)|^p \right] dt \right)^{\frac{1}{p}}.$$

Lemma 3. Under the condition $\omega < v_0 \omega^*$, we have

$$\|u\|_{L^\infty} \leq \left(\frac{p}{2c_0} \right)^{\frac{1}{p}} \|u\|, \quad \forall u \in E, \quad (5)$$

where c_0 is defined in Lemma 2.

Proof. Let $u \in E$, we have for $r \geq 0$

$$\int_{|t| \geq r} [|u'(t)|^p + |u(t)|^p] dt \leq \|u\|_{W^{2,p}}^p,$$

and so

$$\lim_{r \rightarrow \infty} \int_{|t| \geq r} [|u'(t)|^p + |u(t)|^p] dt = 0.$$

It results from [19] that $\lim_{|t| \rightarrow \infty} u(t) = 0$. Hence, by the continuity of u , there exists $t^* \in \mathbb{R}$ such that

$$|u(t^*)| = \max_{t \in \mathbb{R}} |u(t)| = \|u\|_{L^\infty}. \quad (6)$$

Consider two real sequences $(t_k)_{k \in \mathbb{N}}$ and $(t_{-k})_{k \in \mathbb{N}}$ such that

$$\dots < t_{-3} < t_{-2} < t_{-1} < t_1 < t_2 < t_3 < \dots,$$

$$\lim_{k \rightarrow \infty} t_k = +\infty, \quad \lim_{k \rightarrow \infty} t_{-k} = -\infty,$$

and

$$\lim_{k \rightarrow \infty} u(t_k) = 0 = \lim_{k \rightarrow \infty} u(t_{-k}).$$

Let us remark that

$$|u(t^*)|^p = |u(t_k)|^p - p \int_{t^*}^{t_k} |u(s)|^{p-2} u(s) u'(s) ds, \quad (7)$$

and

$$|u(t^*)|^p = |u(t_{-k})|^p + p \int_{t_{-k}}^{t^*} |u(s)|^{p-2} u(s) u'(s) ds. \quad (8)$$

Combining (7), (8) and Young's inequality yields

$$\begin{aligned} |u(t^*)|^p &= \frac{1}{2} \left(|u(t_k)|^p + |u(t_{-k})|^p \right) - \frac{p}{2} \int_{t^*}^{t_k} |u(s)|^{p-2} u(s) u'(s) ds + \frac{p}{2} \int_{t_{-k}}^{t^*} |u(s)|^{p-2} u(s) u'(s) ds \\ &\leq \frac{1}{2} \left(|u(t_k)|^p + |u(t_{-k})|^p \right) + \frac{p}{2} \int_{t_{-k}}^{t_k} \left[\frac{1}{p} |u'(s)|^p + \frac{p-1}{p} |u(s)|^p \right] ds. \end{aligned} \quad (9)$$

Taking $k \rightarrow \infty$ in (9), one gets

$$\|u\|_{L^\infty}^p = |u(t^*)|^p \leq \frac{p}{2} \int_{\mathbb{R}} [|u''(t)|^p + |u'(s)|^p + |u(s)|^p] ds \leq \frac{p}{2c_0} \|u\|^p,$$

which implies (5). $\square$

Remark 3. Noting by $\eta_\infty = \left(\frac{p}{2c_0} \right)^{\frac{1}{p}}$, for $s \geq p$ and $u \in E$, we have

$$\int_{\mathbb{R}} |u(t)|^s dt \leq \|u\|_{L^\infty}^{s-p} \|u\|_{L^p}^p \leq \eta_\infty^{s-p} \|u\|^{s-p} \|u\|_{W^{2,p}}^p \leq \frac{\eta_\infty^{s-p}}{c_0} \|u\|^s = \eta_s^s \|u\|^s, \quad (10)$$

where $\eta_s^s = \frac{\eta_\infty^{s-p}}{c_0}$.

Lemma 4. Assume that (V) is satisfied. Then E is compactly embedded in $L^p(\mathbb{R})$.

Proof. Let $(u_n) \subset E$ be a bounded sequence such that $u_n \rightharpoonup u$ in E . We shall show that $u_n \rightarrow u$ in $L^p(\mathbb{R})$. Suppose, without loss of generality, that $u_n \rightharpoonup 0$ in E . For any $s \in \mathbb{R}$, we denote $I_{r_0}(s)$ the open interval in $\mathbb{R}$ centered at s with radius r_0 , i.e., $I_{r_0}(s) =]s - r_0, s + r_0[$, where r_0 is the constant given in (L). Let $(s_i)_{i \in \mathbb{N}} \subset \mathbb{R}$ be a sequence of points such that $\mathbb{R} = \cup_{i=1}^\infty I_{r_0}(s_i)$ and each $t \in \mathbb{R}$ is contained in at most two such intervals. For any $r > 0$ and $M > 0$, let

$$C(r, M) = \{t \in \mathbb{R} \setminus]-r, r[\mid V(t) \geq M\},$$

$$D(r, M) = \{t \in \mathbb{R} \setminus]-r, r[\mid V(t) < M\}.$$

Choose $M_\epsilon > \frac{4}{\epsilon} \sup_{n \in \mathbb{N}} \|u_n\|^p$, we have

$$\int_{C(r, M_\epsilon)} |u_n|^p dt \leq \frac{1}{M_\epsilon} \int_{C(r, M_\epsilon)} V(t) |u_n(t)|^p dt \leq \frac{1}{M_\epsilon} \|u_n\|^p < \frac{\epsilon}{4}. \quad (11)$$

Now, we have

$$\begin{aligned} \int_{D(r, M_\epsilon)} |u_n|^p dt &\leq \sum_{i=1}^\infty \int_{D(r, M_\epsilon) \cap I_{r_0}(s_i)} |u_n|^p dt \\ &\leq \sum_{i=1}^\infty \left(\int_{D(r, M_\epsilon) \cap I_{r_0}(s_i)} |u_n|^{2p} dt \right)^{\frac{1}{2}} [\text{meas}(D(r, M_\epsilon) \cap I_{r_0}(s_i))]^{\frac{1}{2}} \\ &\leq a_r \sum_{i=1}^\infty \left(\int_{I_{r_0}(s_i)} |u_n|^{2p} dt \right)^{\frac{1}{2}}, \end{aligned}$$

where $a_r = \sup_{i \in \mathbb{N}} [\text{meas}(D(r, M_\epsilon) \cap I_{r_0}(s_i))]^{\frac{1}{2}}$. By the inequality (4), we have

$$\left(\int_{I_{r_0}(s_i)} |u_n|^{2p} dt \right)^{\frac{1}{2p}} = \left(\int_{\mathbb{R}} |\chi_{I_{r_0}(s_i)} u_n|^{2p} dt \right)^{\frac{1}{2p}} \leq \eta_{2p} \|\chi_{I_{r_0}(s_i)} u_n\| \leq \eta_{2p} \|u_n\|.$$

Hence

$$\begin{aligned} \int_{D(r, M_\epsilon)} |u_n|^p dt &\leq \eta_{2p}^p a_r \sum_{i=1}^\infty \|\chi_{I_{r_0}(s_i)} u_n\|^p \\ &= \eta_{2p}^p a_r \sum_{i=1}^\infty \int_{I_{r_0}(s_i)} [|u_n''|^p + |u_n'(t)|^p + |u_n(t)|^p] dt \\ &\leq 2\eta_{2p}^p a_r \sup_{n \in \mathbb{N}} \|u_n\|_{W^{2,p}}^p \leq \frac{2}{c_0} \eta_{2p}^p a_r \sup_{n \in \mathbb{N}} \|u_n\|^p. \end{aligned}$$

By an easy computation, we show that $a_r \rightarrow 0$ as $r \rightarrow \infty$. Therefore there exists $r_\epsilon > 0$ such that

$$\int_{D(r_\epsilon, M_\epsilon)} |u_n|^p dt < \frac{\epsilon}{4}, \quad (12)$$

which together with (11) implies

$$\int_{\mathbb{R} \setminus]-r_\epsilon, r_\epsilon[} |u_n|^p dt = \int_{C(r_\epsilon, M_\epsilon)} |u_n|^p dt + \int_{D(r_\epsilon, M_\epsilon)} |u_n|^p dt < \frac{\epsilon}{2}. \quad (13)$$

By Sobolev's theorem, $u_n \rightarrow 0$ uniformly on $[-r_\epsilon, r_\epsilon]$. Then there exists $n_0 > 0$ such that

$$\int_{[-r_\epsilon, r_\epsilon]} |u_n|^p dt < \frac{\epsilon}{2}, \quad \forall n \geq n_0. \quad (14)$$

Combining (13) with (14), by the arbitrary of ϵ we can obtain that $u_k \rightarrow 0$ in $L^p(\mathbb{R}, \mathbb{R})$. $\square$

To study the existence and multiplicity of homoclinic solutions of $(\mathcal{F})$ under our assumptions, we shall employ the minimization theorem and the generalized Clark's theorem.

Lemma 5. [Minimization theorem [20]]. Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$ satisfying (PS) condition. If J is bounded from below, then $c = \inf_E J$ is a critical point of J .

Lemma 6. [Generalized Clark's theorem [21]]. Let E be a Banach space and $J \in C^1(E, \mathbb{R})$. Assume that J satisfies the (PS) condition, it is even, bounded from below and $J(0) = 0$. If for any $k \in \mathbb{N}$, there exists a k -dimensional subspace E^k of E and $\rho_k > 0$ such that $\sup_{E^k \cap S_{\rho_k}} J < 0$, where $S_{\rho} = \{u \in E / \|u\| = 1\}$, then at least one of the following conditions holds

- There exists a sequence of critical points (u_n) of J satisfying $J(u_n) < 0$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \|u_n\|_E = 0$.
- There exists $r > 0$ such that for any $0 < \alpha < r$ there exists a critical point u of J such that $\|u\|_E = \alpha$ and $J(u) = 0$.

3. Proof of the main results

Let us consider the variational functional $J : E \rightarrow \mathbb{R}$ associated to system $(\mathcal{F})$

$$J(u) = \frac{1}{p} \int_{\mathbb{R}} \left[|u''(t)|^p - \omega |u'(t)|^p + V(t) |u(t)|^p \right] dt - \int_{\mathbb{R}} F(t, u(t)) dt.$$

Lemma 7. Assume that (V) and (F_1) are satisfied. If $u_n \rightharpoonup u$, then $f(., u_n) \rightarrow f(., u)$ in $L^v(\mathbb{R})$ as $n \rightarrow \infty$, where $v = \frac{p}{p-1}$.

Proof. Let $u_n \rightharpoonup u$. Arguing indirectly, by Lemma 4, we may assume that there exists a subsequence (u_{n_k}) such that as $k \rightarrow \infty$

$$u_{n_k} \rightarrow u \text{ in } L^p(\mathbb{R}) \text{ and } u_{n_k} \rightarrow u \text{ a.e. in } \mathbb{R}, \quad (15)$$

and

$$\int_{\mathbb{R}} |f(t, u_{n_k}(t)) - f(t, u(t))|^v dt \geq \epsilon_0, \quad \forall k \in \mathbb{N}, \quad (16)$$

for some positive constant ϵ_0 . Using (16) and up to a subsequence if necessary, we may assume that $\sum_{k=1}^{\infty} \|u_{n_k} - u\|_{L^p} < \infty$. Let $w(t) = \sum_{k=1}^{\infty} |u_{n_k}(t) - u(t)|$ for all $t \in \mathbb{R}$. Then $w \in L^p(\mathbb{R})$. By (F_1) , there holds for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$

$$\begin{aligned} |f(t, u_{n_k}(t)) - f(t, u(t))|^v &\leq [|f(t, u_{n_k}(t))| + |f(t, u(t))|]^v \\ &\leq 2^{v-1} [|f(t, u_{n_k}(t))|^v + |f(t, u(t))|^v] \\ &\leq 2^{v-1} b^v(t) [|u_{n_k}(t)|^{v(\mu-1)} + |u(t)|^{v(\mu-1)}] \\ &\leq 2^{v-1} b^v(t) [(|u_{n_k}(t) - u(t)| + |u(t)|)^{v(\mu-1)} + |u(t)|^{v(\mu-1)}] \\ &\leq 2^{v-1} b^v(t) [(w(t) + |u(t)|)^{v(\mu-1)} + |u(t)|^{v(\mu-1)}] \\ &\leq c_1 b^v(t) [(w(t))^{v(\mu-1)} + |u(t)|^{v(\mu-1)}], \end{aligned} \quad (17)$$

where $c_1 = 2^{v-1} [1 + \sup \{1, 2^{v(\mu-1)-1}\}]$. By Hölder's inequality, we obtain for $\alpha = \frac{p-1}{p-\mu}$ and $\beta = \frac{p-1}{\mu-1}$

$$\begin{aligned} \int_{\mathbb{R}} b^v(t) [(w(t))^{v(\mu-1)} + |u(t)|^{v(\mu-1)}] dt &\leq \left(\int_{\mathbb{R}} b^{\alpha v}(t) dt \right)^{\frac{1}{\alpha}} \left[\left(\int_{\mathbb{R}} (w(t))^{\beta v(\mu-1)} dt \right)^{\frac{1}{\beta}} + \left(\int_{\mathbb{R}} |u(t)|^{\beta v(\mu-1)} dt \right)^{\frac{1}{\beta}} \right] \\ &\leq \left(\int_{\mathbb{R}} b^r(t) dt \right)^{\frac{1}{\alpha}} \left[\left(\int_{\mathbb{R}} (w(t))^p dt \right)^{\frac{1}{\beta}} + \left(\int_{\mathbb{R}} |u(t)|^p dt \right)^{\frac{1}{\beta}} \right]. \end{aligned} \quad (18)$$

Combining (15), (17) and (18), Lebesgue's dominated convergence theorem implies that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}} |f(t, u_{n_k}(t)) - f(t, u(t))|^{\nu} dt = 0,$$

which contradicts (16). Hence $f(\cdot, u_n) \rightarrow f(\cdot, u)$ in $L^{\nu}(\mathbb{R})$ as $n \rightarrow \infty$. $\square$

Lemma 8. Under assumptions (V) and (F_1) , the functional $g(u) = \int_{\mathbb{R}} F(t, u(t)) dt$ is continuously differentiable on E and for all $u, v \in E$

$$g'(u)v = \int_{\mathbb{R}} f(t, u(t))v(t) dt.$$

Proof. First, by (F_1) and Hölder's inequality we have

$$\begin{aligned} \left| \int_{\mathbb{R}} f(t, u(t))v(t) dt \right| &\leq \int_{\mathbb{R}} b(t) |u(t)|^{\mu-1} |v(t)| dt \\ &\leq \left(\int_{\mathbb{R}} b^r(t) dt \right)^{\frac{1}{r}} \left(\int_{\mathbb{R}} |u(t)|^{r'(\mu-1)} |v(t)|^{r'} dt \right)^{\frac{1}{r'}} \\ &= \|b\|_{L^r} \left(\int_{\mathbb{R}} |u(t)|^{\frac{p(\mu-1)}{\mu}} |v(t)|^{\frac{p}{\mu}} dt \right)^{\frac{\mu}{p}} \\ &\leq \|b\|_{L^r} \|u\|_{L^p}^{\mu-1} \|v\|_{L^p}. \end{aligned}$$

Hence the functional $v \mapsto \int_{\mathbb{R}} f(t, u(t))v(t) dt$ is linear and continuous.

Next, let $\epsilon > 0$ be given. Since $b \in L^r(\mathbb{R})$, there exists $T_{\epsilon} > 0$ such that $\left(\int_{|t| \geq T_{\epsilon}} b^r(t) dt \right)^{\frac{1}{r}} \leq \epsilon$. It is well known that the functional $u \mapsto \int_{|t| \leq T_{\epsilon}} F(t, u(t)) dt$ is continuously differentiable on the space

$$W_{T_{\epsilon}}^{2,p} = \{u \in L^p([-T_{\epsilon}, T_{\epsilon}], \mathbb{R}) / u', u'' \in L^p([-T_{\epsilon}, T_{\epsilon}], \mathbb{R})\},$$

with derivative $\int_{[-T_{\epsilon}, T_{\epsilon}]} f(t, u(t))v(t) dt$. Let $u, v \in E$ with $\|v\| \leq 1$, then there exists a constant $0 < \alpha_{\epsilon} < 1$ such that for $\|v\| \leq \alpha_{\epsilon}$,

$$\left| \int_{[-T_{\epsilon}, T_{\epsilon}]} [F(t, u(t) + v(t)) - F(t, u(t)) - f(t, u(t))v(t)] dt \right| \leq \epsilon \|v\|. \quad (19)$$

Now, by the Mean Value Theorem, we have

$$\int_{|t| \geq T_{\epsilon}} [F(t, u(t) + v(t)) - F(t, u(t))] dt = \int_{|t| \geq T_{\epsilon}} [f(t, u(t) + h(t)v(t))v(t)] dt,$$

where $h(t) \in]0, 1[$. Therefore, by (F_1) and Hölder's inequality, we obtain

$$\begin{aligned} &\left| \int_{|t| \geq T_{\epsilon}} [F(t, u(t) + v(t)) - F(t, u(t)) - f(t, u(t))v(t)] dt \right| \\ &= \left| \int_{|t| \geq T_{\epsilon}} [f(t, u(t) + h(t)v(t)) - f(t, u(t))]v(t) dt \right| \\ &\leq \int_{|t| \geq T_{\epsilon}} b(t) [|u(t) + h(t)v(t)|^{\mu-1} + |u(t)|^{\mu-1}] |v(t)| dt \\ &\leq 2 \int_{|t| \geq T_{\epsilon}} b(t) [|u(t)|^{\mu-1} + |v(t)|^{\mu-1}] |v(t)| dt \\ &\leq 2 \left(\int_{|t| \geq T_{\epsilon}} b^r(t) dt \right)^{\frac{1}{r}} \left[\left(\int_{|t| \geq T_{\epsilon}} [|u(t)|^{r'(\mu-1)} |v(t)|^{r'} dt \right)^{\frac{1}{r'}} + \left(\int_{|t| \geq T_{\epsilon}} |v(t)|^{r'\mu} dt \right)^{\frac{1}{r'}} \right] \\ &\leq 2\epsilon \left[\left(\int_{\mathbb{R}} |u(t)|^p dt \right)^{\frac{\mu-1}{p}} \left(\int_{\mathbb{R}} |v(t)|^p dt \right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}} |v(t)|^p dt \right)^{\frac{\mu}{p}} \right] \\ &= 2\epsilon \left[\|u\|_{L^p}^{\mu-1} \|v\|_{L^p} + \|v\|_{L^p}^{\mu} \right] \end{aligned}$$

$$\leq 2\epsilon \left[\|u\|_{L^p}^{\mu-1} \eta_p + \eta_p^{\mu-1} \right] \|v\|. \quad (20)$$

Combining (19) and (20) yields for $\|v\| \leq \alpha_\epsilon$

$$\left| \int_{\mathbb{R}} [F(t, u(t) + v(t)) - F(t, u(t)) - f(t, u(t))v(t)] dt \right| \leq \epsilon \left[1 + 2 \left(\eta_p \|u\|_{L^p}^{\mu-1} + \eta_p^{\mu-1} \right) \right] \|v\|,$$

which implies that g is differentiable on E and $g'(u)v = \int_{\mathbb{R}} f(t, u(t))v(t)dt$ for all $u, v \in E$.

It remains to prove that g' is continuous. Since $u \mapsto \int_{|t| \leq T_\epsilon} f(t, u(t))dt$ is continuous, there is $0 < \beta_\epsilon \leq \alpha_\epsilon$ such that for $\|w\| \leq \beta_\epsilon$

$$\left| \int_{|t| \leq T_\epsilon} [f(t, u(t) + w(t)) - f(t, u(t))] dt \right| \leq \epsilon. \quad (21)$$

On the other hand, we have

$$\begin{aligned} \sup_{\|v\|=1} \left| \int_{|t| \geq T_\epsilon} [f(t, u(t) + w(t)) - f(t, u(t))]v(t)dt \right| &\leq \sup_{\|v\|=1} \int_{|t| \geq T_\epsilon} b(t) [(|u(t)| + |w(t)|)^{\mu-1} + |u(t)|^{\mu-1}] |v(t)| dt \\ &\leq 2 \sup_{\|v\|=1} \int_{|t| \geq T_\epsilon} b(t) [|u(t)|^{\mu-1} + |w(t)|^{\mu-1}] |v(t)| dt \\ &\leq 2 \sup_{\|v\|=1} \left(\int_{|t| \geq T_\epsilon} b^r(t) \right)^{\frac{1}{r}} \left[\left(\int_{|t| \geq T_\epsilon} |u(t)|^{r'(\mu-1)} |v(t)|^{r'} \right)^{\frac{1}{r'}} \right. \\ &\quad \left. + \left(\int_{|t| \geq T_\epsilon} |w(t)|^{r'(\mu-1)} |v(t)|^{r'} \right)^{\frac{1}{r'}} \right] \\ &\leq 2\epsilon \sup_{\|v\|=1} \left[\left(\int_{|t| \geq T_\epsilon} |u(t)|^p \right)^{\frac{\mu-1}{p}} \left(\int_{|t| \geq T_\epsilon} |v(t)|^p \right)^{\frac{1}{p}} \right. \\ &\quad \left. + \left(\int_{|t| \geq T_\epsilon} |w(t)|^p \right)^{\frac{\mu-1}{p}} \left(\int_{|t| \geq T_\epsilon} |v(t)|^p \right)^{\frac{1}{p}} \right] \\ &\leq 2\epsilon \eta_p^\mu \left[\|u\|^{\mu-1} + \|w\|^{\mu-1} \right] \leq 2\epsilon \eta_p^\mu \left[\|u\|^{\mu-1} + 1 \right]. \end{aligned} \quad (22)$$

Combining (21) and (22) yields for $\|w\| \leq \beta_\epsilon$

$$\|g'(u+w) - g'(u)\|_{E^*} \leq \epsilon + 2\epsilon \eta_p^\mu (\|u\|^{\mu-1} + 1).$$

Hence $g \in C^1(E, \mathbb{R})$. $\square$

Therefore, it is well known that the functional J is continuously differentiable on E and we have

$$\begin{aligned} J'(u)v &= \int_{\mathbb{R}} \left[|u''(t)|^{p-2} u''(t)v''(t) - \omega |u'(t)|^{p-2} u'(t)v'(t) + V(t) |u(t)|^{p-2} u(t)v(t) \right] dt \\ &\quad - \int_{\mathbb{R}} f(t, u(t))v(t)dt, \quad \forall u, v \in E. \end{aligned}$$

Moreover the critical points of J on E are the homoclinic solutions of $(\mathcal{F})$.

Lemma 9. Assume that (V) and (F_1) hold. Then J satisfies the (PS) -condition.

Proof. Let (u_n) be a (PS) -sequence, that is $(J(u_n))$ is bounded and $J'(u_n) \rightarrow 0$ as $n \rightarrow \infty$, then there exists a positive constant c_2 such that $|J(u_n)| \leq c_2$. By (F_1) and Hölder's inequality, we obtain

$$\begin{aligned} \|u_n\|^p &= pJ(u_n) + p \int_{\mathbb{R}} F(t, u_n(t))dt \\ &\leq pc_2 + p \int_{\mathbb{R}} b(t) |u_n(t)|^\mu dt \\ &\leq pc_2 + p \left(\int_{\mathbb{R}} b^r(t)dt \right)^{\frac{1}{r}} \left(\int_{\mathbb{R}} |u_n(t)|^p dt \right)^{\frac{\mu}{p}} \end{aligned}$$

$$\begin{aligned} &\leq pc_2 + p \|b\|_{L^r} \|u_n\|_{L^p}^\mu \\ &\leq pc_2 + p\eta_p^\mu \|b\|_{L^r} \|u_n\|^\mu. \end{aligned}$$

Since $1 < \mu < p$, then (u_n) is bounded in E .

Then, going to a subsequence if necessary, we can assume that $u_n \rightharpoonup u$ weakly in E . By Lemma 4, $u_n \rightarrow u$ in $L^p(\mathbb{R})$ and by Lemma 7, we have $f(\cdot, u_n) \rightarrow f(\cdot, u)$ in $L^q(\mathbb{R})$ as $n \rightarrow \infty$. Therefore, by Hölder's inequality, we obtain

$$\begin{aligned} \|u_n - u\|^p &= (J'(u_n) - J'(u))(u_n - u) + \int_{\mathbb{R}} (f(t, u_n(t)) - f(t, u(t)))(u_n(t) - u(t)) dt \\ &\leq (J'(u_n) - J'(u))(u_n - u) + \|f(\cdot, u_n) - f(\cdot, u)\|_{L^q} \|u_n - u\|_{L^p} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence $u_n \rightarrow u$ in E and J satisfies the (PS) condition. $\square$

Proof of Theorem 2. By (F_1) , Hölder's inequality and Remark 3, we have

$$\begin{aligned} \int_{\mathbb{R}} F(t, u(t)) dt &\leq \frac{1}{\mu} \int_{\mathbb{R}} b(t) |u(t)|^\mu dt \\ &\leq \frac{1}{\mu} \left(\int_{\mathbb{R}} b^r(t) dt \right)^{\frac{1}{r}} \left(\int_{\mathbb{R}} |u(t)|^{r'\mu} dt \right)^{\frac{1}{r'}} \\ &= \frac{1}{\mu} \|b\|_{L^r} \|u\|_{L^p}^\mu \leq \frac{\eta_p^\mu}{p} \|b\|_{L^r} \|u\|^\mu. \end{aligned}$$

Hence

$$\begin{aligned} J(u) &= \frac{1}{p} \|u\|^p - \int_{\mathbb{R}} F(t, u(t)) dt \\ &\geq \frac{1}{p} \|u\|^p - \frac{\eta_p^\mu}{\mu} \|b\|_{L^r} \|u\|^\mu. \end{aligned}$$

Since $1 < \mu < p$, we deduce that $J(u) \rightarrow +\infty$ as $\|u\| \rightarrow \infty$, i.e. J is coercive and bounded from below. Since J satisfies the (PS) condition, then the minimization Theorem implies that J achieves its minimum at a point $u_0 \in E$. It remains to prove that u_0 is nontrivial. Let $v_0 \in E$ be such that $v_0(t) = 0$, $t \in \mathbb{R} \setminus I$ be a nonzero function. Then for all $\lambda > 0$, we have by (F_2)

$$\begin{aligned} J(\lambda v_0) &= \frac{\lambda^p}{p} \|v_0\|^p - \int_{\mathbb{R}} F(t, \lambda v_0(t)) dt \\ &= \frac{\lambda^p}{p} \|v_0\|^p - \int_I F(t, \lambda v_0(t)) dt \\ &\leq \frac{\lambda^p}{p} \|v_0\|^p - a_0 \lambda^\mu \int_I |v_0(t)|^\mu dt, \end{aligned}$$

which shows that $J(\lambda v_0) < 0$ for $\lambda > 0$ sufficiently small since $1 < \mu < p$. Therefore $J(u_0) = \min_E J < J(0) = 0$ and u_0 is a nontrivial solution of $(\mathcal{F})$. $\square$

Proof of Theorem 3. Assumptions (F_1) and (F_3) imply that $J(0) = 0$ and J is even. Lemma 9 implies that J satisfies the (PS) condition and by the proof of Theorem 2, J is bounded from below. To apply Lemma 6, it remains to prove that for all $k \in \mathbb{N}$, there exists a k -dimensional subspace E^k of E and $\rho_k > 0$ such that $\sup_{E^k \cap S_{\rho_k}} J < 0$, where $S_{\rho} = \{u \in E / \|u\| = \rho\}$. For $k \in \mathbb{N}$, set $I_j =]x_{j-1}, x_j[$, $j = 1, \dots, k$, where $x_j = c + \frac{j}{k}(d - c)$. We have $\bigcup_{j=1}^k I_j \subset I =]c, d[$. For $j = 1, \dots, k$, let $u_j \in C_0^\infty(I_j, \mathbb{R})$ such that $\|u_j\| = 1$. Consider the k -dimensional subspace

$$E^k = \text{span} \{u_1, \dots, u_k\}.$$

For $u = \sum_{j=1}^k \lambda_j u_j \in E^k$, we have

$$\begin{aligned}\|u\|_{L^\mu}^\mu &= \int_{\mathbb{R}} |u(t)|^\mu dt = \sum_{j=1}^k \int_{I_j} |u(t)|^\mu dt \\ &= \sum_{j=1}^k \int_{I_j} |\lambda_j u_j(t)|^\mu dt \\ &= \sum_{j=1}^k |\lambda_j|^\mu \int_{I_j} |u_j(t)|^\mu dt,\end{aligned}$$

and by (F_2) , we obtain for all $u \in E^k$

$$\begin{aligned}\int_{\mathbb{R}} F(t, u(t)) dt &= \sum_{j=1}^k \int_{I_j} F(t, u(t)) dt = \sum_{j=1}^k \int_{I_j} F(t, \lambda_j u_j(t)) dt \\ &\geq \sum_{j=1}^k \int_{I_j} a_0 |\lambda_j u_j(t)|^\mu dt \\ &= a_0 \sum_{j=1}^k |\lambda_j|^\mu \int_{I_j} |u_j(t)|^\mu dt = a_0 \|u\|_{L^\mu}^\mu.\end{aligned}\quad (23)$$

Since E^k is a finite dimensional subspace, there exists a constant $C_k > 0$ such that

$$C_k \|u\| \leq \|u\|_{L^\mu}, \quad \forall u \in E^k. \quad (24)$$

Combining (23) and (24) yields for all $u \in E^k$

$$J(u) = \frac{1}{p} \|u\|^p - \int_{\mathbb{R}} F(t, u(t)) dt \leq \frac{1}{p} \|u\|^p - a_0 \|u\|_{L^\mu}^\mu \leq \frac{1}{p} \|u\|^p - a_0 C_k^\mu \|u\|^\mu.$$

Hence $J(u) < 0$ for $u \in E^k$ with $\|u\|$ small enough and then there exists a constant $\rho_k > 0$ such that $\sup_{E^k \cap S_{\rho_k}} J < 0$. The functional J satisfies all the conditions of Lemma 6. Therefore J possesses infinitely many pairs of nontrivial critical points $(u_n, -u_n)$ such that $\lim_{n \rightarrow \infty} \|u_n\| = 0$ and then by Remark 3, system $(\mathcal{F})$ has infinitely many pairs of nontrivial homoclinic solutions $(u_n, -u_n)$ such that $\lim_{n \rightarrow \infty} \|u_n\|_{L^\infty} = 0$. The proof of Theorem 3 is completed. $\square$

Proof of Theorem 4. Using assumptions (V) and (F_1) , we have proved above that J achieves its minimum on E at a point u_0 . We will prove that u_0 is nontrivial. By (W'_2) , there exist positive constants r, R, l_0 such that

$$F(t, x) \geq l_0 |x|^\mu, \quad \forall (t, x) \in]t_0 - r, t_0 + r[\times B(0, R). \quad (25)$$

Let $v \in C_0^\infty(]t_0 - r, t_0 + r[, \mathbb{R})$ be a nonzero function. For all $\lambda > 0$ such that $\lambda \|v\|_{L^\infty} < R$, we have

$$J(\lambda v) = \frac{\lambda^p}{p} \|v\|^p - \int_{\mathbb{R}} F(t, \lambda v(t)) dt \leq \frac{\lambda^p}{p} \|v\|^p - l_0 \lambda^\mu \int_{\mathbb{R}} |v(t)|^\mu dt,$$

which implies that $J(\lambda v) < 0$ for $\lambda > 0$ small enough because $1 < \mu < p$. Hence $J(u_0) = \min_E J < 0 = J(0)$ and u_0 is a nontrivial homoclinic solution of $(\mathcal{F})$. $\square$

Proof of Theorem 5. Denote $I =]t_0 - r, t_0 + r[$ the open interval introduced above. For $k \in \mathbb{N}$, we take k disjoint intervals $I_j =]x_{j-1}, x_j[$ where $x_j = t_0 - r + \frac{j}{k}(2r)$, $j = 1, \dots, k$, we have $\bigcup_{j=1}^k I_j \subset I$. For $j = 1, \dots, k$, let $u_j \in C_0^\infty(I_j, \mathbb{R})$ such that $\|u_j\| = 1$. Consider the k -dimensional subspace E^k of E defined by

$$E^k = \text{span} \{u_1, \dots, u_k\}.$$

For $u = \sum_{j=1}^k \lambda_j u_j \in E^k$, we have

$$\|u\|_{L^\mu}^\mu = \sum_{j=1}^k |\lambda_j|^\mu \int_{I_j} |u_j(t)|^\mu dt.$$

Since E^k is a finite dimensional subspace, there exists a constant $C_k > 0$ such that

$$C_k \|u\| \leq \|u\|_{L^\mu}, \quad \forall u \in E^k. \quad (26)$$

Let $u = \sum_{j=1}^k \lambda_j u_j \in E^k$ be such that $\|u\| < \frac{R}{\eta_\infty}$, then we have

$$\|\lambda_j u_j\|_{L^\infty} \leq \|u\|_{L^\infty} \leq \eta_\infty \|u\| < R. \quad (27)$$

Combining (25), (26) and (27) yields for $u = \sum_{j=1}^k \lambda_j u_j \in E^k$ with $\|u\| < \frac{R}{\eta_\infty}$

$$\begin{aligned} J(u) &= \frac{1}{p} \|u\|^p - \int_{\mathbb{R}} F(t, u(t)) dt \\ &= \frac{1}{p} \|u\|^p - \sum_{j=1}^k \int_{I_j} F(t, u(t)) dt \\ &= \frac{1}{p} \|u\|^p - \sum_{j=1}^k \int_{I_j} F(t, \lambda_j u_j(t)) dt \\ &\leq \frac{1}{p} \|u\|^p - \sum_{j=1}^k \int_{I_j} l_0 |\lambda_j u_j(t)|^\mu dt \\ &= \frac{1}{p} \|u\|^p - l_0 \sum_{j=1}^k |\lambda_j|^\mu \int_{I_j} |u_j(t)|^\mu dt \\ &= \frac{1}{p} \|u\|^p - l_0 \|u\|_{L^\mu}^\mu \leq \frac{1}{p} \|u\|^p - l_0 C_k^\mu \|u\|^\mu. \end{aligned}$$

Since $1 < \mu < p$, we deduce that there exists a constant $0 < \rho_k < \frac{R}{\eta_\infty}$ such that $J(u) < 0$ for all $u \in E^k \cap S_{\rho_k}$. We conclude as in the proof of Theorem 3 that system $(\mathcal{F})$ has infinitely many pairs of nontrivial homoclinic solutions $(u_n, -u_n)$ such that $\lim_{n \rightarrow \infty} \|u_n\|_{L^\infty} = 0$. $\square$

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