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Article

Existence and uniqueness of classical solution to the initial-boundary value problem for the unsteady general planar Broadwell model with four velocities

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Abstract: We consider the unsteady problem for the general planar Broadwell model with fourh velocities in a rectangular spatial domain over a finite time interval. We impose a class of non-negative initial and Dirichlet boundary data that are bounded and continuous, along with their first-order partial derivatives. We then prove the existence and uniqueness of a non-negative continuous solution, bounded together with its first-order partial derivatives, to the initial-boundary value problem.

Keywords: discrete velocity(Boltzmann) models, discrete kinetic equations, initial-boundary value problems, existence, uniqueness, fixed point theorems

MSC: 76A02, 76M28.

1. Introduction

The discrete Boltzmann equation (DBE) [1] has long been a fundamental tool for approximating the behavior of dilute gases, offering a simplified alternative to the full Boltzmann equation. Pioneering contributions were made by Carleman [2] and Broadwell [3,4], and later generalized by Gatignol [5]. The key idea is to replace the continuous velocity space with a finite set of discrete velocities, leading to hyperbolic systems of conservation laws with nonlinear collision terms. Such systems have proven instrumental in the analysis of numerous physical and mathematical phenomena, including shock formation, entropy dissipation, and boundary layer behavior.

Historically, the initial-boundary value problem (IBVP) for discrete velocity models has been studied extensively in one-dimensional settings [6–13]. In the multi-dimensional steady case, boundary value problems for discrete velocity models have also been investigated, see for example [14–17].

In contrast, the multi-dimensional theory of the IBVP for DBE models remains much less developed. To the best of our knowledge, prior to the recent work [18], no results had been established on the existence and uniqueness of classical solutions to multi-dimensional IBVPs in bounded domains with Dirichlet boundary conditions—even in simple geometries such as rectangles. In [18], we addressed this gap by proving the existence and uniqueness of classical solutions for the two-dimensional Broadwell model in a rectangular domain with Dirichlet boundary conditions.

The objective of the present paper is to extend these results to the general four-velocity Broadwell model in the plane, one of the canonical multi-dimensional discrete models. This extension represents an important step toward a broader mathematical understanding of discrete kinetic models in higher dimensions.

The paper is organized as follows. In §2, we introduce the initial-boundary value problem and state our main result. In §3, we begin the proof by establishing the positivity of the solution. Finally, in §4, we complete the proof by showing the existence and uniqueness.

2. The initial-boundary value problemma

2.1. The Discrete Boltzmann Equations

Let $d \in \{1,2,3\}$ and p be a non-negative integer. Define the discrete velocities $\overrightarrow{u_i} = \left(u_i^\alpha\right)_{\alpha=1}^d \in \mathbb{R}^d$, $i = 1,\dots,p$, and coefficients $A_{ij}^{kl} \geq 0$ for $i,j,k,l = 1,\dots,p$. The discrete p-velocity model describes a system of particles with velocities $\overrightarrow{u_i}$.

Let $N_i\left(t,(x_\alpha)_{\alpha=1}^d\right)$ denote the number density of particles with velocity $\vec{u}_i$, $i = 1,\dots,p$, at time t and position $M\left((x_\alpha)_{\alpha=1}^d\right)$, with $(x_\alpha)_{\alpha=1}^d \in \mathbb{R}^d$. The corresponding discrete Boltzmann (or kinetic) equations are given by

$$\frac{\partial N_i}{\partial t} + \sum_{\alpha=1}^d u_i^\alpha \frac{\partial N_i}{\partial x_\alpha} = \underbrace{\frac{1}{2}\sum_{j,k,l\neq i} A_{ij}^{kl}\left(N_k N_l - N_i N_j\right)}_{\equiv Q_i(N)}, \qquad \forall i = 1,\dots,p. \tag{1}$$

In the case of the general four-velocity Broadwell model in the plane, we set $N_i \equiv N_i(t,x,y)$, and the system of kinetic equations becomes

$$\begin{cases} \dfrac{\partial N_1}{\partial t} + c\cos\theta\dfrac{\partial N_1}{\partial x} + c\sin\theta\dfrac{\partial N_1}{\partial y} = Q, \\ \dfrac{\partial N_2}{\partial t} - c\sin\theta\dfrac{\partial N_2}{\partial x} + c\cos\theta\dfrac{\partial N_2}{\partial y} = -Q, \\ \dfrac{\partial N_3}{\partial t} + c\sin\theta\dfrac{\partial N_3}{\partial x} - c\cos\theta\dfrac{\partial N_3}{\partial y} = -Q, \\ \dfrac{\partial N_4}{\partial t} - c\cos\theta\dfrac{\partial N_4}{\partial x} - c\sin\theta\dfrac{\partial N_4}{\partial y} = Q, \end{cases} \tag{2}$$

where the collision term is defined by

$$Q = 2cS\left(N_2N_3 - N_1N_4\right),$$

with constants $c, S > 0$ and angle $0 \leq \theta < \dfrac{\pi}{2}$.

2.2. Initial-Boundary Value Problem

Let $[a_1,b_1] \times [a_2,b_2]$ be a rectangular spatial domain and $[0,T]$ a finite time interval. We define the space-time domain

$$\mathscr{P} = [0,T] \times [a_1,b_1] \times [a_2,b_2]\,.$$

In [18] we considered the case $\theta = 0$. In what follows, we study the case $\theta \neq 0$.

The initial-boundary value problem we consider is the system Σ (3)–(27):

$$\frac{\partial N_1}{\partial t} + c\cos\theta\frac{\partial N_1}{\partial x} + c\sin\theta\frac{\partial N_1}{\partial y} = Q(N), \quad (t,x,y) \in \mathring{\mathscr{P}}, \tag{3}$$

$$\frac{\partial N_2}{\partial t} - c\sin\theta\frac{\partial N_2}{\partial x} + c\cos\theta\frac{\partial N_2}{\partial y} = -Q(N), \quad (t,x,y) \in \mathring{\mathscr{P}}, \tag{4}$$

$$\frac{\partial N_3}{\partial t} + c\sin\theta\frac{\partial N_3}{\partial x} - c\cos\theta\frac{\partial N_3}{\partial y} = -Q(N), \quad (t,x,y) \in \mathring{\mathscr{P}}, \tag{5}$$

$$\frac{\partial N_4}{\partial t} - c\cos\theta\frac{\partial N_4}{\partial x} - c\sin\theta\frac{\partial N_4}{\partial y} = Q(N), \quad (t,x,y) \in \mathring{\mathscr{P}}, \tag{6}$$

$$N_i(0,x,y) = N_i^0(x,y), \quad (x,y) \in [a_1,b_1] \times [a_2,b_2],\ i = 1,\dots,4, \tag{7}$$

$$N_1(t,a_1,y) = N_1^-(t,y), \quad (t,y) \in [0,T] \times [a_2,b_2], \tag{8}$$

$$N_1(t,x,a_2) = N_1^{--}(t,x), \quad (t,x) \in [0,T] \times [a_1,b_1], \tag{9}$$

$$N_2(t,b_1,y) = N_2^+(t,y), \quad (t,y) \in [0,T] \times [a_2,b_2], \tag{10}$$
$$N_2(t,x,a_2) = N_2^{--}(t,x), \quad (t,x) \in [0,T] \times [a_1,b_1], \tag{11}$$
$$N_3(t,a_1,y) = N_3^-(t,y), \quad (t,y) \in [0,T] \times [a_2,b_2], \tag{12}$$
$$N_3(t,x,b_2) = N_3^{++}(t,x), \quad (t,x) \in [0,T] \times [a_1,b_1], \tag{13}$$
$$N_4(t,b_1,y) = N_4^+(t,y), \quad (t,y) \in [0,T] \times [a_2,b_2], \tag{14}$$
$$N_4(t,x,b_2) = N_4^{++}(t,x), \quad (t,x) \in [0,T] \times [a_1,b_1], \tag{15}$$

together with the compatibility conditions

$$N_1^0(a_1,y) = N_1^-(0,y), \quad y \in [a_2,b_2], \tag{16}$$
$$N_1^0(x,a_2) = N_1^{--}(0,x), \quad x \in [a_1,b_1], \tag{17}$$
$$N_1^-(t,a_2) = N_1^{--}(t,a_1), \quad t \in [0,T], \tag{18}$$
$$N_2^0(b_1,y) = N_2^+(0,y), \quad y \in [a_2,b_2], \tag{19}$$
$$N_2^0(x,a_2) = N_2^{--}(0,x), \quad x \in [a_1,b_1], \tag{20}$$
$$N_2^+(t,a_2) = N_2^{--}(t,b_1), \quad t \in [0,T], \tag{21}$$
$$N_3^0(a_1,y) = N_3^-(0,y), \quad y \in [a_2,b_2], \tag{22}$$
$$N_3^0(x,b_2) = N_3^{++}(0,x), \quad x \in [a_1,b_1], \tag{23}$$
$$N_3^-(t,b_2) = N_3^{++}(t,a_1), \quad t \in [0,T], \tag{24}$$
$$N_4^0(b_1,y) = N_4^+(0,y), \quad y \in [a_2,b_2], \tag{25}$$
$$N_4^0(x,b_2) = N_4^{++}(0,x), \quad x \in [a_1,b_1], \tag{26}$$
$$N_4^+(t,b_2) = N_4^{++}(t,b_1), \quad t \in [0,T]. \tag{27}$$

The initial conditions are given by N_i^0, for $i = 1,\ldots,4$, while the boundary conditions are specified by $N_1^-, N_1^{--}, N_2^+, N_2^{--}, N_3^-, N_3^{++}, N_4^+$, and N_4^{++}. All these functions are assumed to be non-negative and continuous, with continuous and bounded first-order partial derivatives. Relations (16)–(27) represent compatibility conditions on the data.

Our goal is to prove the existence and uniqueness of a positive, continuous solution to the system Σ.

We use the uniform norm for bounded real-valued functions, and define

$$\|f\| = \max_{1\le i\le 4} \|f_i\|_\infty$$

for $f = (f_i)_{i=1}^4$ with bounded components. For bounded real functions of two variables $g(x,y)$ with bounded partial derivatives, we adopt the C^1 norm

$$\|g\|_1 \equiv \max\left\{ \|g\|_\infty, \left\|\frac{\partial g}{\partial x}\right\|_\infty, \left\|\frac{\partial g}{\partial y}\right\|_\infty \right\}.$$

We prove the following result:

Theorem 1. *There exist two positive parameters p and q, with p depending on the dimensions of the domain $\mathscr{P}$ and q on the data, such that if $pq \le \frac{1}{4}$, then the system Σ admits a unique non-negative continuous solution. This solution is differentiable except possibly on finitely many planes, has bounded derivatives, and satisfies explicit bounds on both the solution and its derivatives.*

3. Non-negativity of the Solution

Theorem 2. *If solutions to the problem Σ (Eqs. (3)–(27)) exist, then they are non-negative.*

Proof. The proof follows classical arguments (see [18]). Let $\sigma > 0$ and define

$$\rho(N) = \sum_{i=1}^{4} N_i, \tag{28}$$

together with

$$\begin{cases} Q_1^\sigma(N) = \sigma\rho(N)N_1 + Q(N), \\ Q_2^\sigma(N) = \sigma\rho(N)N_2 - Q(N), \\ Q_3^\sigma(N) = \sigma\rho(N)N_3 - Q(N), \\ Q_4^\sigma(N) = \sigma\rho(N)N_4 + Q(N). \end{cases} \tag{29}$$

Then, for all $\sigma > 0$, the problem Σ is equivalent to the system Σ_σ:

$$\frac{\partial N_1}{\partial t} + c\cos\theta\frac{\partial N_1}{\partial x} + c\sin\theta\frac{\partial N_1}{\partial y} + \sigma\rho(N)N_1 = Q_1^\sigma(N), \quad (t,x,y) \in \mathring{\mathscr{P}}, \tag{30}$$

$$\frac{\partial N_2}{\partial t} - c\sin\theta\frac{\partial N_2}{\partial x} + c\cos\theta\frac{\partial N_2}{\partial y} + \sigma\rho(N)N_2 = Q_2^\sigma(N), \quad (t,x,y) \in \mathring{\mathscr{P}}, \tag{31}$$

$$\frac{\partial N_3}{\partial t} + c\sin\theta\frac{\partial N_3}{\partial x} - c\cos\theta\frac{\partial N_3}{\partial y} + \sigma\rho(N)N_3 = Q_3^\sigma(N), \quad (t,x,y) \in \mathring{\mathscr{P}}, \tag{32}$$

$$\frac{\partial N_4}{\partial t} - c\cos\theta\frac{\partial N_4}{\partial x} - c\sin\theta\frac{\partial N_4}{\partial y} + \sigma\rho(N)N_4 = Q_4^\sigma(N), \quad (t,x,y) \in \mathring{\mathscr{P}}, \tag{33}$$

together with the conditions (7)–(27).

For $M = (M_1, M_2, M_3, M_4)$, a fixed 4-tuple of continuous functions defined from $\mathscr{P}$ to $\mathbb{R}$, we set

$$|M| = \big(|M_1|, |M_2|, |M_3|, |M_4|\big).$$

Consider now the following linear system $(\Sigma_{\sigma,M})$, given by (34)–(37):

$$\frac{\partial N_1}{\partial t} + c\cos\theta\frac{\partial N_1}{\partial x} + c\sin\theta\frac{\partial N_1}{\partial y} + \sigma\rho(|M|)N_1 = Q_1^\sigma(|M|), \quad (t,x,y) \in \mathring{\mathscr{P}}, \tag{34}$$

$$\frac{\partial N_2}{\partial t} - c\sin\theta\frac{\partial N_2}{\partial x} + c\cos\theta\frac{\partial N_2}{\partial y} + \sigma\rho(|M|)N_2 = Q_2^\sigma(|M|), \quad (t,x,y) \in \mathring{\mathscr{P}}, \tag{35}$$

$$\frac{\partial N_3}{\partial t} + c\sin\theta\frac{\partial N_3}{\partial x} - c\cos\theta\frac{\partial N_3}{\partial y} + \sigma\rho(|M|)N_3 = Q_3^\sigma(|M|), \quad (t,x,y) \in \mathring{\mathscr{P}}, \tag{36}$$

$$\frac{\partial N_4}{\partial t} - c\cos\theta\frac{\partial N_4}{\partial x} - c\sin\theta\frac{\partial N_4}{\partial y} + \sigma\rho(|M|)N_4 = Q_4^\sigma(|M|), \quad (t,x,y) \in \mathring{\mathscr{P}}, \tag{37}$$

with the boundary and initial conditions (7)–(27).

It is straightforward to verify that $(\Sigma_{\sigma,M})$ admits a unique solution $N_M = (N_1, N_2, N_3, N_4)$, defined as follows.

For inequalities $f(t,x,y) \leq 0$ and $g(t,x,y) \leq 0$ defined on $\mathscr{P}$, let

$$\mathbb{I}\begin{cases} f(t,x,y) \leq 0 \\ g(t,x,y) \leq 0 \end{cases}$$

denote the indicator function of the set

$$\big\{(t,x,y) \in \mathscr{P} : f(t,x,y) \leq 0 \text{ and } g(t,x,y) \leq 0\big\}.$$

Then,

$$
\begin{aligned}
N_1(t,x,y) = {} & N_1^A(t,x,y)\cdot \mathbb{I}_{\begin{cases} x-ct\cos\theta \ge a_1 \\ y-ct\sin\theta \ge a_2 \end{cases}}(t,x,y) \\
& + N_1^B(t,x,y)\cdot \mathbb{I}_{\begin{cases} x-ct\cos\theta \le a_1 \\ x\sin\theta - y\cos\theta \le a_1\sin\theta - a_2\cos\theta \end{cases}}(t,x,y) \\
& + N_1^C(t,x,y)\cdot \mathbb{I}_{\begin{cases} y-ct\sin\theta \le a_2 \\ x\sin\theta - y\cos\theta \ge a_1\sin\theta - a_2\cos\theta \end{cases}}(t,x,y),
\end{aligned}
\tag{38}
$$

where

$$
\begin{aligned}
N_1^A(t,x,y) = {} & \Bigg(\int_0^t e^{\sigma\int_0^s \rho(|M|)\left(r,x+c(r-t)\cos\theta,y+c(r-t)\sin\theta\right)dr}\cdot Q_1^\sigma(|M|) \\
& \times \left(s,\, x+c(s-t)\cos\theta,\, y+c(s-t)\sin\theta\right)ds + N_1^0(x-ct\cos\theta, y-ct\sin\theta)\Bigg) \\
& \times e^{-\sigma\int_0^t \rho(|M|)\left(s,x+c(s-t)\cos\theta,y+c(s-t)\sin\theta\right)ds},
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
N_1^B(t,x,y) = {} & \Bigg(\int_0^{\frac{x}{c\cos\theta}-\frac{a_1}{c\cos\theta}} e^{\sigma\int_0^s \rho(|M|)\left(r+t-\frac{x}{c\cos\theta}+\frac{a_1}{c\cos\theta},rc\cos\theta+a_1,rc\sin\theta-\frac{\sin\theta}{\cos\theta}x+y+\frac{\sin\theta}{\cos\theta}a_1\right)dr} \\
& \times Q_1^\sigma(|M|)\left(s+t-\tfrac{x}{c\cos\theta}+\tfrac{a_1}{c\cos\theta},\, sc\cos\theta+a_1,\, sc\sin\theta-\tfrac{\sin\theta}{\cos\theta}x+y+\tfrac{\sin\theta}{\cos\theta}a_1\right)ds \\
& + N_1^-\left(t-\tfrac{x}{c\cos\theta}+\tfrac{a_1}{c\cos\theta}, -\tfrac{\sin\theta}{\cos\theta}x+y+\tfrac{\sin\theta}{\cos\theta}a_1\right)\Bigg) \\
& \times e^{-\sigma\int_0^t \rho(|M|)\left(s+t-\frac{x}{c\cos\theta}+\frac{a_1}{c\cos\theta},sc\cos\theta+a_1,sc\sin\theta-\frac{\sin\theta}{\cos\theta}x+y+\frac{\sin\theta}{\cos\theta}a_1\right)ds},
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
N_1^C(t,x,y) = {} & \Bigg(\int_0^{\frac{y}{c\sin\theta}-\frac{a_2}{c\sin\theta}} e^{\sigma\int_0^s \rho(|M|)\left(r+t-\frac{y}{c\sin\theta}+\frac{a_2}{c\sin\theta},rc\cos\theta+x-\frac{\cos\theta}{\sin\theta}y+\frac{\cos\theta}{\sin\theta}a_2,rc\sin\theta+a_2\right)dr} \\
& \times Q_1^\sigma(|M|)\left(s+t-\tfrac{y}{c\sin\theta}+\tfrac{a_2}{c\sin\theta},\, sc\cos\theta+x-\tfrac{\cos\theta}{\sin\theta}y+\tfrac{\cos\theta}{\sin\theta}a_2,\, sc\sin\theta+a_2\right)ds \\
& + N_1^{--}\left(t-\tfrac{y}{c\sin\theta}+\tfrac{a_2}{c\sin\theta},\, x-\tfrac{\cos\theta}{\sin\theta}y+\tfrac{\cos\theta}{\sin\theta}a_2\right)\Bigg) \\
& \times e^{-\sigma\int_0^{\frac{y}{c\sin\theta}-\frac{a_2}{c\sin\theta}} \rho(|M|)\left(s+t-\frac{y}{c\sin\theta}+\frac{a_2}{c\sin\theta},sc\cos\theta+x-\frac{\cos\theta}{\sin\theta}y+\frac{\cos\theta}{\sin\theta}a_2,sc\sin\theta+a_2\right)ds}.
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
N_2(t,x,y) = {} & N_2^A(t,x,y)\cdot \mathbb{I}_{\begin{cases} x-ct\cos\theta \ge a_1 \\ y-ct\sin\theta \ge a_2 \end{cases}}(t,x,y) \\
& + N_2^B(t,x,y)\cdot \mathbb{I}_{\begin{cases} x-ct\cos\theta \le a_1 \\ x\sin\theta - y\cos\theta \le a_1\sin\theta - a_2\cos\theta \end{cases}}(t,x,y) \\
& + N_2^C(t,x,y)\cdot \mathbb{I}_{\begin{cases} y-ct\sin\theta \le a_2 \\ x\sin\theta - y\cos\theta \ge a_1\sin\theta - a_2\cos\theta \end{cases}}(t,x,y)
\end{aligned}
\tag{42}
$$

where

$$N_2^A(t,x,y) = \left(\int_0^t e^{\sigma \int_0^s \rho(|M|)(r,x-c(r-t)\sin\theta, y+c(r-t)\cos\theta)dr} \times Q_2^\sigma(|M|)(s, x-c(s-t)\sin\theta, y+c(s-t)\cos\theta)\,ds + N_2^0(x+ct\sin\theta, y-ct\cos\theta) \right) \times e^{-\sigma \int_0^t \rho(|M|)(s,x-c(s-t)\sin\theta, y+c(s-t)\cos\theta)ds}, \tag{43}$$

$$N_2^B(t,x,y) = \left(\int_0^{-\frac{1}{c\sin\theta}x+\frac{b_1}{c\sin\theta}} e^{\sigma \int_0^s \rho(|M|)\left(r+t+\frac{1}{c\sin\theta}x-\frac{b_1}{c\sin\theta}, -rc\sin\theta+b_1, rc\cos\theta+\frac{\cos\theta}{\sin\theta}x+y-\frac{\cos\theta}{\sin\theta}b_1\right)dr} \times Q_2^\sigma(|M|)\left(s+t+\frac{1}{c\sin\theta}x-\frac{b_1}{c\sin\theta}, -sc\sin\theta+b_1, sc\cos\theta+\frac{\cos\theta}{\sin\theta}x+y-\frac{\cos\theta}{\sin\theta}b_1\right)ds + N_2^+\left(t+\frac{1}{c\sin\theta}x-\frac{b_1}{c\sin\theta}, \frac{\cos\theta}{\sin\theta}x+y-\frac{\cos\theta}{\sin\theta}b_1\right) \right) \times e^{-\sigma \int_0^{-\frac{1}{c\sin\theta}x+\frac{b_1}{c\sin\theta}} \rho(|M|)\left(s+t+\frac{1}{c\sin\theta}x-\frac{b_1}{c\sin\theta}, -sc\sin\theta+b_1, sc\cos\theta+\frac{\cos\theta}{\sin\theta}x+y-\frac{\cos\theta}{\sin\theta}b_1\right)ds}, \tag{44}$$

$$N_2^C(t,x,y) = \left(\int_0^{\frac{1}{c\cos\theta}y-\frac{a_2}{c\cos\theta}} e^{\sigma \int_0^s \rho(|M|)\left(r+t-\frac{1}{c\cos\theta}y+\frac{a_2}{c\cos\theta}, -rc\sin\theta+x+\frac{\sin\theta}{\cos\theta}y-\frac{\sin\theta}{\cos\theta}a_2, rc\cos\theta+a_2\right)dr} \times Q_2^\sigma(|M|)\left(s+t-\frac{1}{c\cos\theta}y+\frac{a_2}{c\cos\theta}, -sc\sin\theta+x+\frac{\sin\theta}{\cos\theta}y-\frac{\sin\theta}{\cos\theta}a_2, sc\cos\theta+a_2\right)ds + N_2^{--}\left(t-\frac{1}{c\cos\theta}y+\frac{a_2}{c\cos\theta}, x+\frac{\sin\theta}{\cos\theta}y-\frac{\sin\theta}{\cos\theta}a_2\right) \right) \times e^{-\sigma \int_0^{\frac{1}{c\cos\theta}y-\frac{a_2}{c\cos\theta}} \rho(|M|)\left(s+t-\frac{1}{c\cos\theta}y+\frac{a_2}{c\cos\theta}, -sc\sin\theta+x+\frac{\sin\theta}{\cos\theta}y-\frac{\sin\theta}{\cos\theta}a_2, sc\cos\theta+a_2\right)ds}. \tag{45}$$

$$N_3(t,x,y) = N_3^A(t,x,y)\cdot \mathbb{I}_{\begin{cases} x-ct\sin\theta \geq a_1 \\ y+ct\cos\theta \leq b_2 \end{cases}}(t,x,y) + N_3^B(t,x,y)\cdot \mathbb{I}_{\begin{cases} x-ct\sin\theta \leq a_1 \\ x\cos\theta+y\sin\theta \leq a_1\cos\theta+b_2\sin\theta \end{cases}}(t,x,y) + N_3^C(t,x,y)\cdot \mathbb{I}_{\begin{cases} y+ct\cos\theta \geq b_2 \\ x\cos\theta+y\sin\theta \geq a_1\cos\theta+b_2\sin\theta \end{cases}}(t,x,y) \tag{46}$$

where

$$N_3^A(t,x,y) = \left(\int_0^t e^{\sigma \int_0^s \rho(|M|)(r,x+c(r-t)\sin\theta, y-c(r-t)\cos\theta)dr} \cdot Q_3^\sigma(|M|) \times (s, x+c(s-t)\sin\theta, y-c(s-t)\cos\theta)\,ds + N_3^0(x-ct\sin\theta, y+ct\cos\theta) \right)$$

$$\times e^{-\sigma\int_0^t \rho(|M|)(s,x+c(s-t)\sin\theta,y-c(s-t)\cos\theta)ds}, \tag{47}$$

$$\begin{aligned} N_3^B(t,x,y) =& \left(\int_0^{\frac{1}{c\sin\theta}x-\frac{a_1}{c\sin\theta}} e^{\sigma\int_0^s \rho(|M|)\left(r+t-\frac{1}{c\sin\theta}x+\frac{a_1}{c\sin\theta},rc\sin\theta+a_1,-rc\cos\theta+\frac{\cos\theta}{\sin\theta}x+y-\frac{\cos\theta}{\sin\theta}a_1\right)dr} \right. \\ & \times Q_3^\sigma(|M|)\left(s+t-\frac{1}{c\sin\theta}x+\frac{a_1}{c\sin\theta}, sc\sin\theta+a_1, -sc\cos\theta+\frac{\cos\theta}{\sin\theta}x+y-\frac{\cos\theta}{\sin\theta}a_1\right)ds \\ & \left. + N_3^-\left(t-\frac{1}{c\sin\theta}x+\frac{a_1}{c\sin\theta}, \frac{\cos\theta}{\sin\theta}x+y-\frac{\cos\theta}{\sin\theta}a_1\right)\right) \\ & \times e^{-\sigma\int_0^{\frac{1}{c\sin\theta}x-\frac{a_1}{c\sin\theta}} \rho(|M|)\left(s+t-\frac{1}{c\sin\theta}x+\frac{a_1}{c\sin\theta},sc\sin\theta+a_1,-sc\cos\theta+\frac{\cos\theta}{\sin\theta}x+y-\frac{\cos\theta}{\sin\theta}a_1\right)ds}, \end{aligned} \tag{48}$$

$$\begin{aligned} N_3^C(t,x,y) =& \left(\int_0^{-\frac{1}{c\cos\theta}y+\frac{b_2}{c\cos\theta}} e^{\sigma\int_0^s \rho(|M|)\left(r+t+\frac{1}{c\cos\theta}y-\frac{b_2}{c\cos\theta},rc\sin\theta+x+\frac{\sin\theta}{\cos\theta}y-\frac{\sin\theta}{\cos\theta}b_2,-rc\cos\theta+b_2\right)dr} \right. \\ & \times Q_3^\sigma(|M|)\left(s+t+\frac{1}{c\cos\theta}y-\frac{b_2}{c\cos\theta}, sc\sin\theta+x+\frac{\sin\theta}{\cos\theta}y-\frac{\sin\theta}{\cos\theta}b_2, -sc\cos\theta+b_2\right)ds \\ & \left. + N_3^{++}\left(t+\frac{1}{c\cos\theta}y-\frac{b_2}{c\cos\theta}, x+\frac{\sin\theta}{\cos\theta}y-\frac{\sin\theta}{\cos\theta}b_2\right)\right) \\ & \times e^{-\sigma\int_0^{-\frac{1}{c\cos\theta}y+\frac{b_2}{c\cos\theta}} \rho(|M|)\left(s+t+\frac{1}{c\cos\theta}y-\frac{b_2}{c\cos\theta},sc\sin\theta+x+\frac{\sin\theta}{\cos\theta}y-\frac{\sin\theta}{\cos\theta}b_2,-sc\cos\theta+b_2\right)ds}. \end{aligned} \tag{49}$$

$$\begin{aligned} N_4(t,x,y) =& N_4^A(t,x,y)\cdot \mathbb{I}_{\begin{cases} x+ct\cos\theta \le b_1 \\ y+ct\sin\theta \le b_2 \end{cases}}(t,x,y) \\ & + N_4^B(t,x,y)\cdot \mathbb{I}_{\begin{cases} x+ct\cos\theta \ge b_1 \\ x\sin\theta - y\cos\theta \ge b_1\sin\theta - b_2\cos\theta \end{cases}}(t,x,y) \\ & + N_4^C(t,x,y)\cdot \mathbb{I}_{\begin{cases} y+ct\sin\theta \ge b_2 \\ x\sin\theta - y\cos\theta \le b_1\sin\theta - b_2\cos\theta \end{cases}}(t,x,y) \end{aligned} \tag{50}$$

where

$$\begin{aligned} N_4^A(t,x,y) =& \left(\int_0^t e^{\sigma\int_0^s \rho(|M|)(r,x-c(r-t)\cos\theta,y-c(r-t)\sin\theta)dr} \cdot Q_4^\sigma(|M|) \right. \\ & \left. \times (s, x-c(s-t)\cos\theta, y-c(s-t)\sin\theta)\,ds + N_4^0(x+ct\cos\theta, y+ct\sin\theta)\right) \\ & \times e^{-\sigma\int_0^t \rho(|M|)(s,x-c(s-t)\cos\theta,y-c(s-t)\sin\theta)ds}, \end{aligned} \tag{51}$$

$$\begin{aligned} N_4^B(t,x,y) =& \left(\int_0^{-\frac{1}{c\cos\theta}x+\frac{b_1}{c\cos\theta}} e^{\sigma\int_0^s \rho(|M|)\left(r+t+\frac{1}{c\cos\theta}x-\frac{b_1}{c\cos\theta},-rc\cos\theta+b_1,-rc\sin\theta-\frac{\sin\theta}{\cos\theta}x+y+\frac{\sin\theta}{\cos\theta}b_1\right)dr} \right. \\ & \times Q_4^\sigma(|M|)\left(s+t+\frac{1}{c\cos\theta}x-\frac{b_1}{c\cos\theta}, -sc\cos\theta+b_1, -sc\sin\theta-\frac{\sin\theta}{\cos\theta}x+y+\frac{\sin\theta}{\cos\theta}b_1\right)ds \end{aligned}$$

$$+ N_4^{+}\left(t + \frac{1}{c\cos\theta}x - \frac{b_1}{c\cos\theta}, -\frac{\sin\theta}{\cos\theta}x + y + \frac{\sin\theta}{\cos\theta}b_1\right)\Bigg)$$
$$\times e^{-\sigma\int_0^{-\frac{1}{c\cos\theta}x+\frac{b_1}{c\cos\theta}} \rho(|M|)\left(s+t+\frac{1}{c\cos\theta}x-\frac{b_1}{c\cos\theta}, -sc\cos\theta+b_1, -sc\sin\theta-\frac{\sin\theta}{\cos\theta}x+y+\frac{\sin\theta}{\cos\theta}b_1\right)ds}, \tag{52}$$

$$N_4^C(t,x,y) = \Bigg(\int_0^{-\frac{1}{c\sin\theta}y+\frac{b_2}{c\sin\theta}} e^{\sigma\int_0^s \rho(|M|)\left(r+t+\frac{1}{c\sin\theta}y-\frac{b_2}{c\sin\theta}, -rc\cos\theta+x-\frac{\cos\theta}{\sin\theta}y+\frac{\cos\theta}{\sin\theta}b_2, -rc\sin\theta+b_2\right)dr}$$
$$\times Q_4^{\sigma}(|M|)\left(s+t+\frac{1}{c\sin\theta}y - \frac{b_2}{c\sin\theta}, -sc\cos\theta + x - \frac{\cos\theta}{\sin\theta}y + \frac{\cos\theta}{\sin\theta}b_2, -sc\sin\theta + b_2\right)ds$$
$$+ N_4^{++}\left(t + \frac{1}{c\sin\theta}y - \frac{b_2}{c\sin\theta}, x - \frac{\cos\theta}{\sin\theta}y + \frac{\cos\theta}{\sin\theta}b_2\right)\Bigg)$$
$$\times e^{-\sigma\int_0^{-\frac{1}{c\sin\theta}y+\frac{b_2}{c\sin\theta}} \rho(|M|)\left(s+t+\frac{1}{c\sin\theta}y-\frac{b_2}{c\sin\theta}, -sc\cos\theta+x-\frac{\cos\theta}{\sin\theta}y+\frac{\cos\theta}{\sin\theta}b_2, -sc\sin\theta+b_2\right)ds}. \tag{53}$$

Now we have

$$\begin{aligned}
Q_1^{\sigma}(|M|) &= \sigma(|M_1| + |M_2| + |M_3|)|M_1| + 2cS|M_2||M_3| + (\sigma - 2cS)|M_1||M_4|, \\
Q_2^{\sigma}(|M|) &= \sigma(|M_1| + |M_2| + |M_4|)|M_2| + 2cS|M_1||M_4| + (\sigma - 2cS)|M_2||M_3|, \\
Q_3^{\sigma}(|M|) &= \sigma(|M_1| + |M_3| + |M_4|)|M_3| + 2cS|M_1||M_4| + (\sigma - 2cS)|M_2||M_3|, \\
Q_4^{\sigma}(|M|) &= \sigma(|M_2| + |M_3| + |M_4|)|M_4| + 2cS|M_2||M_3| + (\sigma - 2cS)|M_1||M_4|.
\end{aligned}$$

and we conclude that for $\sigma \geq 2cS$, the solution $N = (N_1, N_2, N_3, N_4)$ of $(\Sigma_{\sigma,M})$ is non-negative.

Let us consider the operator $\mathcal{T}^{\sigma} : M \longrightarrow N_M$ where N_M is the unique non-negative solution of the problem $(\Sigma_{\sigma,M})$ for sufficiently large σ.

We easily verify from the statements of (Σ_{σ}) (30)-(33) and $(\Sigma_{\sigma,M})$ (34)-(37) that for M possessing all first-order partial derivatives, $\mathcal{T}^{\sigma}(M) = M$ iff M is a solution of (Σ_{σ}) for sufficiently large σ i.e. M is a solution of Σ as (Σ_{σ}) is equivalent to Σ. As $\mathcal{T}^{\sigma}(M)$ is non-negative for sufficiently large σ , so is any solution M of Σ. □

4. Existence and Uniqueness

4.1. Auxiliary Operator

Let $M = (M_1, M_2, M_3, M_4)$ be fixed. Replacing N by M in the right-hand side of Σ (Eqs. (3)–(11)), we obtain the linear system Σ_M defined by

$$\frac{\partial N_1}{\partial t} + c\cos\theta\frac{\partial N_1}{\partial x} + c\sin\theta\frac{\partial N_1}{\partial y} = Q(M), \quad (t,x,y) \in \mathring{\mathscr{P}}, \tag{54}$$
$$\frac{\partial N_2}{\partial t} - c\sin\theta\frac{\partial N_2}{\partial x} + c\cos\theta\frac{\partial N_2}{\partial y} = -Q(M), \quad (t,x,y) \in \mathring{\mathscr{P}}, \tag{55}$$
$$\frac{\partial N_3}{\partial t} + c\sin\theta\frac{\partial N_3}{\partial x} - c\cos\theta\frac{\partial N_3}{\partial y} = -Q(M), \quad (t,x,y) \in \mathring{\mathscr{P}}, \tag{56}$$
$$\frac{\partial N_4}{\partial t} - c\cos\theta\frac{\partial N_4}{\partial x} - c\sin\theta\frac{\partial N_4}{\partial y} = Q(M), \quad (t,x,y) \in \mathring{\mathscr{P}}, \tag{57}$$

subject to conditions (7)–(27).

The system Σ_M admits a unique continuous solution

$$\mathcal{T}(M) = (\mathcal{T}_i(M))_{i=1}^4,$$

whose explicit representation is given in Appendix 5, see formulas (94)–(109).

It follows immediately that the solutions of Σ correspond to the fixed points of the operator

$$\begin{aligned} \mathcal{T} : C(\mathscr{P};\mathbb{R}^4) &\longrightarrow C(\mathscr{P};\mathbb{R}^4), \\ M &\longmapsto \mathcal{T}(M) = \big(\mathcal{T}_i(M)\big)_{i=1}^4, \end{aligned} \tag{58}$$

where the fixed points are assumed to possess all first-order partial derivatives.

We can show, following [18], the following result.

Lemma 1. *The operator $\mathcal{T}$ is continuous.*

Proof. From Eqs. (94)–(97), for $M, N \in C(\mathscr{P}, \mathbb{R}^4)$ we obtain

$$\begin{aligned} \|\mathcal{T}_1(M) - \mathcal{T}_1(N)\|_\infty \leq \max\Bigg\{ &\sup_{(t,x,y)\in\mathscr{P}} \left| \int_0^t \big[Q(M) - Q(N)\big](s, x + c(s-t)\cos\theta, y + c(s-t)\sin\theta)\; ds \right|; \\ &\sup_{(t,x,y)\in\mathscr{P}} \Bigg| \int_0^{\frac{1}{c\cos\theta}x - \frac{a_1}{c\cos\theta}} \big[Q(M) - Q(N)\big] \\ &\Big(s + t - \tfrac{1}{c\cos\theta}x + \tfrac{a_1}{c\cos\theta},\, sc\cos\theta + a_1,\, sc\sin\theta - \tfrac{\sin\theta}{\cos\theta}x + y + \tfrac{\sin\theta}{\cos\theta}a_1\Big) ds \Bigg|; \\ &\sup_{(t,x,y)\in\mathscr{P}} \Bigg| \int_0^{\frac{1}{c\sin\theta}y - \frac{a_2}{c\sin\theta}} \big[Q(M) - Q(N)\big] \\ &\Big(s + t - \tfrac{1}{c\sin\theta}y + \tfrac{a_2}{c\sin\theta},\, sc\cos\theta + x - \tfrac{\cos\theta}{\sin\theta}y + \tfrac{\cos\theta}{\sin\theta}a_2,\, sc\sin\theta + a_2\Big) ds \Bigg| \Bigg\}. \end{aligned} \tag{59}$$

Moreover,

$$\begin{aligned} Q(M) - Q(N) = &\, 2cS(M_2 - N_2)M_3 + 2cS\, N_2(M_3 - N_3) \\ &- 2cS(M_1 - N_1)M_4 - 2cS\, N_1(M_4 - N_4), \end{aligned} \tag{60}$$

which implies

$$\begin{aligned} \|Q(M) - Q(N)\|_\infty \leq &\, 2cS\|M_2 - N_2\|_\infty \|M_3\|_\infty + 2cS\|N_2\|_\infty \|M_3 - N_3\|_\infty \\ &+ 2cS\|M_1 - N_1\|_\infty \|M_4\|_\infty + 2cS\|N_1\|_\infty \|M_4 - N_4\|_\infty. \end{aligned} \tag{61}$$

Hence,

$$\begin{aligned} \|Q(M) - Q(N)\|_\infty &\leq 4cS\|M - N\|\|M\| + 4cS\|N\|\|M - N\| \\ &\leq 4cS(\|M\| + \|N\|)\|M - N\|. \end{aligned} \tag{62}$$

Substituting into (59) yields

$$\|\mathcal{T}_1(M) - \mathcal{T}_1(N)\|_\infty \leq \max\left\{ T, \tfrac{b_1 - a_1}{c\cos\theta}, \tfrac{b_2 - a_2}{c\sin\theta} \right\} \cdot 4cS(\|M\| + \|N\|)\|M - N\|. \tag{63}$$

Analogous inequalities follow:

$$\|\mathcal{T}_2(M) - \mathcal{T}_2(N)\|_\infty \leq \max\left\{ T, \tfrac{b_1 - a_1}{c\sin\theta}, \tfrac{b_2 - a_2}{c\cos\theta} \right\} \cdot 4cS(\|M\| + \|N\|)\|M - N\|, \tag{64}$$

$$\|\mathcal{T}_3(M) - \mathcal{T}_3(N)\|_\infty \leq \max\left\{ T, \tfrac{b_1 - a_1}{c\sin\theta}, \tfrac{b_2 - a_2}{c\cos\theta} \right\} \cdot 4cS(\|M\| + \|N\|)\|M - N\|, \tag{65}$$

$$\|\mathcal{T}_4(M)-\mathcal{T}_4(N)\|_\infty \le \max\left\{T, \tfrac{b_1-a_1}{c\cos\theta}, \tfrac{b_2-a_2}{c\sin\theta}\right\}\cdot 4cS(\|M\|+\|N\|)\|M-N\|. \tag{66}$$

Therefore, combining these results we obtain

$$\begin{aligned}\|\mathcal{T}(M)-\mathcal{T}(N)\| &= \max_{1\le i\le 4}\|\mathcal{T}_i(M)-\mathcal{T}_i(N)\|_\infty \\ &\le \underbrace{\max\left\{T, \tfrac{b_1-a_1}{c\cos\theta}, \tfrac{b_2-a_2}{c\sin\theta}, \tfrac{b_1-a_1}{c\sin\theta}, \tfrac{b_2-a_2}{c\cos\theta}\right\}\cdot 4cS}_{\equiv p'}(\|M\|+\|N\|)\|M-N\|. \end{aligned} \tag{67}$$

This proves that $\mathcal{T}$ is continuous. □

Proposition 1. *Suppose $M=(M_1,M_2,M_3,M_4)\in C(\mathscr{P};\mathbb{R}^4)$ is such that $\dfrac{\partial M_i}{\partial t}, \dfrac{\partial M_i}{\partial x}, \dfrac{\partial M_i}{\partial y}$ exist in $\mathring{\mathscr{P}}$, except possibly on a finite number of planes, and are continuous and bounded for all $i=1,2,3,4$. Then each of the derivatives*

$$\frac{\partial \mathcal{T}_i(M)}{\partial t}, \qquad \frac{\partial \mathcal{T}_i(M)}{\partial x}, \qquad \frac{\partial \mathcal{T}_i(M)}{\partial y}, \qquad (i=1,2,3,4)$$

are also defined in $\mathring{\mathscr{P}}$, except possibly on a finite number of planes, and are continuous and bounded.

In other words, if we denote by $\mathscr{E}$ the subspace of $C(\mathscr{P};\mathbb{R})$ consisting of functions u that are continuous on $\mathscr{P}$ and whose partial derivatives $\dfrac{\partial u}{\partial t}, \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$ exist in $\mathring{\mathscr{P}}$, except possibly on finitely many planes, and are continuous and bounded, then we have

$$\mathcal{T}(\mathscr{E}^4)\subset\mathscr{E}^4.$$

Proof. The result follows directly from the explicit expressions of the derivatives of $\mathcal{T}_i(M)$, $i=1,2,3,4$, given in the appendix (5). □

For $N=(N_i)_{i=1}^4\in\mathscr{E}^4$, define

$$\mathscr{V}(N)\equiv\max\left\{\|N\|, \left\|\frac{\partial N}{\partial t}\right\|, \left\|\frac{\partial N}{\partial x}\right\|, \left\|\frac{\partial N}{\partial y}\right\|\right\}.$$

For $R>0$, set

$$\mathscr{B}_R\equiv\left\{N\in\mathscr{E}^4 : \mathscr{V}(N)\le R\right\}.$$

Proposition 2. *For every $R>0$, the sets $\mathscr{B}_R$ are non-empty convex subsets of $C(\mathscr{P};\mathbb{R}^4)$ and are relatively compact in $\big(C(\mathscr{P};\mathbb{R}^4),\|\cdot\|\big)$.*

Proof. First, $\mathscr{B}_R$ is non-empty since it contains the zero function. Let $M,N\in\mathscr{B}_R$ and $\lambda\in[0,1]$. Then $\lambda M+(1-\lambda)N\in\mathscr{E}^4$ and

$$\mathscr{V}\big(\lambda M+(1-\lambda)N\big)\le R,$$

so $\mathscr{B}_R$ is convex.

By construction, $\mathscr{B}_R$ is bounded in $\big(C(\mathscr{P};\mathbb{R}^4),\|\cdot\|\big)$. Moreover, for each $M\in\mathscr{B}_R$ the functions

$$\frac{\partial M_i}{\partial t}, \quad \frac{\partial M_i}{\partial x}, \quad \frac{\partial M_i}{\partial y}, \qquad i=1,2,3,4,$$

are continuous and uniformly bounded on their domain. Hence, for each i there exists a constant $k^i>0$, independent of M, such that

$$\|d(M_i)(t,x,y)\|_{\mathscr{L}(\mathbb{R}^3,\mathbb{R})}\le k^i, \qquad \forall(t,x,y)\in\mathscr{P}. \tag{68}$$

Take $M \in \mathscr{B}_R$ and $(t,x,y),(t',x',y') \in \mathscr{P}$ such that $\left(\partial M/\partial t, \partial M/\partial x, \partial M/\partial y\right)$ is defined on the segment

$$\left[(t,x,y),(t',x',y')\right] \equiv \{(t,x,y)+\alpha(t'-t,x'-x,y'-y) : 0 \le \alpha \le 1\}.$$

Since $\mathscr{P}$ is convex, by the mean value inequality, (68) yields for all $i = 1,2,3,4$:

$$|M_i(t,x,y) - M_i(t',x',y')| \le k^i \|(t,x,y)-(t',x',y')\|_{\mathbb{R}^3}. \tag{69}$$

Given $\varepsilon > 0$, set

$$\alpha_\varepsilon \equiv \min_{1\le i\le 4} \frac{\varepsilon}{k^i}.$$

Then from (69) we obtain

$$\|(t,x,y)-(t',x',y')\|_{\mathbb{R}^3} < \alpha_\varepsilon \implies \|M(t,x,y)-M(t',x',y')\|_{\mathbb{R}^4} \le \varepsilon. \tag{70}$$

Since the points where derivatives of M are undefined lie on finitely many planes, and M is continuous on $\mathscr{P}$, inequality (70) extends to all of $\mathscr{P}$. Thus, $\mathscr{B}_R$ is equicontinuous in $\left(C(\mathscr{P};\mathbb{R}^4), \|\cdot\|\right)$.

By the Arzelà–Ascoli theorem, we conclude that $\mathscr{B}_R$ is relatively compact in $\left(C(\mathscr{P};\mathbb{R}^4), \|\cdot\|\right)$. □

Proposition 3. *There exists $p,q > 0$ such that*

$$\forall R > 0, \mathcal{T}\left(\mathscr{B}_R\right) \subset \mathscr{B}_{pR^2+q}. \tag{71}$$

Proof. By Proposition (1), if $M \in \mathscr{B}_R \subset \mathscr{E}^4$ then $\mathcal{T}(M) \in \mathscr{E}^4$. For any $M \in \mathscr{E}^4$, $Q(M) = 2cS\left(M_2M_3 - M_1M_4\right)$ implies

$$\|Q(M)\|_\infty \le 4cS\left(\mathscr{V}(M)\right)^2. \tag{72}$$

We then have from the expressions (94)-(109) of the solution of the system (Σ_M), the following inequalities for $M \in \mathscr{E}^4$:

$$\|\mathcal{T}_1(M)\|_\infty \le 4cS \cdot \max\left\{T, \frac{b_1-a_1}{c\cos\theta}, \frac{b_2-a_2}{c\sin\theta}\right\}\left(\mathscr{V}(M)\right)^2 + \max\left\{\left\|N_1^0\right\|_1, \|N_1^-\|_1, \|N_1^{--}\|_1\right\}, \tag{73}$$

$$\|\mathcal{T}_2(M)\|_\infty \le 4cS \cdot \max\left\{T, \frac{b_1-a_1}{c\sin\theta}, \frac{b_2-a_2}{c\cos\theta}\right\}\left(\mathscr{V}(M)\right)^2 + \max\left\{\left\|N_2^0\right\|_1, \|N_2^+\|_1, \|N_2^{--}\|_1\right\}, \tag{74}$$

$$\|\mathcal{T}_3(M)\|_\infty \le 4cS \cdot \max\left\{T, \frac{b_1-a_1}{c\sin\theta}, \frac{b_2-a_2}{c\cos\theta}\right\}\left(\mathscr{V}(M)\right)^2 + \max\left\{\left\|N_3^0\right\|_1, \|N_3^-\|_1, \|N_3^{++}\|_1\right\}, \tag{75}$$

$$\|\mathcal{T}_4(M)\|_\infty \le 4cS \cdot \max\left\{T, \frac{b_1-a_1}{c\cos\theta}, \frac{b_2-a_2}{c\sin\theta}\right\}\left(\mathscr{V}(M)\right)^2 + \max\left\{\left\|N_4^0\right\|_1, \|N_4^+\|_1, \|N_4^{++}\|_1\right\}, \tag{76}$$

and

$$\begin{aligned}\|\mathcal{T}(M)\| \le & 4cS \cdot \max\left\{T, \frac{b_1-a_1}{c\cos\theta}, \frac{b_2-a_2}{c\sin\theta}, \frac{b_1-a_1}{c\sin\theta}, \frac{b_2-a_2}{c\cos\theta}\right\}\left(\mathscr{V}(M)\right)^2 \\ & + \max_{1\le i\le 4}\left\{\left\|N_i^0\right\|_1, \|N_1^-\|_1, \|N_1^{--}\|_1, \|N_2^+\|_1, \|N_2^{--}\|_1, \|N_3^-\|_1, \|N_3^{++}\|_1, \|N_4^+\|_1, \|N_4^{++}\|_1\right\}.\end{aligned} \tag{77}$$

We have $\dfrac{\partial Q(M)}{\partial t} = 2cS\left(\dfrac{\partial M_2}{\partial t}M_3 + M_2\dfrac{\partial M_3}{\partial t} - \dfrac{\partial M_1}{\partial t}M_4 - M_1\dfrac{\partial M_4}{\partial t}\right)$ and similar expressions for $\dfrac{\partial Q(M)}{\partial x}, \dfrac{\partial Q(M)}{\partial y}$. Therefore

$$\left\|\frac{\partial Q(M)}{\partial t}\right\|_\infty, \left\|\frac{\partial Q(M)}{\partial x}\right\|_\infty, \left\|\frac{\partial Q(M)}{\partial y}\right\|_\infty \leq 2cS \cdot 4\left(\mathscr{V}(M)\right)^2 \leq 8cS\left(\mathscr{V}(M)\right)^2. \tag{78}$$

Eq. (78) with explicit formula of the derivatives $\dfrac{\partial \mathcal{T}_i(M)}{\partial t}, \dfrac{\partial \mathcal{T}_i(M)}{\partial x}, \dfrac{\partial \mathcal{T}_i(M)}{\partial y}, (i = 1,2,3,4)$ in the appendix Eq. (5) imply:

$$\begin{aligned}\left\|\frac{\partial \mathcal{T}_1(M)}{\partial t}\right\|_\infty \leq & 4cS\max\left\{1 + 2T(c\cos\theta + c\sin\theta); 2\frac{b_1 - a_1}{c\cos\theta}; 2\frac{b_2 - a_2}{c\sin\theta}\right\}\left(\mathscr{V}(M)\right)^2 \\ & \max\left\{(c\cos\theta + c\sin\theta)\left\|N_1^0\right\|_1; \|N_1^-\|_1; \|N_1^{--}\|_1\right\},\end{aligned} \tag{79}$$

$$\begin{aligned}\left\|\frac{\partial \mathcal{T}_2(M)}{\partial t}\right\|_\infty \leq & 4cS\max\left\{1 + 2T(c\cos\theta + c\sin\theta); 2\frac{b_1 - a_1}{c\sin\theta}; 2\frac{b_2 - a_2}{c\cos\theta}\right\}\left(\mathscr{V}(M)\right)^2 \\ & \max\left\{(c\cos\theta + c\sin\theta)\left\|N_2^0\right\|_1; \|N_2^+\|_1; \|N_2^{--}\|_1\right\},\end{aligned} \tag{80}$$

$$\begin{aligned}\left\|\frac{\partial \mathcal{T}_3(M)}{\partial t}\right\|_\infty \leq & 4cS\max\left\{1 + 2T(c\cos\theta + c\sin\theta); 2\frac{b_1 - a_1}{c\sin\theta}; 2\frac{b_2 - a_2}{c\cos\theta}\right\}\left(\mathscr{V}(M)\right)^2 \\ & \max\left\{(c\cos\theta + c\sin\theta)\left\|N_3^0\right\|_1; \|N_3^-\|_1; \|N_3^{++}\|_1\right\},\end{aligned} \tag{81}$$

$$\begin{aligned}\left\|\frac{\partial \mathcal{T}_4(M)}{\partial t}\right\|_\infty \leq & 4cS\max\left\{1 + 2T(c\cos\theta + c\sin\theta); 2\frac{b_1 - a_1}{c\cos\theta}; 2\frac{b_2 - a_2}{c\sin\theta}\right\}\left(\mathscr{V}(M)\right)^2 \\ & \max\left\{(c\cos\theta + c\sin\theta)\left\|N_4^0\right\|_1; \|N_4^+\|_1; \|N_4^{++}\|_1\right\},\end{aligned} \tag{82}$$

$$\begin{aligned}\left\|\frac{\partial \mathcal{T}_1(M)}{\partial x}\right\|_\infty \leq & 4cS\max\left\{2T; \frac{1}{c\cos\theta} + 2\frac{b_1 - a_1}{c\cos\theta}\left(\frac{1}{c\cos\theta} + \frac{\sin\theta}{\cos\theta}\right); 2\frac{b_2 - a_2}{c\sin\theta}\right\}\left(\mathscr{V}(M)\right)^2 \\ & \max\left\{\left\|N_1^0\right\|_1; \left(\frac{1}{c\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)\|N_1^-\|_1; \|N_1^{--}\|_1\right\},\end{aligned} \tag{83}$$

$$\begin{aligned}\left\|\frac{\partial \mathcal{T}_2(M)}{\partial x}\right\|_\infty \leq & 4cS\max\left\{2T; \frac{1}{c\sin\theta} + 2\frac{b_1 - a_1}{c\sin\theta}\left(\frac{1}{c\sin\theta} + \frac{\cos\theta}{\sin\theta}\right); 2\frac{b_2 - a_2}{c\cos\theta}\right\}\left(\mathscr{V}(M)\right)^2 \\ & \max\left\{\left\|N_2^0\right\|_1; \left(\frac{1}{c\sin\theta} + \frac{\cos\theta}{\sin\theta}\right)\|N_2^+\|_1; \|N_2^{--}\|_1\right\},\end{aligned} \tag{84}$$

$$\begin{aligned}\left\|\frac{\partial \mathcal{T}_3(M)}{\partial x}\right\|_\infty \leq & 4cS\max\left\{2T; \frac{1}{c\sin\theta} + 2\frac{b_1 - a_1}{c\sin\theta}\left(\frac{1}{c\sin\theta} + \frac{\cos\theta}{\sin\theta}\right); 2\frac{b_2 - a_2}{c\cos\theta}\right\}\left(\mathscr{V}(M)\right)^2 \\ & \max\left\{\left\|N_3^0\right\|_1; \left(\frac{1}{c\sin\theta} + \frac{\cos\theta}{\sin\theta}\right)\|N_3^-\|_1; \|N_3^{++}\|_1\right\},\end{aligned} \tag{85}$$

$$\left\| \frac{\partial \mathcal{T}_4(M)}{\partial x} \right\|_\infty \leq 4cS \max\left\{2T; \frac{1}{c\cos\theta} + 2\frac{b_1-a_1}{c\cos\theta}\left(\frac{1}{c\cos\theta} + \frac{\sin\theta}{\cos\theta}\right); 2\frac{b_2-a_2}{c\sin\theta}\right\}(\mathscr{V}(M))^2$$
$$\max\left\{\left\|N_4^0\right\|_1; \left(\frac{1}{c\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)\|N_4^+\|_1; \|N_4^{++}\|_1\right\}. \tag{86}$$

(79)-(82) imply

$$\left\| \frac{\partial \mathcal{T}(M)}{\partial t} \right\| \leq 4cS \max\left\{1 + 2T(c\cos\theta + c\sin\theta); 2\max\left\{\frac{1}{c\cos\theta}; \frac{1}{c\sin\theta}\right\}(b_1-a_1);\right.$$
$$\left. 2\max\left\{\frac{1}{c\cos\theta}; \frac{1}{c\sin\theta}\right\}(b_2-a_2)\right\}(\mathscr{V}(M))^2$$
$$+\max_{1\leq i\leq 4}\left\{(c\cos\theta + c\sin\theta)\left\|N_i^0\right\|_1, \|N_1^-\|_1, \|N_1^{--}\|_1,\right.$$
$$\left.\|N_2^+\|_1, \|N_2^{--}\|_1, \|N_3^-\|_1, \|N_3^{++}\|_1, \|N_4^+\|_1, \|N_4^{++}\|_1\right\}. \tag{87}$$

(83)-(86) imply

$$\left\| \frac{\partial \mathcal{T}(M)}{\partial x} \right\| \leq 4cS \max\left\{2T; \max\left\{\frac{1}{c\cos\theta}; \frac{1}{c\sin\theta}\right\} + 2(b_1-a_1)\max\left\{\frac{1}{c\cos\theta}\left(\frac{1}{c\cos\theta} + \frac{\sin\theta}{\cos\theta}\right);\right.\right.$$
$$\left.\left.\frac{1}{c\sin\theta}\left(\frac{1}{c\sin\theta} + \frac{\cos\theta}{\sin\theta}\right)\right\}; 2(b_2-a_2)\max\left\{\frac{1}{c\cos\theta}; \frac{1}{c\sin\theta}\right\}\right\}(\mathscr{V}(M))^2$$
$$+\max_{1\leq i\leq 4}\left\{\left\|N_i^0\right\|_1, \left(\frac{1}{c\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)\|N_1^-\|_1, \|N_1^{--}\|_1, \left(\frac{1}{c\sin\theta} + \frac{\cos\theta}{\sin\theta}\right)\|N_2^+\|_1, \|N_2^{--}\|_1,\right.$$
$$\left.\left(\frac{1}{c\sin\theta} + \frac{\cos\theta}{\sin\theta}\right)\|N_3^-\|_1, \|N_3^{++}\|_1, \left(\frac{1}{c\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)\|N_4^+\|_1, \|N_4^{++}\|_1\right\}. \tag{88}$$

We similarly obtain

$$\left\| \frac{\partial \mathcal{T}(M)}{\partial y} \right\| \leq 4cS \max\left\{2T; 2(b_1-a_1)\max\left\{\frac{1}{c\cos\theta}; \frac{1}{c\sin\theta}\right\};\right.$$
$$\max\left\{\frac{1}{c\cos\theta}; \frac{1}{c\sin\theta}\right\} + 2(b_2-a_2)\max\left\{\frac{1}{c\cos\theta}\left(\frac{1}{c\cos\theta} + \frac{\sin\theta}{\cos\theta}\right);\right.$$
$$\left.\left.\frac{1}{c\sin\theta}\left(\frac{1}{c\sin\theta} + \frac{\cos\theta}{\sin\theta}\right)\right\}\right\}(\mathscr{V}(M))^2$$
$$+\max_{1\leq i\leq 4}\left\{\left\|N_i^0\right\|_1, \|N_1^-\|_1, \left(\frac{1}{c\sin\theta} + \frac{\cos\theta}{\sin\theta}\right)\|N_1^{--}\|_1, \|N_2^+\|_1, \left(\frac{1}{c\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)\|N_2^{--}\|_1,\right.$$
$$\left.\|N_3^-\|_1, \left(\frac{1}{c\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)\|N_3^{++}\|_1, \|N_4^+\|_1, \left(\frac{1}{c\sin\theta} + \frac{\cos\theta}{\sin\theta}\right)\|N_4^{++}\|_1\right\}. \tag{89}$$

(77), (87), (88) and (89) imply that for $M \in \mathscr{E}^4$,

$$\mathscr{V}(\mathcal{T}(M)) \leq 4cS \max\left\{\max\{1 + 2T(c\cos\theta + c\sin\theta); 2T\}; \max\left\{\frac{1}{c\cos\theta}; \frac{1}{c\sin\theta}\right\} + 2\max\left\{\frac{1}{c\cos\theta}; \frac{1}{c\sin\theta};\right.\right.$$
$$\left.\left.\frac{1}{c\cos\theta}\left(\frac{1}{c\cos\theta} + \frac{\sin\theta}{\cos\theta}\right); \frac{1}{c\sin\theta}\left(\frac{1}{c\sin\theta} + \frac{\cos\theta}{\sin\theta}\right)\right\}\max\{b_1-a_1; b_2-a_2\}\right\}(\mathscr{V}(M))^2$$

$$+\max_{1\le i\le 4}\left\{\max\left\{1;c\cos\theta+c\sin\theta\right\}\left\|N_i^0\right\|_1;\right.$$
$$\max\left\{1;\frac{1}{c\cos\theta}+\frac{\sin\theta}{\cos\theta}\right\}\|N_1^-\|_1;\max\left\{1;\frac{1}{c\sin\theta}+\frac{\cos\theta}{\sin\theta}\right\}\|N_1^{--}\|_1;$$
$$\max\left\{1;\frac{1}{c\sin\theta}+\frac{\cos\theta}{\sin\theta}\right\}\|N_2^+\|_1;\max\left\{1;\frac{1}{c\cos\theta}+\frac{\sin\theta}{\cos\theta}\right\}\|N_2^{--}\|_1;$$
$$\max\left\{1;\frac{1}{c\sin\theta}+\frac{\cos\theta}{\sin\theta}\right\}\|N_3^-\|_1;\max\left\{1;\frac{1}{c\cos\theta}+\frac{\sin\theta}{\cos\theta}\right\}\|N_3^{++}\|_1;$$
$$\left.\max\left\{1;\frac{1}{c\cos\theta}+\frac{\sin\theta}{\cos\theta}\right\}\|N_4^+\|_1;\max\left\{1;\frac{1}{c\sin\theta}+\frac{\cos\theta}{\sin\theta}\right\}\|N_4^{++}\|_1\right\}, \tag{90}$$

that is $\mathscr{V}(\mathcal{T}(M))\le p(\mathscr{V}(M))^2+q$ where

$$p\equiv 4cs\max\left\{\max\left\{1+2T\left(c\cos\theta+c\sin\theta\right);2T\right\};\right.$$
$$\max\left\{\frac{1}{c\cos\theta};\frac{1}{c\sin\theta}\right\}+2\max\left\{\frac{1}{c\cos\theta};\frac{1}{c\sin\theta};\right.$$
$$\left.\left.\frac{1}{c\cos\theta}\left(\frac{1}{c\cos\theta}+\frac{\sin\theta}{\cos\theta}\right);\frac{1}{c\sin\theta}\left(\frac{1}{c\sin\theta}+\frac{\cos\theta}{\sin\theta}\right)\right\}\max\left\{b_1-a_1;b_2-a_2\right\}\right\}, \tag{91}$$

$$q\equiv\max_{1\le i\le 4}\left\{\max\left\{1;c\cos\theta+c\sin\theta\right\}\left\|N_i^0\right\|_1;\right.$$
$$\max\left\{1;\frac{1}{c\cos\theta}+\frac{\sin\theta}{\cos\theta}\right\}\|N_1^-\|_1;\max\left\{1;\frac{1}{c\sin\theta}+\frac{\cos\theta}{\sin\theta}\right\}\|N_1^{--}\|_1;$$
$$\max\left\{1;\frac{1}{c\sin\theta}+\frac{\cos\theta}{\sin\theta}\right\}\|N_2^+\|_1;\max\left\{1;\frac{1}{c\cos\theta}+\frac{\sin\theta}{\cos\theta}\right\}\|N_2^{--}\|_1;$$
$$\max\left\{1;\frac{1}{c\sin\theta}+\frac{\cos\theta}{\sin\theta}\right\}\|N_3^-\|_1;\max\left\{1;\frac{1}{c\cos\theta}+\frac{\sin\theta}{\cos\theta}\right\}\|N_3^{++}\|_1;$$
$$\left.\max\left\{1;\frac{1}{c\cos\theta}+\frac{\sin\theta}{\cos\theta}\right\}\|N_4^+\|_1;\max\left\{1;\frac{1}{c\sin\theta}+\frac{\cos\theta}{\sin\theta}\right\}\|N_4^{++}\|_1\right\}. \tag{92}$$

If $M\in\mathscr{B}_R$ then $\mathscr{V}(\mathcal{T}(M))\le pR^2+q$. So $\mathcal{T}(\mathscr{B}_R)\subset\mathscr{B}_{pR^2+q}$. □

Lemma 2. *The operator $\mathcal{T}$ is compact on $\mathscr{B}_R$ for all $R>0$.*

Proof. We have $\mathcal{T}(\mathscr{B}_R)\subset\mathscr{B}_{pR^2+q}$, and $\mathscr{B}_{pR^2+q}$ is relatively compact in $\left(C(\mathscr{P};\mathbb{R}^4),\|\cdot\|\right)$. Thus, $\mathcal{T}$ is compact on $\mathscr{B}_R$. □

Lemma 3. *If $pq\le\dfrac{1}{4}$ and*

$$\frac{1-\sqrt{1-4pq}}{2p}\le R\le\frac{1+\sqrt{1-4pq}}{2p},$$

then $\mathcal{T}(\mathscr{B}_R)\subset\mathscr{B}_R$.

Proof. Since $pR^2-R+q\le 0$, it follows that $pR^2+q\le R$. But $\mathcal{T}(\mathscr{B}_R)\subset\mathscr{B}_{pR^2+q}$, hence $\mathcal{T}(\mathscr{B}_R)\subset\mathscr{B}_R$. □

Theorem 3 (Schauder [19])**.** *Let $\mathcal{M}$ be a non-empty convex subset of a normed space $\mathscr{X}$, and let $\mathcal{T}$ be a continuous and compact mapping from $\mathcal{M}$ into $\mathcal{M}$. Then $\mathcal{T}$ has a fixed point.*

Proof of Theorem 1. Let $\frac{1-\sqrt{1-4pq}}{2p} \leq R \leq \frac{1+\sqrt{1-4pq}}{2p}$. From Proposition 2, $\mathscr{B}_R$ is a non-empty convex subset of $\big(C(\mathscr{P};\mathbb{R}^4), \|\cdot\|\big)$. By Lemmas 1, 2, and 3, $\mathcal{T}$ is continuous and compact from $\mathscr{B}_R$ into itself. Hence, by Schauder's Theorem 3, $\mathcal{T}$ has a fixed point $N \in \mathscr{B}_R$, which is a solution of problem Σ. By Theorem 2, this solution is non-negative. Furthermore, $N \in \mathscr{E}^4$ and $\mathscr{V}(N) \leq R \leq \frac{1+\sqrt{1-4pq}}{2p}$. Thus $N \in C(\mathscr{P};\mathbb{R}^4)$, and

$$\frac{\partial N_i}{\partial t}, \quad \frac{\partial N_i}{\partial x}, \quad \frac{\partial N_i}{\partial y} \quad (i = 1,2,3,4)$$

are defined in $\mathring{\mathscr{P}}$, except possibly on a finite number of planes, and remain continuous and bounded with

$$\|N\|, \ \left\|\frac{\partial N}{\partial t}\right\|, \ \left\|\frac{\partial N}{\partial x}\right\|, \ \left\|\frac{\partial N}{\partial y}\right\| \leq \frac{1+\sqrt{1-4pq}}{2p}.$$

Suppose now that the problem Σ admits two solutions M and N satisfying $\|M\|, \|N\| \leq \frac{1+\sqrt{1-4pq}}{2p}$. From relation (67), we obtain

$$\begin{aligned} \|\mathcal{T}(M) - \mathcal{T}(N)\| &\leq p' \cdot 2 \cdot \frac{1+\sqrt{1-4pq}}{2p} \|M-N\| \\ &\leq \frac{p'}{p}\big(1+\sqrt{1-4pq}\big)\|M-N\|. \end{aligned} \tag{93}$$

Since M and N are fixed points of $\mathcal{T}$, (93) reduces to

$$\|M-N\| \leq \frac{p'}{p}\big(1+\sqrt{1-4pq}\big)\|M-N\|,$$

i.e.,

$$\left(1 - \frac{p'}{p}\big(1+\sqrt{1-4pq}\big)\right)\|M-N\| \leq 0.$$

From (67) and (91), we have

$$\frac{p'}{p} = \frac{\max\{T,\alpha\}}{\max\{\beta_T,\beta\}} \leq \frac{1}{2},$$

with

$$\begin{aligned} \alpha &= \max\left\{\frac{b_1-a_1}{c\cos\theta}, \frac{b_2-a_2}{c\sin\theta}, \frac{b_1-a_1}{c\sin\theta}, \frac{b_2-a_2}{c\cos\theta}\right\}, \\ \beta_T &= \max\big\{1+2T(c\cos\theta + c\sin\theta), 2T\big\}, \\ \beta &= \max\left\{\frac{1}{c\cos\theta}, \frac{1}{c\sin\theta}\right\} \\ &\quad + 2\max\left\{\frac{1}{c\cos\theta}, \frac{1}{c\sin\theta}, \frac{1}{c\cos\theta}\left(\frac{1}{c\cos\theta} + \frac{\sin\theta}{\cos\theta}\right), \frac{1}{c\sin\theta}\left(\frac{1}{c\sin\theta} + \frac{\cos\theta}{\sin\theta}\right)\right\}\max\{b_1-a_1, b_2-a_2\}. \end{aligned}$$

Since $1+\sqrt{1-4pq} < 2$, we obtain

$$\frac{p'}{p}\big(1+\sqrt{1-4pq}\big) < 1,$$

so the prefactor in the inequality is strictly positive. Therefore, $\|M-N\| \leq 0$, implying $M = N$. Hence uniqueness. □

5. Conclusion

In this paper, we have extended the results of [18] to the general four-velocity Broadwell model in the plane. Specifically, we established the existence and uniqueness of classical positive solutions to the initial–boundary value problem for this model in a rectangular domain with Dirichlet boundary conditions. This study represents an important step toward a deeper mathematical understanding of initial–boundary value problems for unsteady discrete kinetic models in higher dimensions.

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Appendix

Solution of the system (Σ_M)

$$\begin{aligned}\mathcal{T}_1(M)(t,x,y) &= \mathcal{T}_1^A(M)(t,x,y)\,\mathbb{I}_{\substack{x-ct\cos\theta\geq a_1\\ y-ct\sin\theta\geq a_2}}(t,x,y)\\ &\quad+\mathcal{T}_1^B(M)(t,x,y)\,\mathbb{I}_{\substack{x-ct\cos\theta\leq a_1\\ x\sin\theta-y\cos\theta\leq a_1\sin\theta-a_2\cos\theta}}(t,x,y)\\ &\quad+\mathcal{T}_1^C(M)(t,x,y)\,\mathbb{I}_{\substack{y-ct\sin\theta\leq a_2\\ x\sin\theta-y\cos\theta\geq a_1\sin\theta-a_2\cos\theta}}(t,x,y). \end{aligned}\tag{94}$$

where

$$\begin{aligned}\mathcal{T}_1^A(M)(t,x,y) &= \int_0^t Q(M)(s,\,x+c(s-t)\cos\theta,\,y+c(s-t)\sin\theta)\;ds\\ &\quad+N_1^0(x-ct\cos\theta,\,y-ct\sin\theta)\,,\end{aligned}\tag{95}$$

$$\begin{aligned}\mathcal{T}_1^B(M)(t,x,y) &= \int_0^{\frac{x}{c\cos\theta}-\frac{a_1}{c\cos\theta}} Q(M)\Big(s+t-\tfrac{x}{c\cos\theta}+\tfrac{a_1}{c\cos\theta},\,sc\cos\theta+a_1,\,sc\sin\theta-\tfrac{\sin\theta}{\cos\theta}x+y+\tfrac{\sin\theta}{\cos\theta}a_1\Big)\,ds\\ &\quad+N_1^-\Big(t-\tfrac{x}{c\cos\theta}+\tfrac{a_1}{c\cos\theta},\,-\tfrac{\sin\theta}{\cos\theta}x+y+\tfrac{\sin\theta}{\cos\theta}a_1\Big)\,,\end{aligned}\tag{96}$$

$$\begin{aligned}\mathcal{T}_1^C(M)(t,x,y) &= \int_0^{\frac{y}{c\sin\theta}-\frac{a_2}{c\sin\theta}} Q(M)\Big(s+t-\tfrac{y}{c\sin\theta}+\tfrac{a_2}{c\sin\theta},\,sc\cos\theta+x-\tfrac{\cos\theta}{\sin\theta}y+\tfrac{\cos\theta}{\sin\theta}a_2,\,sc\sin\theta+a_2\Big)\,ds\\ &\quad+N_1^{--}\Big(t-\tfrac{y}{c\sin\theta}+\tfrac{a_2}{c\sin\theta},\,x-\tfrac{\cos\theta}{\sin\theta}y+\tfrac{\cos\theta}{\sin\theta}a_2\Big)\,.\end{aligned}\tag{97}$$

$$\begin{aligned}\mathcal{T}_2(M)(t,x,y) &= \mathcal{T}_2^A(M)(t,x,y)\,\mathbb{I}_{\substack{x-ct\cos\theta\geq a_1\\ y-ct\sin\theta\geq a_2}}(t,x,y)\\ &\quad+\mathcal{T}_2^B(M)(t,x,y)\,\mathbb{I}_{\substack{x-ct\cos\theta\leq a_1\\ x\sin\theta-y\cos\theta\leq a_1\sin\theta-a_2\cos\theta}}(t,x,y)\\ &\quad+\mathcal{T}_2^C(M)(t,x,y)\,\mathbb{I}_{\substack{y-ct\sin\theta\leq a_2\\ x\sin\theta-y\cos\theta\geq a_1\sin\theta-a_2\cos\theta}}(t,x,y). \end{aligned}\tag{98}$$

where

$$\begin{aligned}\mathcal{T}_2^A(M)(t,x,y) &= \int_0^t -Q(M)(s,\,x-c(s-t)\sin\theta,\,y+c(s-t)\cos\theta)\;ds\\ &\quad+N_2^0(x+ct\sin\theta,\,y-ct\cos\theta)\,,\end{aligned}\tag{99}$$

$$\begin{aligned}\mathcal{T}_2^B(M)(t,x,y) &= \int_0^{-\frac{x}{c\sin\theta}+\frac{b_1}{c\sin\theta}} -Q(M)\Big(s+t+\tfrac{x}{c\sin\theta}-\tfrac{b_1}{c\sin\theta},\,-sc\sin\theta+b_1,\,sc\cos\theta+\tfrac{\cos\theta}{\sin\theta}x+y-\tfrac{\cos\theta}{\sin\theta}b_1\Big)\,ds\\ &\quad+N_2^+\Big(t+\tfrac{x}{c\sin\theta}-\tfrac{b_1}{c\sin\theta},\,\tfrac{\cos\theta}{\sin\theta}x+y-\tfrac{\cos\theta}{\sin\theta}b_1\Big)\,,\end{aligned}\tag{100}$$

$$\mathcal{T}_2^C(M)(t,x,y) = \int_0^{\frac{y}{c\cos\theta}-\frac{a_2}{c\cos\theta}} -Q(M)\Big(s+t-\tfrac{y}{c\cos\theta}+\tfrac{a_2}{c\cos\theta},\ -sc\sin\theta+x+\tfrac{\sin\theta}{\cos\theta}y-\tfrac{\sin\theta}{\cos\theta}a_2,\ sc\cos\theta+a_2\Big)\,ds$$
$$+N_2^{--}\Big(t-\tfrac{y}{c\cos\theta}+\tfrac{a_2}{c\cos\theta},\ x+\tfrac{\sin\theta}{\cos\theta}y-\tfrac{\sin\theta}{\cos\theta}a_2\Big)\,. \tag{101}$$

$$\begin{aligned}\mathcal{T}_3(M)(t,x,y) &= \mathcal{T}_3^A(M)(t,x,y)\,\mathbb{I}_{\substack{x-ct\sin\theta\geq a_1\\ y+ct\cos\theta\leq b_2}}(t,x,y)\\ &\quad+\mathcal{T}_3^B(M)(t,x,y)\,\mathbb{I}_{\substack{x-ct\sin\theta\leq a_1\\ x\cos\theta+y\sin\theta\leq a_1\cos\theta+b_2\sin\theta}}(t,x,y)\\ &\quad+\mathcal{T}_3^C(M)(t,x,y)\,\mathbb{I}_{\substack{y+ct\cos\theta\geq b_2\\ x\cos\theta+y\sin\theta\geq a_1\cos\theta+b_2\sin\theta}}(t,x,y).\end{aligned} \tag{102}$$

where

$$\mathcal{T}_3^A(M)(t,x,y) = \int_0^t -Q(M)(s,\ x+c(s-t)\sin\theta,\ y-c(s-t)\cos\theta)\ ds$$
$$+N_3^0(x-ct\sin\theta,\ y+ct\cos\theta)\,, \tag{103}$$

$$\mathcal{T}_3^B(M)(t,x,y) = \int_0^{\frac{x}{c\sin\theta}-\frac{a_1}{c\sin\theta}} -Q(M)\Big(s+t-\tfrac{x}{c\sin\theta}+\tfrac{a_1}{c\sin\theta},\ sc\sin\theta+a_1,\ -sc\cos\theta+\tfrac{\cos\theta}{\sin\theta}x+y-\tfrac{\cos\theta}{\sin\theta}a_1\Big)\,ds$$
$$+N_3^{-}\Big(t-\tfrac{x}{c\sin\theta}+\tfrac{a_1}{c\sin\theta},\ \tfrac{\cos\theta}{\sin\theta}x+y-\tfrac{\cos\theta}{\sin\theta}a_1\Big)\,, \tag{104}$$

$$\mathcal{T}_3^C(M)(t,x,y) = \int_0^{-\frac{y}{c\cos\theta}+\frac{b_2}{c\cos\theta}} -Q(M)\Big(s+t+\tfrac{y}{c\cos\theta}-\tfrac{b_2}{c\cos\theta},\ sc\sin\theta+x+\tfrac{\sin\theta}{\cos\theta}y-\tfrac{\sin\theta}{\cos\theta}b_2,\ -sc\cos\theta+b_2\Big)\,ds$$
$$+N_3^{++}\Big(t+\tfrac{y}{c\cos\theta}-\tfrac{b_2}{c\cos\theta},\ x+\tfrac{\sin\theta}{\cos\theta}y-\tfrac{\sin\theta}{\cos\theta}b_2\Big)\,. \tag{105}$$

$$\begin{aligned}\mathcal{T}_4(M)(t,x,y) &= \mathcal{T}_4^A(M)(t,x,y)\,\mathbb{I}_{\substack{x+ct\cos\theta\leq b_1\\ y+ct\sin\theta\leq b_2}}(t,x,y)\\ &\quad+\mathcal{T}_4^B(M)(t,x,y)\,\mathbb{I}_{\substack{x+ct\cos\theta\geq b_1\\ x\sin\theta-y\cos\theta\geq b_1\sin\theta-b_2\cos\theta}}(t,x,y)\\ &\quad+\mathcal{T}_4^C(M)(t,x,y)\,\mathbb{I}_{\substack{y+ct\sin\theta\geq b_2\\ x\sin\theta-y\cos\theta\leq b_1\sin\theta-b_2\cos\theta}}(t,x,y).\end{aligned} \tag{106}$$

where

$$\mathcal{T}_4^A(M)(t,x,y) = \int_0^t Q(M)(s,\ x-c(s-t)\cos\theta,\ y-c(s-t)\sin\theta)\ ds$$
$$+N_4^0(x+ct\cos\theta,\ y+ct\sin\theta)\,, \tag{107}$$

$$\mathcal{T}_4^B(M)(t,x,y) = \int_0^{-\frac{x}{c\cos\theta}+\frac{b_1}{c\cos\theta}} Q(M)\Big(s+t+\tfrac{x}{c\cos\theta}-\tfrac{b_1}{c\cos\theta},\ -sc\cos\theta+b_1,\ -sc\sin\theta-\tfrac{\sin\theta}{\cos\theta}x+y+\tfrac{\sin\theta}{\cos\theta}b_1\Big)\,ds$$
$$+N_4^{+}\Big(t+\tfrac{x}{c\cos\theta}-\tfrac{b_1}{c\cos\theta},\ -\tfrac{\sin\theta}{\cos\theta}x+y+\tfrac{\sin\theta}{\cos\theta}b_1\Big)\,, \tag{108}$$

$$\mathcal{T}_4^C(M)(t,x,y) = \int_0^{-\frac{y}{c\sin\theta}+\frac{b_2}{c\sin\theta}} Q(M)\Big(s+t+\tfrac{y}{c\sin\theta}-\tfrac{b_2}{c\sin\theta},\, -sc\cos\theta+x-\tfrac{\cos\theta}{\sin\theta}y+\tfrac{\cos\theta}{\sin\theta}b_2,\, -sc\sin\theta+b_2\Big)\,ds$$
$$+N_4^{++}\Big(t+\tfrac{y}{c\sin\theta}-\tfrac{b_2}{c\sin\theta},\, x-\tfrac{\cos\theta}{\sin\theta}y+\tfrac{\cos\theta}{\sin\theta}b_2\Big). \tag{109}$$

Derivatives $\dfrac{\partial\mathcal{T}_i(M)}{\partial t}, \dfrac{\partial\mathcal{T}_i(M)}{\partial x}, \dfrac{\partial\mathcal{T}_i(M)}{\partial y}, (i=1,2,3,4)$

$\dfrac{\partial\mathcal{T}_1(M)}{\partial t}, \dfrac{\partial\mathcal{T}_1(M)}{\partial x}, \dfrac{\partial\mathcal{T}_1(M)}{\partial y}$ are defined in $\mathring{\mathscr{P}}$ except perhaps on the planes given respectively by the equations $x-ct\cos\theta=a_1; y-ct\sin\theta=a_2; x\sin\theta-y\cos\theta=a_1\sin\theta-a_2\cos\theta$ and

$$\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\mathcal{T}_1(M)(t,x,y) = \left(\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\mathcal{T}_1^A(M)(t,x,y)\cdot\mathbb{I}_{\begin{cases}x-ct\cos\theta>a_1\\ y-ct\sin\theta>a_2\end{cases}}(t,x,y)$$
$$+\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\mathcal{T}_1^B(M)(t,x,y)\cdot\mathbb{I}_{\begin{cases}x-ct\cos\theta<a_1\\ x\sin\theta-y\cos\theta<a_1\sin\theta-a_2\cos\theta\end{cases}}(t,x,y)$$
$$+\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\mathcal{T}_1^C(M)(t,x,y)\cdot\mathbb{I}_{\begin{cases}y-ct\sin\theta<a_2\\ x\sin\theta-y\cos\theta>a_1\sin\theta-a_2\cos\theta\end{cases}}(t,x,y) \tag{110}$$

where

$$\frac{\partial\mathcal{T}_1^A(M)}{\partial t}(t,x,y) = Q(M)(t,x,y)+\int_0^t\Big[-c\cos\theta\frac{\partial Q(M)}{\partial x}(s,x+c(s-t)\cos\theta,y+c(s-t)\sin\theta)-c\sin\theta\frac{\partial Q(M)}{\partial y}(s,x+c(s-t)\cos\theta,y+c(s-t)\sin\theta)\Big]ds$$
$$-c\cos\theta\frac{\partial N_1^0}{\partial x}(x-ct\cos\theta,y-ct\sin\theta)-c\sin\theta\frac{\partial N_1^0}{\partial y}(x-ct\cos\theta,y-ct\sin\theta), \tag{111}$$

$$\frac{\partial\mathcal{T}_1^B(M)}{\partial t}(t,x,y) = \int_0^{\frac{1}{c\cos\theta}x-\frac{a_1}{c\cos\theta}}\frac{\partial Q(M)}{\partial t}\Big(s+t-\tfrac{1}{c\cos\theta}x+\tfrac{a_1}{c\cos\theta}, sc\cos\theta+a_1, sc\sin\theta-\tfrac{\sin\theta}{\cos\theta}x+y+\tfrac{\sin\theta}{\cos\theta}a_1\Big)ds$$
$$+\frac{\partial N_1^-}{\partial t}\Big(t-\tfrac{1}{c\cos\theta}x+\tfrac{a_1}{c\cos\theta}, -\tfrac{\sin\theta}{\cos\theta}x+y+\tfrac{\sin\theta}{\cos\theta}a_1\Big), \tag{112}$$

$$\frac{\partial\mathcal{T}_1^C(M)}{\partial t}(t,x,y) = \int_0^{\frac{1}{c\sin\theta}y-\frac{a_2}{c\sin\theta}}\frac{\partial Q(M)}{\partial t}\Big(s+t-\tfrac{1}{c\sin\theta}y+\tfrac{a_2}{c\sin\theta}, sc\cos\theta+x-\tfrac{\cos\theta}{\sin\theta}y+\tfrac{\cos\theta}{\sin\theta}a_2, sc\sin\theta+a_2\Big)ds$$
$$+\frac{\partial N_1^{--}}{\partial t}\Big(t-\tfrac{1}{c\sin\theta}y+\tfrac{a_2}{c\sin\theta}, x-\tfrac{\cos\theta}{\sin\theta}y+\tfrac{\cos\theta}{\sin\theta}a_2\Big), \tag{113}$$

$$\frac{\partial\mathcal{T}_1^A(M)}{\partial x}(t,x,y) = \int_0^t\Big[\frac{\partial Q(M)}{\partial x}(s,x+c(s-t)\cos\theta,y+c(s-t)\sin\theta)ds$$
$$+\frac{\partial N_1^0}{\partial x}(x-ct\cos\theta,y-ct\sin\theta), \tag{114}$$

$$\frac{\partial \mathcal{T}_1^B(M)}{\partial x}(t,x,y) = \tfrac{1}{c\cos\theta} Q(M)(t,x,y) + \int_0^{\frac{1}{c\cos\theta}x - \frac{a_1}{c\cos\theta}} - \tfrac{1}{c\cos\theta}\frac{\partial Q(M)}{\partial t}\left(s + t - \tfrac{1}{c\cos\theta}x + \tfrac{a_1}{c\cos\theta},\right.$$
$$sc\cos\theta + a_1, sc\sin\theta - \tfrac{\sin\theta}{\cos\theta}x + y + \tfrac{\sin\theta}{\cos\theta}a_1$$
$$- \tfrac{\sin\theta}{\cos\theta}\frac{\partial Q(M)}{\partial y}\left(s + t - \tfrac{1}{c\cos\theta}x + \tfrac{a_1}{c\cos\theta}, sc\cos\theta + a_1, sc\sin\theta - \tfrac{\sin\theta}{\cos\theta}x + y + \tfrac{\sin\theta}{\cos\theta}a_1\right) ds +$$
$$- \frac{1}{c\cos\theta}\frac{\partial N_1^-}{\partial t}\left(t - \tfrac{1}{c\cos\theta}x + \tfrac{a_1}{c\cos\theta}, -\tfrac{\sin\theta}{\cos\theta}x + y + \tfrac{\sin\theta}{\cos\theta}a_1\right)$$
$$- \frac{\sin\theta}{\cos\theta}\frac{\partial N_1^-}{\partial t}\left(t - \frac{1}{c\cos\theta}x + \frac{a_1}{c\cos\theta}, -\frac{\sin\theta}{\cos\theta}x + y + \frac{\sin\theta}{\cos\theta}a_1\right), \tag{115}$$

$$\frac{\partial \mathcal{T}_1^C(M)}{\partial x}(t,x,y) = \int_0^{\frac{1}{c\sin\theta}y - \frac{a_2}{c\sin\theta}} \frac{\partial Q(M)}{\partial y}\left(s + t - \tfrac{1}{c\sin\theta}y + \tfrac{a_2}{c\sin\theta}, sc\cos\theta + x - \tfrac{\cos\theta}{\sin\theta}y\right.$$
$$\left. + \tfrac{\cos\theta}{\sin\theta}a_2, sc\sin\theta + a_2\right) ds + \frac{\partial N_1^{--}}{\partial x}\left(t - \tfrac{1}{c\sin\theta}y + \tfrac{a_2}{c\sin\theta}, x - \tfrac{\cos\theta}{\sin\theta}y + \tfrac{\cos\theta}{\sin\theta}a_2\right), \tag{116}$$

$$\frac{\partial \mathcal{T}_1^A(M)}{\partial y}(t,x,y) = \int_0^t \frac{\partial Q(M)}{\partial y}\left(s, x + c(s-t)\cos\theta, y + c(s-t)\sin\theta \right) ds$$
$$+ \frac{\partial N_1^0}{\partial y}(x - ct\cos\theta, y - ct\sin\theta), \tag{117}$$

$$\frac{\partial \mathcal{T}_1^B(M)}{\partial y}(t,x,y) = \int_0^{\frac{1}{c\cos\theta}x - \frac{a_1}{c\cos\theta}} \frac{\partial Q(M)}{\partial y}\left(s + t - \tfrac{1}{c\cos\theta}x + \tfrac{a_1}{c\cos\theta}, sc\cos\theta + a_1, sc\sin\theta\right.$$
$$\left. - \tfrac{\sin\theta}{\cos\theta}x + y + \tfrac{\sin\theta}{\cos\theta}a_1\right) ds + \frac{\partial N_1^-}{\partial y}\left(t - \tfrac{1}{c\cos\theta}x + \tfrac{a_1}{c\cos\theta}, -\tfrac{\sin\theta}{\cos\theta}x + y + \tfrac{\sin\theta}{\cos\theta}a_1\right), \tag{118}$$

$$\frac{\partial \mathcal{T}_1^C(M)}{\partial y}(t,x,y) = \tfrac{1}{c\sin\theta}Q(M)(t,x,y) + \int_0^{\frac{1}{c\sin\theta}y - \frac{a_1}{c\sin\theta}} \left(-\tfrac{1}{c\sin\theta}\frac{\partial Q(M)}{\partial t}\left(s + t - \tfrac{1}{c\sin\theta}y + \tfrac{a_2}{c\sin\theta},\right.\right.$$
$$\left. sc\cos\theta + x - \tfrac{\cos\theta}{\sin\theta}y + \tfrac{\cos\theta}{\sin\theta}a_2, sc\sin\theta + a_2\right) - \tfrac{\cos\theta}{\sin\theta}\frac{\partial Q(M)}{\partial y}\left(s + t - \tfrac{1}{c\sin\theta}y\right.$$
$$\left.\left. + \tfrac{a_2}{c\sin\theta}, sc\cos\theta + x - \tfrac{\cos\theta}{\sin\theta}y + \tfrac{\cos\theta}{\sin\theta}a_2, sc\sin\theta + a_2\right)\right) ds$$
$$+ -\frac{1}{c\sin\theta}\frac{\partial N_1^{--}}{\partial t}\left(t - \tfrac{1}{c\sin\theta}y + \tfrac{a_2}{c\sin\theta}, x - \tfrac{\cos\theta}{\sin\theta}y + \tfrac{\cos\theta}{\sin\theta}a_2\right)$$
$$- \frac{\cos\theta}{\sin\theta}\frac{\partial N_1^{--}}{\partial x}\left(t - \frac{1}{c\sin\theta}y + \frac{a_2}{c\sin\theta}, x - \frac{\cos\theta}{\sin\theta}y + \frac{\cos\theta}{\sin\theta}a_2\right). \tag{119}$$

Now $\frac{\partial \mathcal{T}_2(M)}{\partial t}, \frac{\partial \mathcal{T}_2(M)}{\partial x}, \frac{\partial \mathcal{T}_2(M)}{\partial y}$ are defined in $\mathring{\mathscr{P}}$ except perhaps on the planes given respectively by the equations $x + ct\sin\theta = b_1; y - ct\cos\theta = a_2; x\cos\theta + y\sin\theta = b_1\cos\theta + a_2\sin\theta$ and

$$\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\mathcal{T}_2(M)(t,x,y) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\mathcal{T}_2^A(M)(t,x,y)\cdot \mathbb{I}_{\begin{cases} x + ct\sin\theta < b_1 \\ y - ct\cos\theta > a_2 \end{cases}}(t,x,y)$$
$$+ \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\mathcal{T}_2^B(M)(t,x,y)\cdot \mathbb{I}_{\begin{cases} x + ct\sin\theta > b_1 \\ x\cos\theta + y\sin\theta > b_1\cos\theta + a_2\sin\theta \end{cases}}(t,x,y)$$

$$+\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\mathcal{T}_2^C\left(M\right)\left(t,x,y\right)\cdot\mathbb{I}_{\begin{cases}y-ct\cos\theta<a_2\\ x\cos\theta+y\sin\theta<b_1\cos\theta+a_2\sin\theta\end{cases}}\left(t,x,y\right) \tag{120}$$

where

$$\begin{aligned}\frac{\partial\mathcal{T}_2^A\left(M\right)}{\partial t}\left(t,x,y\right)=&-Q\left(M\right)\left(t,x,y\right)+\int_0^t\Big[-c\sin\theta\frac{\partial Q\left(M\right)}{\partial x}\left(\,s,x-c\left(s-t\right)\sin\theta,y+c\left(s-t\right)\cos\theta\,\right)\\&+c\cos\theta\frac{\partial Q\left(M\right)}{\partial y}\left(\,s,x-c\left(s-t\right)\sin\theta,y+c\left(s-t\right)\cos\theta\,\right)\Big]ds\\&+c\sin\theta\frac{\partial N_2^0}{\partial x}\left(x+ct\sin\theta,y-ct\cos\theta\right)-c\cos\theta\frac{\partial N_2^0}{\partial y}\left(x+ct\sin\theta,y-ct\cos\theta\right),\end{aligned} \tag{121}$$

$$\begin{aligned}\frac{\partial\mathcal{T}_2^B\left(M\right)}{\partial t}\left(t,x,y\right)=&\int_0^{-\frac{1}{c\sin\theta}x+\frac{b_1}{c\sin\theta}}-\frac{\partial Q\left(M\right)}{\partial t}\Big(s+t+\frac{1}{c\sin\theta}x-\frac{b_1}{c\sin\theta},-sc\sin\theta+b_1,sc\cos\theta\\&+\frac{\cos\theta}{\sin\theta}x+y-\frac{\cos\theta}{\sin\theta}b_1\Big)ds+\frac{\partial N_2^+}{\partial t}\left(t+\tfrac{1}{c\sin\theta}x-\tfrac{b_1}{c\sin\theta},\tfrac{\cos\theta}{\sin\theta}x+y-\tfrac{\cos\theta}{\sin\theta}b_1\right),\end{aligned} \tag{122}$$

$$\begin{aligned}\frac{\partial\mathcal{T}_2^C\left(M\right)}{\partial t}\left(t,x,y\right)=&\int_0^{\frac{1}{c\cos\theta}y-\frac{a_2}{c\cos\theta}}-\frac{\partial Q\left(M\right)}{\partial t}\Big(s+t-\frac{1}{c\cos\theta}y+\frac{a_2}{c\cos\theta},-sc\sin\theta+x+\frac{\sin\theta}{\cos\theta}y\\&-\frac{\sin\theta}{\cos\theta}a_2,sc\cos\theta+a_2\Big)ds+\frac{\partial N_2^{--}}{\partial t}\left(t-\tfrac{1}{c\cos\theta}y+\tfrac{a_2}{c\cos\theta},x+\tfrac{\sin\theta}{\cos\theta}y-\tfrac{\sin\theta}{\cos\theta}a_2\right),\end{aligned} \tag{123}$$

$$\begin{aligned}\frac{\partial\mathcal{T}_2^A\left(M\right)}{\partial x}\left(t,x,y\right)=&\int_0^t-\frac{\partial Q\left(M\right)}{\partial x}\left(s,x-c\left(s-t\right)\sin\theta,y+c\left(s-t\right)\cos\theta\right)ds\\&+\frac{\partial N_2^0}{\partial x}\left(x+ct\sin\theta,y-ct\cos\theta\right),\end{aligned} \tag{124}$$

$$\begin{aligned}\frac{\partial\mathcal{T}_2^B\left(M\right)}{\partial x}\left(t,x,y\right)=&\tfrac{1}{c\sin\theta}Q\left(M\right)\left(t,x,y\right)+\int_0^{-\frac{1}{c\sin\theta}x+\frac{b_1}{c\sin\theta}}\Bigg(-\tfrac{1}{c\sin\theta}\frac{\partial Q\left(M\right)}{\partial t}\Big(s+t+\frac{1}{c\sin\theta}x-\frac{b_1}{c\sin\theta},\\&-sc\sin\theta+b_1,sc\cos\theta+\frac{\cos\theta}{\sin\theta}x+y-\frac{\cos\theta}{\sin\theta}b_1\Big)\\&-\tfrac{\cos\theta}{\sin\theta}\frac{\partial Q\left(M\right)}{\partial y}\Big(s+t+\frac{1}{c\sin\theta}x-\frac{b_1}{c\sin\theta},-sc\sin\theta+b_1,\\&sc\cos\theta+\frac{\cos\theta}{\sin\theta}x+y-\frac{\cos\theta}{\sin\theta}b_1\Big)\Bigg)ds\\&+\frac{1}{c\sin\theta}\frac{\partial N_2^+}{\partial t}\left(t+\tfrac{1}{c\sin\theta}x-\tfrac{b_1}{c\sin\theta},\tfrac{\cos\theta}{\sin\theta}x+y-\tfrac{\cos\theta}{\sin\theta}b_1\right)\\&+\frac{\cos\theta}{\sin\theta}\frac{\partial N_2^+}{\partial t}\left(t+\frac{1}{c\sin\theta}x-\frac{b_1}{c\sin\theta},\frac{\cos\theta}{\sin\theta}x+y-\frac{\cos\theta}{\sin\theta}b_1\right),\end{aligned} \tag{125}$$

$$\begin{aligned}\frac{\partial\mathcal{T}_2^C\left(M\right)}{\partial x}\left(t,x,y\right)=&\int_0^{\frac{1}{c\cos\theta}y-\frac{a_2}{c\cos\theta}}-\frac{\partial Q\left(M\right)}{\partial x}\Big(s+t-\frac{1}{c\cos\theta}y+\frac{a_2}{c\cos\theta},\\&-sc\sin\theta+x+\frac{\sin\theta}{\cos\theta}y-\frac{\sin\theta}{\cos\theta}a_2,sc\cos\theta+a_2\Big)ds\\&+\frac{\partial N_2^{--}}{\partial x}\left(t-\tfrac{1}{c\cos\theta}y+\tfrac{a_2}{c\cos\theta},x+\tfrac{\sin\theta}{\cos\theta}y-\tfrac{\sin\theta}{\cos\theta}a_2\right),\end{aligned} \tag{126}$$

$$\frac{\partial \mathcal{T}_2^A(M)}{\partial y}(t,x,y) = \int_0^t -\frac{\partial Q(M)}{\partial y}\left(s, x - c(s-t)\sin\theta, y + c(s-t)\cos\theta\right) ds + \frac{\partial N_2^0}{\partial y}\left(x + ct\sin\theta, y - ct\cos\theta\right), \tag{127}$$

$$\begin{aligned}\frac{\partial \mathcal{T}_2^B(M)}{\partial y}(t,x,y) =& \int_0^{-\frac{1}{c\sin\theta}x+\frac{b_1}{c\sin\theta}} -\frac{\partial Q(M)}{\partial y}\Big(s+t+\frac{1}{c\sin\theta}x-\frac{b_1}{c\sin\theta},\\ & -sc\sin\theta+b_1, sc\cos\theta+\frac{\cos\theta}{\sin\theta}x+y-\frac{\cos\theta}{\sin\theta}b_1\Big)ds\\ & +\frac{\partial N_2^+}{\partial y}\left(t+\frac{1}{c\sin\theta}x-\frac{b_1}{c\sin\theta}, \frac{\cos\theta}{\sin\theta}x+y-\frac{\cos\theta}{\sin\theta}b_1\right),\end{aligned} \tag{128}$$

$$\begin{aligned}\frac{\partial \mathcal{T}_2^C(M)}{\partial y}(t,x,y) =& -\frac{1}{c\cos\theta}Q(M)(t,x,y) + \int_0^{\frac{1}{c\cos\theta}y-\frac{a_2}{c\cos\theta}}\Big(\frac{1}{c\cos\theta}\frac{\partial Q(M)}{\partial t}\Big(s+t-\frac{1}{c\cos\theta}y+\frac{a_2}{c\cos\theta},\\ & -sc\sin\theta+x+\frac{\sin\theta}{\cos\theta}y-\frac{\sin\theta}{\cos\theta}a_2, sc\cos\theta+a_2\Big)\\ & -\frac{\sin\theta}{\cos\theta}\frac{\partial Q(M)}{\partial x}\Big(s+t-\frac{1}{c\cos\theta}y+\frac{a_2}{c\cos\theta}, -sc\sin\theta+x+\frac{\sin\theta}{\cos\theta}y-\frac{\sin\theta}{\cos\theta}a_2, sc\cos\theta+a_2\Big)\Big)ds\\ & -\frac{1}{c\cos\theta}\frac{\partial N_2^{--}}{\partial t}\left(t-\frac{1}{c\cos\theta}y+\frac{a_2}{c\cos\theta}, x+\frac{\sin\theta}{\cos\theta}y-\frac{\sin\theta}{\cos\theta}a_2\right)\\ & +\frac{\sin\theta}{\cos\theta}\frac{\partial N_2^{--}}{\partial x}\left(t-\frac{1}{c\cos\theta}y+\frac{a_2}{c\cos\theta}, x+\frac{\sin\theta}{\cos\theta}y-\frac{\sin\theta}{\cos\theta}a_2\right).\end{aligned} \tag{129}$$

Now $\frac{\partial \mathcal{T}_3(M)}{\partial t}, \frac{\partial \mathcal{T}_3(M)}{\partial x}, \frac{\partial \mathcal{T}_3(M)}{\partial y}$ are defined in $\mathring{\mathscr{P}}$ except perhaps on the planes given respectively by the equations $x - ct\sin\theta = a_1; y + ct\cos\theta = b_2; x\cos\theta + y\sin\theta = a_1\cos\theta + b_2\sin\theta$ and

$$\begin{aligned}\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\mathcal{T}_3(M)(t,x,y) =& \left(\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\mathcal{T}_3^A(M)(t,x,y)\cdot\mathbb{I}_{\begin{cases}x-ct\sin\theta>a_1\\ y+ct\cos\theta<b_2\end{cases}}(t,x,y)\\ & +\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\mathcal{T}_3^B(M)(t,x,y)\cdot\mathbb{I}_{\begin{cases}x-ct\sin\theta<a_1\\ x\cos\theta+y\sin\theta<a_1\cos\theta+b_2\sin\theta\end{cases}}(t,x,y)\\ & +\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\mathcal{T}_3^C(M)(t,x,y)\cdot\mathbb{I}_{\begin{cases}y+ct\cos\theta>b_2\\ x\cos\theta+y\sin\theta>a_1\cos\theta+b_2\sin\theta\end{cases}}(t,x,y)\end{aligned} \tag{130}$$

where

$$\begin{aligned}\frac{\partial \mathcal{T}_3^A(M)}{\partial t}(t,x,y) =& -Q(M)(t,x,y) + \int_0^t\Big[c\sin\theta\frac{\partial Q(M)}{\partial x}\left(s, x+c(s-t)\sin\theta, y-c(s-t)\cos\theta\right)\\ & -c\cos\theta\frac{\partial Q(M)}{\partial y}\left(s, x+c(s-t)\sin\theta, y-c(s-t)\cos\theta\right)\Big]ds\\ & -c\sin\theta\frac{\partial N_3^0}{\partial x}\left(x-ct\sin\theta, y+ct\cos\theta\right)+c\cos\theta\frac{\partial N_3^0}{\partial y}\left(x-ct\sin\theta, y+ct\cos\theta\right),\end{aligned} \tag{131}$$

$$\frac{\partial \mathcal{T}_3^B(M)}{\partial t}(t,x,y) = \int_0^{\frac{1}{c\sin\theta}x-\frac{a_1}{c\sin\theta}} -\frac{\partial Q(M)}{\partial t}\Big(s+t-\frac{1}{c\sin\theta}x+\frac{a_1}{c\sin\theta}, sc\sin\theta+a_1, -sc\cos\theta$$

$$+\frac{\cos\theta}{\sin\theta}x+y-\frac{\cos\theta}{\sin\theta}a_1\ )\,ds+\frac{\partial N_3^-}{\partial t}\left(t-\tfrac{1}{c\sin\theta}x+\tfrac{a_1}{c\sin\theta},\tfrac{\cos\theta}{\sin\theta}x+y-\tfrac{\cos\theta}{\sin\theta}a_1\right), \tag{132}$$

$$\frac{\partial\mathcal{T}_3^C(M)}{\partial t}(t,x,y)=\int_0^{-\frac{1}{c\cos\theta}y+\frac{b_2}{c\cos\theta}}-\frac{\partial Q(M)}{\partial t}\left(s+t+\frac{1}{c\cos\theta}y-\frac{b_2}{c\cos\theta},sc\sin\theta+x+\frac{\sin\theta}{\cos\theta}y\right.$$
$$\left.-\frac{\sin\theta}{\cos\theta}b_2,-sc\cos\theta+b_2\right)ds+\frac{\partial N_3^{++}}{\partial t}\left(t+\tfrac{1}{c\cos\theta}y-\tfrac{b_2}{c\cos\theta},x+\tfrac{\sin\theta}{\cos\theta}y-\tfrac{\sin\theta}{\cos\theta}b_2\right), \tag{133}$$

$$\frac{\partial\mathcal{T}_3^A(M)}{\partial x}(t,x,y)=\int_0^t-\frac{\partial Q(M)}{\partial x}(\ s,x+c(s-t)\sin\theta,y-c(s-t)\cos\theta\ )\,ds+\frac{\partial N_3^0}{\partial x}(x-ct\sin\theta,y+ct\cos\theta), \tag{134}$$

$$\frac{\partial\mathcal{T}_3^B(M)}{\partial x}(t,x,y)=-\tfrac{1}{c\sin\theta}Q(M)(t,x,y)+\int_0^{\frac{1}{c\sin\theta}x-\frac{a_1}{c\sin\theta}}\left(\tfrac{1}{c\sin\theta}\frac{\partial Q(M)}{\partial t}(\ s+t-\frac{1}{c\sin\theta}x+\frac{a_1}{c\sin\theta},\right.$$
$$\left.sc\sin\theta+a_1,-sc\cos\theta+\frac{\cos\theta}{\sin\theta}x+y-\frac{\cos\theta}{\sin\theta}a_1\right)-\tfrac{\cos\theta}{\sin\theta}\frac{\partial Q(M)}{\partial y}\left(s+t-\right.$$
$$\left.\left.\frac{1}{c\sin\theta}x+\frac{a_1}{c\sin\theta},sc\sin\theta+a_1,-sc\cos\theta+\frac{\cos\theta}{\sin\theta}x+y-\frac{\cos\theta}{\sin\theta}a_1\right)\right)ds$$
$$-\frac{1}{c\sin\theta}\frac{\partial N_3^-}{\partial t}\left(t-\tfrac{1}{c\sin\theta}x+\tfrac{a_1}{c\sin\theta},\tfrac{\cos\theta}{\sin\theta}x+y-\tfrac{\cos\theta}{\sin\theta}a_1\right)$$
$$+\frac{\cos\theta}{\sin\theta}\frac{\partial N_3^-}{\partial t}\left(t-\frac{1}{c\sin\theta}x+\frac{a_1}{c\sin\theta},\frac{\cos\theta}{\sin\theta}x+y-\frac{\cos\theta}{\sin\theta}a_1\right), \tag{135}$$

$$\frac{\partial\mathcal{T}_3^C(M)}{\partial x}(t,x,y)=\int_0^{-\frac{1}{c\cos\theta}y+\frac{b_2}{c\cos\theta}}-\frac{\partial Q(M)}{\partial x}(\ s+t+\frac{1}{c\cos\theta}y-\frac{b_2}{c\cos\theta},$$
$$sc\sin\theta+x+\frac{\sin\theta}{\cos\theta}y-\frac{\sin\theta}{\cos\theta}b_2,-sc\cos\theta+b_2\ )\,ds$$
$$+\frac{\partial N_3^{++}}{\partial x}\left(t+\tfrac{1}{c\cos\theta}y-\tfrac{b_2}{c\cos\theta},x+\tfrac{\sin\theta}{\cos\theta}y-\tfrac{\sin\theta}{\cos\theta}b_2\right), \tag{136}$$

$$\frac{\partial\mathcal{T}_3^A(M)}{\partial y}(t,x,y)=\int_0^t[-\frac{\partial Q(M)}{\partial y}(\ s,x+c(s-t)\sin\theta,y-c(s-t)\cos\theta\ )\,ds+\frac{\partial N_3^0}{\partial y}(x-ct\sin\theta,y+ct\cos\theta), \tag{137}$$

$$\frac{\partial\mathcal{T}_3^B(M)}{\partial y}(t,x,y)=\int_0^{\frac{1}{c\sin\theta}x-\frac{a_1}{c\sin\theta}}-\frac{\partial Q(M)}{\partial y}\left(s+t-\frac{1}{c\sin\theta}x+\frac{a_1}{c\sin\theta},sc\sin\theta+a_1,-sc\cos\theta+\frac{\cos\theta}{\sin\theta}x\right.$$
$$\left.+y-\frac{\cos\theta}{\sin\theta}a_1\right)ds+\frac{\partial N_3^-}{\partial y}\left(t-\tfrac{1}{c\sin\theta}x+\tfrac{a_1}{c\sin\theta},\tfrac{\cos\theta}{\sin\theta}x+y-\tfrac{\cos\theta}{\sin\theta}a_1\right), \tag{138}$$

$$\frac{\partial\mathcal{T}_3^C(M)}{\partial y}(t,x,y)=\tfrac{1}{c\cos\theta}Q(M)(t,x,y)+\int_0^{-\frac{1}{c\cos\theta}y+\frac{b_2}{c\cos\theta}}\left(-\tfrac{1}{c\cos\theta}\frac{\partial Q(M)}{\partial t}\left(s+t+\frac{1}{c\cos\theta}y-\frac{b_2}{c\cos\theta},\right.\right.$$
$$\left.sc\sin\theta+x+\frac{\sin\theta}{\cos\theta}y-\frac{\sin\theta}{\cos\theta}b_2,-sc\cos\theta+b_2\right)-\tfrac{\sin\theta}{\cos\theta}\frac{\partial Q(M)}{\partial x}\left(s+t+\frac{1}{c\cos\theta}y\right.$$
$$\left.\left.-\frac{b_2}{c\cos\theta},sc\sin\theta+x+\frac{\sin\theta}{\cos\theta}y-\frac{\sin\theta}{\cos\theta}b_2,-sc\cos\theta+b_2\right)\right)ds$$
$$+\frac{1}{c\cos\theta}\frac{\partial N_3^{++}}{\partial t}\left(t+\tfrac{1}{c\cos\theta}y-\tfrac{b_2}{c\cos\theta},x+\tfrac{\sin\theta}{\cos\theta}y-\tfrac{\sin\theta}{\cos\theta}b_2\right)$$

$$+\frac{\sin\theta}{\cos\theta}\frac{\partial N_3^{++}}{\partial x}\left(t+\frac{1}{c\cos\theta}y-\frac{b_2}{c\cos\theta},x+\frac{\sin\theta}{\cos\theta}y-\frac{\sin\theta}{\cos\theta}b_2\right). \quad (139)$$

Now $\frac{\partial\mathcal{T}_4(M)}{\partial t},\frac{\partial\mathcal{T}_4(M)}{\partial x},\frac{\partial\mathcal{T}_4(M)}{\partial y}$ are defined in $\mathring{\mathscr{P}}$ except perhaps on the planes given respectively by the equations $x+ct\cos\theta=b_1; y+ct\sin\theta=b_2; x\sin\theta-y\cos\theta=b_1\sin\theta-b_2\cos\theta$ and

$$\begin{aligned}\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\mathcal{T}_4(M)(t,x,y)=&\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\mathcal{T}_4^A(M)(t,x,y)\cdot\mathbb{I}_{\begin{cases}x+ct\cos\theta<b_1\\ y+ct\sin\theta<b_2\end{cases}}(t,x,y)\\&+\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\mathcal{T}_4^B(M)(t,x,y)\cdot\mathbb{I}_{\begin{cases}x+ct\cos\theta>b_1\\ x\sin\theta-y\cos\theta>b_1\sin\theta-b_2\cos\theta\end{cases}}(t,x,y)\\&+\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\mathcal{T}_4^C(M)(t,x,y)\cdot\mathbb{I}_{\begin{cases}y+ct\sin\theta>b_2\\ x\sin\theta-y\cos\theta<b_1\sin\theta-b_2\cos\theta\end{cases}}(t,x,y)\end{aligned} \quad (140)$$

where

$$\begin{aligned}\frac{\partial\mathcal{T}_4^A(M)}{\partial t}(t,x,y)=&Q(M)(t,x,y)+\int_0^t\left[c\cos\theta\frac{\partial Q(M)}{\partial x}(s,x-c(s-t)\cos\theta,y-c(s-t)\sin\theta)\right.\\&\left.+c\sin\theta\frac{\partial Q(M)}{\partial y}(s,x-c(s-t)\cos\theta,y-c(s-t)\sin\theta)\right]ds\\&+c\cos\theta\frac{\partial N_4^0}{\partial x}(x+ct\cos\theta,y+ct\sin\theta)+c\sin\theta\frac{\partial N_4^0}{\partial y}(x+ct\cos\theta,y+ct\sin\theta),\end{aligned} \quad (141)$$

$$\begin{aligned}\frac{\partial\mathcal{T}_4^B(M)}{\partial t}(t,x,y)=&\int_0^{-\frac{1}{c\cos\theta}x+\frac{b_1}{c\cos\theta}}\frac{\partial Q(M)}{\partial t}\left(s+t+\frac{1}{c\cos\theta}x-\frac{b_1}{c\cos\theta},-sc\cos\theta+b_1,-sc\sin\theta\right.\\&\left.-\frac{\sin\theta}{\cos\theta}x+y+\frac{\sin\theta}{\cos\theta}b_1\right)ds+\frac{\partial N_4^+}{\partial t}\left(t+\frac{1}{c\cos\theta}x-\frac{b_1}{c\cos\theta},-\frac{\sin\theta}{\cos\theta}x+y+\frac{\sin\theta}{\cos\theta}b_1\right),\end{aligned} \quad (142)$$

$$\begin{aligned}\frac{\partial\mathcal{T}_4^C(M)}{\partial t}(t,x,y)=&\int_0^{-\frac{1}{c\sin\theta}y+\frac{b_2}{c\sin\theta}}\frac{\partial Q(M)}{\partial t}\left(s+t+\frac{1}{c\sin\theta}y-\frac{b_2}{c\sin\theta},-sc\cos\theta+x-\frac{\cos\theta}{\sin\theta}y\right.\\&\left.+\frac{\cos\theta}{\sin\theta}b_2,-sc\sin\theta+b_2\right)ds+\frac{\partial N_4^{++}}{\partial t}\left(t+\frac{1}{c\sin\theta}y-\frac{b_2}{c\sin\theta},x-\frac{\cos\theta}{\sin\theta}y+\frac{\cos\theta}{\sin\theta}b_2\right),\end{aligned} \quad (143)$$

$$\frac{\partial\mathcal{T}_4^A(M)}{\partial x}(t,x,y)=\int_0^t[\frac{\partial Q(M)}{\partial x}(s,x-c(s-t)\cos\theta,y-c(s-t)\sin\theta)ds+\frac{\partial N_4^0}{\partial x}(x+ct\cos\theta,y+ct\sin\theta), \quad (144)$$

$$\begin{aligned}\frac{\partial\mathcal{T}_4^B(M)}{\partial x}(t,x,y)=&-\frac{1}{c\cos\theta}Q(M)(t,x,y)+\int_0^{-\frac{1}{c\cos\theta}x+\frac{b_1}{c\cos\theta}}\left(\frac{1}{c\cos\theta}\frac{\partial Q(M)}{\partial t}\left(s+t+\frac{1}{c\cos\theta}x-\frac{b_1}{c\cos\theta},\right.\right.\\&\left.-sc\cos\theta+b_1,-sc\sin\theta-\frac{\sin\theta}{\cos\theta}x+y+\frac{\sin\theta}{\cos\theta}b_1\right)-\frac{\sin\theta}{\cos\theta}\frac{\partial Q(M)}{\partial y}\left(s+t\right.\\&\left.\left.+\frac{1}{c\cos\theta}x-\frac{b_1}{c\cos\theta},-sc\cos\theta+b_1,-sc\sin\theta-\frac{\sin\theta}{\cos\theta}x+y+\frac{\sin\theta}{\cos\theta}b_1\right)\right)ds\\&+\frac{1}{c\cos\theta}\frac{\partial N_4^+}{\partial t}\left(t+\frac{1}{c\cos\theta}x-\frac{b_1}{c\cos\theta},-\frac{\sin\theta}{\cos\theta}x+y+\frac{\sin\theta}{\cos\theta}b_1\right)\end{aligned}$$

$$-\frac{\sin\theta}{\cos\theta}\frac{\partial N_4^{+}}{\partial y}\left(t+\frac{1}{c\cos\theta}x-\frac{b_1}{c\cos\theta},-\frac{\sin\theta}{\cos\theta}x+y+\frac{\sin\theta}{\cos\theta}b_1\right), \tag{145}$$

$$\frac{\partial \mathcal{T}_4^{C}(M)}{\partial x}(t,x,y)=\int_0^{-\frac{1}{c\sin\theta}y+\frac{b_2}{c\sin\theta}}\frac{\partial Q(M)}{\partial x}\left(s+t+\frac{1}{c\sin\theta}y-\frac{b_2}{c\sin\theta},-sc\cos\theta+x-\frac{\cos\theta}{\sin\theta}y\right.$$
$$\left.+\frac{\cos\theta}{\sin\theta}b_2,-sc\sin\theta+b_2\right)ds\frac{\partial N_4^{++}}{\partial x}\left(t+\tfrac{1}{c\sin\theta}y-\tfrac{b_2}{c\sin\theta},x-\tfrac{\cos\theta}{\sin\theta}y+\tfrac{\cos\theta}{\sin\theta}b_2\right), \tag{146}$$

$$\frac{\partial \mathcal{T}_4^{A}(M)}{\partial y}(t,x,y)=\int_0^{t}\frac{\partial Q(M)}{\partial y}\left(s,x-c(s-t)\cos\theta,y-c(s-t)\sin\theta\right)ds+\frac{\partial N_4^{0}}{\partial y}\left(x+ct\cos\theta,y+ct\sin\theta\right), \tag{147}$$

$$\frac{\partial \mathcal{T}_4^{B}(M)}{\partial y}(t,x,y)=\int_0^{-\frac{1}{c\cos\theta}x+\frac{b_1}{c\cos\theta}}\frac{\partial Q(M)}{\partial y}\left(s+t+\frac{1}{c\cos\theta}x-\frac{b_1}{c\cos\theta},-sc\cos\theta+b_1,-sc\sin\theta\right.$$
$$\left.-\frac{\sin\theta}{\cos\theta}x+y+\frac{\sin\theta}{\cos\theta}b_1\right)ds+\frac{\partial N_4^{+}}{\partial y}\left(t+\tfrac{1}{c\cos\theta}x-\tfrac{b_1}{c\cos\theta},-\tfrac{\sin\theta}{\cos\theta}x+y+\tfrac{\sin\theta}{\cos\theta}b_1\right), \tag{148}$$

$$\frac{\partial \mathcal{T}_4^{C}(M)}{\partial y}(t,x,y)=-\tfrac{1}{c\sin\theta}Q(M)(t,x,y)+\int_0^{-\frac{1}{c\sin\theta}y+\frac{b_2}{c\sin\theta}}\left(\tfrac{1}{c\sin\theta}\frac{\partial Q(M)}{\partial t}\left(s+t+\frac{1}{c\sin\theta}y-\frac{b_2}{c\sin\theta},\right.\right.$$
$$\left.-sc\cos\theta+x-\frac{\cos\theta}{\sin\theta}y+\frac{\cos\theta}{\sin\theta}b_2,-sc\sin\theta+b_2\right)-\tfrac{\cos\theta}{\sin\theta}\frac{\partial Q(M)}{\partial x}\left(s+t+\frac{1}{c\sin\theta}y\right.$$
$$\left.\left.-\frac{b_2}{c\sin\theta},-sc\cos\theta+x-\frac{\cos\theta}{\sin\theta}y+\frac{\cos\theta}{\sin\theta}b_2,-sc\sin\theta+b_2\right)\right)ds+$$
$$+\frac{1}{c\sin\theta}\frac{\partial N_4^{++}}{\partial t}\left(t+\tfrac{1}{c\sin\theta}y-\tfrac{b_2}{c\sin\theta},x-\tfrac{\cos\theta}{\sin\theta}y+\tfrac{\cos\theta}{\sin\theta}b_2\right)$$
$$-\frac{\cos\theta}{\sin\theta}\frac{\partial N_4^{++}}{\partial x}\left(t+\frac{1}{c\sin\theta}y-\frac{b_2}{c\sin\theta},x-\frac{\cos\theta}{\sin\theta}y+\frac{\cos\theta}{\sin\theta}b_2\right). \tag{149}$$