Open Journal of Mathematical Analysis

PSR Press

Article

Multivariable Jensen integral inequality and fractional Hermite-Hadamard type inequalities

Mehmet Zeki Sarıkaya

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey; sarikayamz@gmail.com

Received: 11 July 2025; Accepted: 24 August 2025; Published: 07 September 2025.

Abstract: This study examines the multivariable Jensen integral inequality for convexity in coordinates. Using the co-ordinate convex functions, we give some results to develop new fractional Hermite-Hadamard type inequalities.

Keywords: convex function, Hermite-Hadamard inequalities, Jensen inequality, fractional integrals

MSC: 26A33, 26A51, 26D15, 26D10

---

1. Introduction

The inequalities discovered by Hermite and Hadamard for convex functions state that if $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave [1].

Let $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be nonnegative weights such that $\sum_{j=1}^{n} \lambda_j = 1$. The well-known Jensen inequality [2] states that if f is convex on an interval containing x_n, then

$$f\left(\sum_{j=1}^{n} \lambda_j x_j\right) \leq \sum_{j=1}^{n} \lambda_j f(x_j). \tag{2}$$

A central tool in the applied literature is Jensen's weighted integral inequality. Assume that I is an arbitrary interval in $\mathbb{R}$ and $w, g : [a, b] \to I$ are integrable functions such that $w > 0$. If $F : I \to \mathbb{R}$ is convex and $F \circ g$ is integrable on $[a, b]$, then

$$F\left(\frac{1}{W}\int_a^b w(x)g(x)dx\right) \leq \frac{1}{W}\int_a^b w(x)F(g(x))dx, \tag{3}$$

where

$$W = \int_a^b w(x)dx.$$

Inequality (3) is reversed if F is concave.

Inequalities like Hermite-Hadamard and Jensen play a central role in mathematical analysis, providing foundational tools in optimization, probability theory, and applied mathematics. The ongoing research in their extensions and applications underscores their relevance in modern mathematics.

The extension of classical inequalities to fractional integrals has been a prominent area of research. These studies aim to adapt convexity properties to fractional integral operators like the Riemann-Liouville and Hadamard fractional integrals. The fractional Jensen inequality explores how convexity is preserved under fractional integrals, often extending to generalized convex functions such as s-convex or h-convex functions. Similarly, the fractional Hermite-Hadamard inequality provides bounds for fractional integrals of convex and quasi-convex functions. These fractional inequalities have been further generalized to weighted fractional integrals and special classes of convex functions, such as harmonic and preinvex functions (see [1,3–9]).

Recall that $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ is convex on Δ if

$$f(tx + (1-t)y, tu + (1-t)v) \leq tf(x, u) + (1-t)f(y, v), \quad \forall (x, u), (y, v) \in \Delta,\ t \in [0, 1].$$

Definition 1 (Co-ordinated convex function)**.** A function $f : \Delta \to \mathbb{R}$ is called *co-ordinated convex* on Δ if for all $(x, u), (y, v) \in \Delta$ and $t, s \in [0, 1]$,

$$f(tx + (1-t)y, su + (1-s)v) \leq tsf(x, u) + t(1-s)f(x, v) + s(1-t)f(y, u) + (1-t)(1-s)f(y, v). \quad (4)$$

The function is co-ordinated concave if the inequality holds in the reversed direction.

A function $f : \Delta \to \mathbb{R}$ is convex on the coordinates if the partial mappings

$$f_y(u) = f(u, y) \text{ on } [a, b], \quad f_x(v) = f(x, v) \text{ on } [c, d],$$

are convex for all $x \in [a, b]$ and $y \in [c, d]$ [10]. Every convex function is co-ordinated convex, but not vice versa.

Dragomir [10] proved the following Hermite-Hadamard type inequalities for co-ordinated convex functions on a rectangle:

Theorem 1. *Let $f : \Delta \to \mathbb{R}$ be co-ordinated convex. Then*

$$f\Big(\frac{a+b}{2}, \frac{c+d}{2}\Big) \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y)\, dy\, dx \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \quad (5)$$

These inequalities are sharp. If f is co-ordinated concave, the inequalities reverse.

Definition 2 (Riemann-Liouville fractional integral)**.** Let $f \in L_1[a, b]$. For $\alpha > 0$, define

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a,$$

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b.$$

Here $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$. Also, $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

Sarikaya et al. [6] proved the following fractional Hermite-Hadamard inequality:

Theorem 2. *Let $f : [a, b] \to [0, \infty)$ be convex and integrable on $[a, b]$ with $a < b$. Then, for $\alpha > 0$,*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{f(a) + f(b)}{2}. \quad (6)$$

For two-variable functions, the Riemann-Liouville fractional integrals are defined as follows [11]:

Definition 3. Let $f \in L_1([a, b] \times [c, d])$. Then

$$J_{a+,c+}^{\alpha,\beta} f(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y (x-t)^{\alpha-1} (y-s)^{\beta-1} f(t, s)\, ds\, dt, \quad x > a, y > c,$$

and similarly for $J_{a+,d-}^{\alpha,\beta}, J_{b-,c+}^{\alpha,\beta}, J_{b-,d-}^{\alpha,\beta}$ as given in [11].

Sarikaya [11] also proved the following Hermite-Hadamard type inequality for two-variable co-ordinated convex functions:

Theorem 3. *Let $f : [a,b] \times [c,d] \to [0,\infty)$ be co-ordinated convex with $0 \le a < b$, $0 \le c < d$, and $f \in L_1(\Delta)$. Then, for $\alpha, \beta > 0$,*

$$f\Big(\frac{a+b}{2}, \frac{c+d}{2}\Big) \le \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}} \Big[J^{\alpha,\beta}_{a+,c+} f(b,d) + J^{\alpha,\beta}_{a+,d-} f(b,c) + J^{\alpha,\beta}_{b-,c+} f(a,d) + J^{\alpha,\beta}_{b-,d-} f(a,c) \Big]$$
$$\le \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}.$$

This study examines multivariable Jensen integral inequalities for convexity in coordinates and develops new fractional Hermite-Hadamard type inequalities generalizing the classical results.

2. Fractional Hermite–Hadamard inequalities

To establish our main result, we recall the following standard facts (see, e.g., [12]).

A set $I \subset \mathbb{R}^n$ is convex if

$$t\mathbf{x} + (1-t)\mathbf{y} \in I \quad \text{for all } \mathbf{x}, \mathbf{y} \in I \text{ and } t \in [0,1].$$

A function $F : I \to \mathbb{R}$ is convex if

$$F\big(t\mathbf{x} + (1-t)\mathbf{y}\big) \le tF(\mathbf{x}) + (1-t)F(\mathbf{y}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in I,\ t \in [0,1].$$

We also say that F admits a supporting hyperplane at $\mathbf{y} \in I$ if there exists $\boldsymbol{\lambda} \in \mathbb{R}^n$ such that

$$F(\mathbf{x}) - F(\mathbf{y}) \ge \sum_{k=1}^{n} \lambda_k\,(x_k - y_k) \quad \text{for all } \mathbf{x} \in I.$$

Lemma 1. *Let $I \subset \mathbb{R}^n$ be a convex set and $F : I \to \mathbb{R}$. Then the following are equivalent:*

1. *F is convex on I.*
2. *F is differentiable on I and, for each $\mathbf{y} \in I$,*

$$F(\mathbf{x}) - F(\mathbf{y}) \ \ge\ \nabla F(\mathbf{y})^{\top}(\mathbf{x} - \mathbf{y}) \quad \textit{for all } \mathbf{x} \in I.$$

If, in addition, I has nonempty interior and $F \in C^2(I)$, then the above are also equivalent to:

3. *The Hessian matrix*

$$\nabla^2 F(\mathbf{y}) = \left(\frac{\partial^2 F}{\partial x_i\, \partial x_j}(\mathbf{y}) \right)_{1 \le i,j \le n},$$

is positive semidefinite for every $\mathbf{y} \in I$.

Moreover, if any (hence all) of the above hold, then at every $\mathbf{y} \in I$ there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}^n$ (e.g., $\boldsymbol{\lambda} = \nabla F(\mathbf{y})$ when F is differentiable) such that

$$F(\mathbf{x}) - F(\mathbf{y}) \ \ge\ \sum_{k=1}^{n} \lambda_k (x_k - y_k) \quad \textit{for all } \mathbf{x} \in I,$$

i.e., F admits a supporting hyperplane at every point of I.

Theorem 4 (Jensen inequality with multiple variables)**.** *Suppose that $F : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is continuous and coordinate-wise convex on Δ. Let $h_1, g_1 : [a,b] \to \mathbb{R}$ and $h_2, g_2 : [c,d] \to \mathbb{R}$ be integrable functions with $h_1, h_2 > 0$. Define*

$$H_1 = \int_a^b h_1(x)\,dx, \quad H_2 = \int_c^d h_2(y)\,dy.$$

Then, we have

$$F\left(\frac{1}{H_1} \int_a^b h_1(x) g_1(x)\,dx,\ \frac{1}{H_2} \int_c^d h_2(y) g_2(y)\,dy \right) \le \frac{1}{H_1 H_2} \int_a^b \int_c^d h_1(x) h_2(y) F(g_1(x), g_2(y))\,dy\,dx.$$

Proof. Let

$$(x_1, y_1) = \left(\frac{1}{H_1} \int_a^b h_1(x) g_1(x)\, dx,\ \frac{1}{H_2} \int_c^d h_2(y) g_2(y)\, dy \right) \in \Delta.$$

Since F is convex and differentiable, from Lemma 1 (3) there exists

$$\nabla F(x_1, y_1) = \left(\frac{\partial F}{\partial x}(x_1, y_1), \frac{\partial F}{\partial y}(x_1, y_1) \right),$$

such that

$$F(x, y) - F(x_1, y_1) \geq \frac{\partial F}{\partial x}(x_1, y_1)(x - x_1) + \frac{\partial F}{\partial y}(x_1, y_1)(y - y_1), \tag{7}$$

for all $(x, y) \in \Delta$.

Multiplying both sides of (7) by $h_1(x)h_2(y)$ and integrating over $[a, b] \times [c, d]$, we obtain

$$\begin{aligned}
&\int_a^b \int_c^d h_1(x) h_2(y) \left[F(g_1(x), g_2(y)) - F(x_1, y_1) \right] dy\, dx \\
&\quad \geq \int_a^b \int_c^d h_1(x) h_2(y) \left[\frac{\partial F}{\partial x}(x_1, y_1)(g_1(x) - x_1) + \frac{\partial F}{\partial y}(x_1, y_1)(g_2(y) - y_1) \right] dy\, dx \\
&\quad = \frac{\partial F}{\partial x}(x_1, y_1) \underbrace{\int_a^b h_1(x)(g_1(x) - x_1) dx}_{=0} \underbrace{\int_c^d h_2(y) dy}_{=H_2} + \frac{\partial F}{\partial y}(x_1, y_1) \underbrace{\int_c^d h_2(y)(g_2(y) - y_1) dy}_{=0} \underbrace{\int_a^b h_1(x) dx}_{=H_1} \\
&\quad = 0.
\end{aligned}$$

Dividing both sides by $H_1 H_2$, we obtain the desired inequality:

$$F(x_1, y_1) \leq \frac{1}{H_1 H_2} \int_a^b \int_c^d h_1(x) h_2(y) F(g_1(x), g_2(y))\, dy\, dx.$$

□

Remark 1. Theorem 4 states a generalization of the classical Jensen inequality to functions of two variables that are convex in each coordinate.

In particular, if we take $h_1(x) = H_1/(b - a)$ and $h_2(y) = H_2/(d - c)$ as constant functions, then the weighted averages reduce to the standard arithmetic averages:

$$\frac{1}{H_1} \int_a^b h_1(x) g_1(x)\, dx = \frac{1}{b - a} \int_a^b g_1(x)\, dx, \quad \frac{1}{H_2} \int_c^d h_2(y) g_2(y)\, dy = \frac{1}{d - c} \int_c^d g_2(y)\, dy.$$

Consequently, Theorem 4 recovers the classical two-variable Jensen inequality for coordinate-wise convex functions:

$$F\left(\frac{1}{b - a} \int_a^b g_1(x)\, dx, \frac{1}{d - c} \int_c^d g_2(y)\, dy \right) \leq \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d F(g_1(x), g_2(y))\, dy\, dx.$$

Theorem 5. *Let $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ be a co-ordinated convex function on Δ. Then, for $\alpha, \beta > 0$, the following inequalities hold:*

$$\begin{aligned}
&f\Big(\frac{b + \alpha a}{\alpha + 1}, \frac{d + \beta c}{\beta + 1}\Big) + f\Big(\frac{b + \alpha a}{\alpha + 1}, \frac{\beta d + c}{\beta + 1}\Big) + f\Big(\frac{\alpha b + a}{\alpha + 1}, \frac{d + \beta c}{\beta + 1}\Big) + f\Big(\frac{\alpha b + a}{\alpha + 1}, \frac{\beta d + c}{\beta + 1}\Big) \\
&\quad \leq \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{(b - a)^\alpha (d - c)^\beta} \left[J^{\alpha,\beta}_{a+,c+} f(b, d) + J^{\alpha,\beta}_{a+,d-} f(b, c) + J^{\alpha,\beta}_{b-,c+} f(a, d) + J^{\alpha,\beta}_{b-,d-} f(a, c) \right] \\
&\quad \leq f(a, c) + f(a, d) + f(b, c) + f(b, d).
\end{aligned} \tag{8}$$

Proof. By the Jensen inequality for weighted integrals with multiple variables, we have

$$f\left(\frac{\alpha}{(b-a)^{\alpha}}\int_a^b (b-x)^{\alpha-1}x\,dx,\ \frac{\beta}{(d-c)^{\beta}}\int_c^d (d-y)^{\beta-1}y\,dy\right)$$
$$\leq \frac{\alpha\beta}{(b-a)^{\alpha}(d-c)^{\beta}}\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1}f(x,y)\,dy\,dx,$$

and similarly for the three other combinations:

$$(x-a)^{\alpha-1},\ (b-x)^{\alpha-1} \quad \text{and} \quad (y-c)^{\beta-1},\ (d-y)^{\beta-1}.$$

Adding these four inequalities, we obtain

$$f\Big(\frac{b+\alpha a}{\alpha+1},\frac{d+\beta c}{\beta+1}\Big)+f\Big(\frac{b+\alpha a}{\alpha+1},\frac{\beta d+c}{\beta+1}\Big)+f\Big(\frac{\alpha b+a}{\alpha+1},\frac{d+\beta c}{\beta+1}\Big)+f\Big(\frac{\alpha b+a}{\alpha+1},\frac{\beta d+c}{\beta+1}\Big)$$
$$\leq \frac{\alpha\beta}{(b-a)^{\alpha}(d-c)^{\beta}}\Bigg[\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1}f(x,y)\,dy\,dx$$
$$+\int_a^b\int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1}f(x,y)\,dy\,dx \quad +\int_a^b\int_c^d (x-a)^{\alpha-1}(d-y)^{\beta-1}f(x,y)\,dy\,dx$$
$$+\int_a^b\int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1}f(x,y)\,dy\,dx\Bigg].$$

Now, applying the change of variables

$$x=ta+(1-t)b,\quad dx=(b-a)dt,\quad y=sc+(1-s)d,\quad dy=(d-c)ds,\quad t,s\in[0,1],$$

and using the co-ordinated convexity of f, we obtain

$$f\Big(\frac{b+\alpha a}{\alpha+1},\frac{d+\beta c}{\beta+1}\Big)+f\Big(\frac{b+\alpha a}{\alpha+1},\frac{\beta d+c}{\beta+1}\Big)+f\Big(\frac{\alpha b+a}{\alpha+1},\frac{d+\beta c}{\beta+1}\Big)+f\Big(\frac{\alpha b+a}{\alpha+1},\frac{\beta d+c}{\beta+1}\Big)$$
$$\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha}(d-c)^{\beta}}\Big[J^{\alpha,\beta}_{a+,c+}f(b,d)+J^{\alpha,\beta}_{a+,d-}f(b,c)+J^{\alpha,\beta}_{b-,c+}f(a,d)+J^{\alpha,\beta}_{b-,d-}f(a,c)\Big]$$
$$\leq f(a,c)+f(a,d)+f(b,c)+f(b,d),$$

which completes the proof. □

Remark 2. Under the assumptions of Theorem 5, if we take $\alpha=\beta=1$, then the fractional integral inequality (8) reduces to the classical Hermite–Hadamard inequality for functions that are convex on each coordinate.

Indeed, for $\alpha=\beta=1$, the Riemann–Liouville fractional integrals become the standard double integrals, and (8) simplifies to

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx \leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.$$

Thus, Theorem 5 can be seen as a generalization of the classical Hermite–Hadamard inequality to the setting of Riemann–Liouville fractional integrals with arbitrary positive parameters α and β.

Similarly, we obtain the following generalized Hermite-Hadamard inequalities for fractional operator:

Theorem 6. *Let $\Delta=[a,b]\times[c,d]$ with $0\leq a<b$ and $0\leq c<d$, fix $n\in\mathbb{N}$, and let $\alpha,\beta>0$. Assume $f:\Delta\to\mathbb{R}$ is co-ordinated convex and*

$$(t,s)\mapsto f(t^n,s^n)\in L^1(\Delta).$$

(Equivalently, the assumption on $a, b, c, d \geq 0$ may be dropped if n is even.) Then the following inequalities hold:

$$
\begin{aligned}
& f\left(\frac{\alpha}{(b-a)^{\alpha}} \int_a^b (b-x)^{\alpha-1}\left(x^n\right) dx, \frac{\beta}{(d-c)^{\beta}} \int_c^d (d-y)^{\beta-1}\left(y^n\right) dy\right) \\
& \quad + f\left(\frac{\alpha}{(b-a)^{\alpha}} \int_a^b (b-x)^{\alpha-1}\left(x^n\right) dx, \frac{\beta}{(d-c)^{\beta}} \int_c^d (y-c)^{\beta-1}\left(y^n\right) dy\right) \\
& \quad + f\left(\frac{\alpha}{(b-a)^{\alpha}} \int_a^b (x-a)^{\alpha-1}\left(x^n\right) dx, \frac{\beta}{(d-c)^{\beta}} \int_c^d (d-y)^{\beta-1}\left(y^n\right) dy\right) \\
& \quad + f\left(\frac{\alpha}{(b-a)^{\alpha}} \int_a^b (x-a)^{\alpha-1}\left(x^n\right) dx, \frac{\beta}{(d-c)^{\beta}} \int_c^d (y-c)^{\beta-1}\left(y^n\right) dy\right) \\
& \leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha}(d-c)^{\beta}}\left[J_{a+,c+}^{\alpha,\beta} f(b^n, d^n) + J_{a+,d-}^{\alpha,\beta} f(b^n, c^n) + J_{b-,c+}^{\alpha,\beta} f(a^n, b^n) + J_{b-,d-}^{\alpha,\beta} f(a^n, c^n)\right] \\
& \leq \alpha\beta \sum_{k_1=0}^{n} \sum_{k_2=0}^{n}\left[\frac{(\alpha+n-k_1-1)!n!}{(n-k_1)!(\alpha+n)!} + \frac{(\alpha+k_2-1)!n!}{k_1!(\alpha+n)!}\right] \\
& \quad \times\left[\frac{(\beta+n-k_2-1)!n!}{(n-k_2)!(\beta+n)!} + \frac{(\beta+k_2-1)!n!}{k_2!(\beta+n)!}\right] f\left(a^{n-k_1} b^{k_1}, c^{n-k_2} d^{k_2}\right). \qquad (9)
\end{aligned}
$$

Proof. Using Jensen integral inequality with multible variables, we have

$$
\begin{aligned}
& f\left(\frac{\alpha}{(b-a)^{\alpha}} \int_a^b (b-x)^{\alpha-1}\left(x^n\right) dx, \frac{\beta}{(d-c)^{\beta}} \int_c^d (d-y)^{\beta-1}\left(y^n\right) dy\right) \\
& \quad \leq \frac{\alpha\beta}{(b-a)^{\alpha}(d-c)^{\beta}} \int_a^b \int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1} f\left(x^n, y^n\right) dy dx, \\
& f\left(\frac{\alpha}{(b-a)^{\alpha}} \int_a^b (b-x)^{\alpha-1}\left(x^n\right) dx, \frac{\beta}{(d-c)^{\beta}} \int_c^d (y-c)^{\beta-1}\left(y^n\right) dy\right) \\
& \quad \leq \frac{\alpha\beta}{(b-a)^{\alpha}(d-c)^{\beta}} \int_a^b \int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1} f\left(x^n, y^n\right) dy dx \\
& f\left(\frac{\alpha}{(b-a)^{\alpha}} \int_a^b (x-a)^{\alpha-1}\left(x^n\right) dx, \frac{\beta}{(d-c)^{\beta}} \int_c^d (d-y)^{\beta-1}\left(y^n\right) dy\right) \\
& \quad \leq \frac{\alpha\beta}{(b-a)^{\alpha}(d-c)^{\beta}} \int_a^b \int_c^d (x-a)^{\alpha-1}(d-y)^{\beta-1} f\left(x^n, y^n\right) dy dx,
\end{aligned}
$$

and

$$
\begin{aligned}
& f\left(\frac{\alpha}{(b-a)^{\alpha}} \int_a^b (x-a)^{\alpha-1}\left(x^n\right) dx, \frac{\beta}{(d-c)^{\beta}} \int_c^d (y-c)^{\beta-1}\left(y^n\right) dy\right) \\
& \quad \leq \frac{\alpha\beta}{(b-a)^{\alpha}(d-c)^{\beta}} \int_a^b \int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1} f\left(x^n, y^n\right) dy dx,
\end{aligned}
$$

for all $(x, y) \in \Delta$. By adding these inequalities, we get

$$
\begin{aligned}
& f\left(\frac{\alpha}{(b-a)^{\alpha}} \int_a^b (b-x)^{\alpha-1}\left(x^n\right) dx, \frac{\beta}{(d-c)^{\beta}} \int_c^d (d-y)^{\beta-1}\left(y^n\right) dy\right) \\
& \quad + f\left(\frac{\alpha}{(b-a)^{\alpha}} \int_a^b (b-x)^{\alpha-1}\left(x^n\right) dx, \frac{\beta}{(d-c)^{\beta}} \int_c^d (y-c)^{\beta-1}\left(y^n\right) dy\right) \\
& \quad + f\left(\frac{\alpha}{(b-a)^{\alpha}} \int_a^b (x-a)^{\alpha-1}\left(x^n\right) dx, \frac{\beta}{(d-c)^{\beta}} \int_c^d (d-y)^{\beta-1}\left(y^n\right) dy\right)
\end{aligned}
$$

$$+ f\left(\frac{\alpha}{(b-a)^{\alpha}}\int_a^b (x-a)^{\alpha-1}(x^n)\,dx, \frac{\beta}{(d-c)^{\beta}}\int_c^d (y-c)^{\beta-1}(y^n)\,dy\right)$$
$$\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha}(d-c)^{\beta}}\left[J^{\alpha,\beta}_{a+,c+}f(b^n,d^n)+J^{\alpha,\beta}_{a+,d-}f(b^n,c^n)+J^{\alpha,\beta}_{b-,c+}f(a^n,b^n)+J^{\alpha,\beta}_{b-,d-}f(a^n,c^n)\right]. \quad (10)$$

By changing of the variables $x = ta + (1-t)b, y = sc + (1-s)d$ and $x = (1-t)a + tb, y = (1-s)c + sd$ for $(x,y) \in \Delta$ and $(t,s) \in [0,1]^2$ in (10) respectively, we obtain

$$f\left(\alpha\int_0^1 t^{\alpha-1}(ta+(1-t)b)^n\,dt, \beta\int_0^1 s^{\beta-1}(sc+(1-s)d)^n\,ds\right)$$
$$+ f\left(\alpha\int_0^1 t^{\alpha-1}(ta+(1-t)b)^n\,dt, \beta\int_0^1 s^{\beta-1}((1-s)c+sd)^n\,ds\right)$$
$$+ f\left(\alpha\int_0^1 t^{\alpha-1}((1-t)a+tb)^n\,dt, \beta\int_0^1 s^{\beta-1}(sc+(1-s)d)^n\,ds\right)$$
$$+ f\left(\alpha\int_0^1 t^{\alpha-1}((1-t)a+tb)^n\,dt, \beta\int_0^1 s^{\beta-1}((1-s)c+sd)^n\,ds\right)$$
$$\leq \alpha\beta\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}f\left[(ta+(1-t)b)^n, (sc+(1-s)d)^n\right]dsdt$$
$$+ \alpha\beta\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}f\left[(ta+(1-t)b)^n, ((1-s)c+sd)^n\right]dsdt$$
$$+ \alpha\beta\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}f\left[((1-t)a+tb)^n, (sc+(1-s)d)^n\right]dsdt$$
$$+ \alpha\beta\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}f\left[((1-t)a+tb)^n, ((1-s)c+sd)^n\right]dsdt,$$

and so

$$f\left(\alpha\int_0^1 t^{\alpha-1}\sum_{k=0}^{n}\binom{n}{k}t^{n-k}a^{n-k}(1-t)^k b^k dt, \beta\int_0^1 s^{\beta-1}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}c^{n-k}(1-s)^k d^k ds\right)$$
$$+ f\left(\alpha\int_0^1 t^{\alpha-1}\sum_{k=0}^{n}\binom{n}{k}t^{n-k}a^{n-k}(1-t)^k b^k dt, \beta\int_0^1 s^{\beta-1}\sum_{k=0}^{n}\binom{n}{k}(1-s)^{n-k}c^{n-k}s^k d^k ds\right)$$
$$+ f\left(\alpha\int_0^1 t^{\alpha-1}\sum_{k=0}^{n}\binom{n}{k}(1-t)^{n-k}a^{n-k}t^k b^k dt, \beta\int_0^1 s^{\beta-1}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}c^{n-k}(1-s)^k d^k ds\right)$$
$$+ f\left(\alpha\int_0^1 t^{\alpha-1}\sum_{k=0}^{n}\binom{n}{k}(1-t)^{n-k}a^{n-k}t^k b^k dt, \beta\int_0^1 s^{\beta-1}\sum_{k=0}^{n}\binom{n}{k}(1-s)^{n-k}c^{n-k}s^k d^k ds\right)$$
$$\leq \alpha\beta\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}f\left[\sum_{k=0}^{n}\binom{n}{k}t^{n-k}a^{n-k}(1-t)^k b^k, \sum_{k=0}^{n}\binom{n}{k}s^{n-k}c^{n-k}(1-s)^k d^k\right]dsdt$$
$$+ \alpha\beta\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}f\left[\sum_{k=0}^{n}\binom{n}{k}t^{n-k}a^{n-k}(1-t)^k b^k, \sum_{k=0}^{n}\binom{n}{k}(1-s)^{n-k}c^{n-k}s^k d^k\right]dsdt$$
$$+ \alpha\beta\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}f\left[\sum_{k=0}^{n}\binom{n}{k}(1-t)^{n-k}a^{n-k}t^k b^k, \sum_{k=0}^{n}\binom{n}{k}s^{n-k}c^{n-k}(1-s)^k d^k\right]dsdt$$
$$+ \alpha\beta\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}f\left[\sum_{k=0}^{n}\binom{n}{k}(1-t)^{n-k}a^{n-k}t^k b^k, \sum_{k=0}^{n}\binom{n}{k}(1-s)^{n-k}c^{n-k}s^k d^k\right]dsdt.$$

Since $\sum_{k=0}^{n}\binom{n}{k}t^k(1-t)^{n-k} = 1$ and $\sum_{k=0}^{n}\binom{n}{k}s^k(1-s)^{n-k} = 1$, by the co-ordinated convexity of f, and considering

$$\int_0^1 t^{m-1}(1-t)^{n-1}dt = B(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!},$$

it follows that

$$
\begin{aligned}
& f\left(\alpha \int_0^1 \sum_{k=0}^{n} \frac{n!\,(\alpha+n-k-1)!}{(n-k)!\,(\alpha+n)!} a^{n-k} b^k dt, \beta \int_0^1 \sum_{k=0}^{n} \frac{n!\,(\beta+n-k-1)!}{(n-k)!\,(\beta+n)!} c^{n-k} d^k ds\right) \\
&\quad + f\left(\alpha \int_0^1 \sum_{k=0}^{n} \frac{n!\,(\alpha+n-k-1)!}{(n-k)!\,(\alpha+n)!} a^{n-k} b^k dt, \beta \int_0^1 \sum_{k=0}^{n} \frac{n!\,(\beta+k-1)!}{k!\,(\beta+n)!} c^{n-k} d^k ds\right) \\
&\quad + f\left(\alpha \int_0^1 \sum_{k=0}^{n} \frac{n!\,(\alpha+k-1)!}{k!\,(\alpha+n)!} a^{n-k} b^k dt, \beta \int_0^1 \sum_{k=0}^{n} \frac{n!\,(\beta+n-k-1)!}{(n-k)!\,(\beta+n)!} c^{n-k} d^k ds\right) \\
&\quad + f\left(\alpha \int_0^1 \sum_{k=0}^{n} \frac{n!\,(\alpha+k-1)!}{k!\,(\alpha+n)!} a^{n-k} b^k dt, \beta \int_0^1 \sum_{k=0}^{n} \frac{n!\,(\beta+k-1)!}{k!\,(\beta+n)!} c^{n-k} d^k ds\right) \\
&\le \alpha\beta \int_0^1 \int_0^1 t^{\alpha-1} s^{\beta-1} f\left[\sum_{k=0}^{n} \binom{n}{k} t^{n-k} (1-t)^k a^{n-k} b^k, \sum_{k=0}^{n} \binom{n}{k} s^{n-k} (1-s)^k c^{n-k} d^k\right] ds dt \\
&\quad + \alpha\beta \int_0^1 \int_0^1 t^{\alpha-1} s^{\beta-1} f\left[\sum_{k=0}^{n} \binom{n}{k} t^{n-k} (1-t)^k a^{n-k} b^k, \sum_{k=0}^{n} \binom{n}{k} s^{k} (1-s)^{n-k} c^{n-k} d^k\right] ds dt \\
&\quad + \alpha\beta \int_0^1 \int_0^1 t^{\alpha-1} s^{\beta-1} f\left[\sum_{k=0}^{n} \binom{n}{k} (1-t)^{n-k} t^k a^{n-k} b^k, \sum_{k=0}^{n} \binom{n}{k} s^{n-k} (1-s)^k c^{n-k} d^k\right] ds dt \\
&\quad + \alpha\beta \int_0^1 \int_0^1 t^{\alpha-1} s^{\beta-1} f\left[\sum_{k=0}^{n} \binom{n}{k} (1-t)^{n-k} t^k a^{n-k} b^k, \sum_{k=0}^{n} \binom{n}{k} s^{k} (1-s)^{n-k} c^{n-k} d^k\right] ds dt \\
&\le \alpha\beta \int_0^1 \int_0^1 \left(\sum_{k_1=0}^{n} \binom{n}{k_1} t^{\alpha+n-k_1-1} (1-t)^{k_1}\right) \\
&\quad \times \left(\sum_{k_2=0}^{n} \binom{n}{k_2} s^{\beta+n-k_2-1} (1-s)^{k_2}\right) f\left(a^{n-k_1} b^{k_1}, c^{n-k_2} d^{k_2}\right) ds dt \\
&\quad + \alpha\beta \int_0^1 \int_0^1 \left(\sum_{k_1=0}^{n} \binom{n}{k_1} t^{\alpha+n-k_1-1} (1-t)^{k_1}\right) \\
&\quad \times \left(\sum_{k_2=0}^{n} \binom{n}{k_2} s^{\beta+k_2-1} (1-s)^{n-k_2}\right) f\left(a^{n-k_1} b^{k_1}, c^{n-k_2} d^{k_2}\right) ds dt \\
&\quad + \alpha\beta \int_0^1 \int_0^1 \left(\sum_{k_1=0}^{n} \binom{n}{k_1} t^{\alpha+k_1-1} (1-t)^{n-k_1}\right) \\
&\quad \times \left(\sum_{k_2=0}^{n} \binom{n}{k_2} s^{\beta+n-k_2-1} (1-s)^{k_2}\right) f\left(a^{n-k_1} b^{k_1}, c^{n-k_2} d^{k_2}\right) ds dt \\
&\quad + \alpha\beta \int_0^1 \int_0^1 \left(\sum_{k_1=0}^{n} \binom{n}{k_1} t^{\alpha+k_1-1} (1-t)^{n-k_1}\right) \\
&\quad \times \left(\sum_{k_2=0}^{n} \binom{n}{k_2} s^{\beta+k_2-1} (1-s)^{n-k_2}\right) f\left(a^{n-k_1} b^{k_1}, c^{n-k_2} d^{k_2}\right) ds dt \\
&= \alpha\beta \left[\sum_{k_1=0}^{n} \binom{n}{k_1} \int_0^1 \left(t^{\alpha+n-k_1-1} (1-t)^{k_1}\right) dt\right] \\
&\quad \times \left[\sum_{k_2=0}^{n} \binom{n}{k_2} \int_0^1 \left(s^{\beta+n-k_2-1} (1-s)^{k_2}\right) ds\right] f\left(a^{n-k_1} b^{k_1}, c^{n-k_2} d^{k_2}\right) \\
&\quad + \alpha\beta \left[\sum_{k_1=0}^{n} \binom{n}{k_1} \int_0^1 \left(t^{\alpha+n-k_1-1} (1-t)^{k_1}\right) dt\right] \\
&\quad \times \left[\sum_{k_2=0}^{n} \binom{n}{k_2} \int_0^1 \left(s^{\beta+k_2-1} (1-s)^{n-k_2}\right) ds\right] f\left(a^{n-k_1} b^{k_1}, c^{n-k_2} d^{k_2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha\beta\left[\sum_{k_1=0}^{n}\left(\begin{array}{c} n \\ k_1 \end{array}\right)\int_0^1 t^{\alpha+k_1-1}(1-t)^{n-k_1}\,dt\right] \\
&\times\left[\sum_{k_2=0}^{n}\left(\begin{array}{c} n \\ k_2 \end{array}\right)\int_0^1 s^{\beta+n-k_2-1}(1-s)^{k_2}\,ds\right] f\left(a^{n-k_1}b^{k_1}, c^{n-k_2}d^{k_2}\right) \\
&+\alpha\beta\left[\sum_{k_1=0}^{n}\left(\begin{array}{c} n \\ k_1 \end{array}\right)\int_0^1 t^{\alpha+k_1-1}(1-t)^{n-k_1}\,dt\right] \\
&\times\left[\sum_{k_2=0}^{n}\left(\begin{array}{c} n \\ k_2 \end{array}\right)\int_0^1 s^{\beta+k_2-1}(1-s)^{n-k_2}\,ds\right] f\left(a^{n-k_1}b^{k_1}, c^{n-k_2}d^{k_2}\right) \\
=&\ \alpha\beta\left[\sum_{k_1=0}^{n}\frac{(\alpha+n-k_1-1)!n!}{(n-k_1)!\,(\alpha+n)!}\right]\left[\sum_{k_2=0}^{n}\frac{(\beta+n-k_2-1)!n!}{(n-k_2)!\,(\beta+n)!}\right] f\left(a^{n-k_1}b^{k_1}, c^{n-k_2}d^{k_2}\right) \\
&+\alpha\beta\left[\sum_{k_1=0}^{n}\frac{(\alpha+n-k_1-1)!n!}{(n-k_1)!\,(\alpha+n)!}\right]\left[\sum_{k_2=0}^{n}\frac{(\beta+k_2-1)!n!}{k_2!\,(\beta+n)!}\right] f\left(a^{n-k_1}b^{k_1}, c^{n-k_2}d^{k_2}\right) \\
&+\alpha\beta\left[\sum_{k_1=0}^{n}\frac{(\alpha+k_2-1)!n!}{k_1!\,(\alpha+n)!}\right]\left[\sum_{k_2=0}^{n}\frac{(\beta+n-k_2-1)!n!}{(n-k_2)!\,(\beta+n)!}\right] f\left(a^{n-k_1}b^{k_1}, c^{n-k_2}d^{k_2}\right) \\
&+\alpha\beta\left[\sum_{k_1=0}^{n}\frac{(\alpha+k_2-1)!n!}{k_1!\,(\alpha+n)!}\right]\left[\sum_{k_2=0}^{n}\frac{(\beta+k_2-1)!n!}{k_2!\,(\beta+n)!}\right] f\left(a^{n-k_1}b^{k_1}, c^{n-k_2}d^{k_2}\right) \\
=&\ \alpha\beta\sum_{k_1=0}^{n}\sum_{k_2=0}^{n}\left[\frac{(\alpha+n-k_1-1)!n!}{(n-k_1)!\,(\alpha+n)!}+\frac{(\alpha+k_2-1)!n!}{k_1!\,(\alpha+n)!}\right] \\
&\times\left[\frac{(\beta+n-k_2-1)!n!}{(n-k_2)!\,(\beta+n)!}+\frac{(\beta+k_2-1)!n!}{k_2!\,(\beta+n)!}\right] f\left(a^{n-k_1}b^{k_1}, c^{n-k_2}d^{k_2}\right)
\end{aligned}
$$

which is completed the inequalities of (9). □

Remark 3. Under the assumptions of Theorem 6, if we take $n = 1$, then the inequality (9) reduces to the fractional integral inequality (8). Indeed, for $n = 1$, we have $x^n = x$ and $y^n = y$, so all terms in (9) involving x^n and y^n become the standard terms in (8).

Moreover, when $n > 1$, the requirement that f is positive or that n is even ensures that the expressions x^n and y^n are well-defined over the domain Δ even if a, b, c, or d are negative. Hence, Theorem 6 generalizes (8) to powers n of the variables while maintaining the validity of the inequality under these conditions.

Now, we give the new following result.

Theorem 7. *Let $f : \Delta \to \mathbb{R}$ be a co-ordinated convex function on Δ. Let $n \in \mathbb{N}$, and assume either $f(x,y) \geq 0$ for all $(x,y) \in \Delta$ or n is an even integer. Then the following inequalities hold for $\alpha, \beta > 0$:*

$$
\begin{aligned}
& f^n\left(\frac{b+\alpha a}{\alpha+1}, \frac{d+\beta c}{\beta+1}\right)+f^n\left(\frac{b+\alpha a}{\alpha+1}, \frac{\beta d+c}{\beta+1}\right)+f^n\left(\frac{\alpha b+a}{\alpha+1}, \frac{d+\beta c}{\beta+1}\right)+f^n\left(\frac{\alpha b+a}{\alpha+1}, \frac{\beta d+c}{\beta+1}\right) \\
\leq &\left[\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta} J^{\alpha,\beta}_{a+,c+} f(b,d)\right]^n+\left[\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta} J^{\alpha,\beta}_{a+,d-} f(b,c)\right]^n \\
&+\left[\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta} J^{\alpha,\beta}_{b-,c+} f(a,b)\right]^n+\left[\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta} J^{\alpha,\beta}_{b-,d-} f(a,c)\right]^n \\
\leq &\ \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta}\left[J^{\alpha,\beta}_{a+,c+} f^n(b,d)+J^{\alpha,\beta}_{a+,d-} f^n(b,c)+J^{\alpha,\beta}_{b-,c+} f^n(a,b)+J^{\alpha,\beta}_{b-,d-} f^n(a,c)\right] \\
\leq &\ \alpha\beta\sum_{k_1+k_2+k_3+k_4=n}\frac{n!}{k_1!k_2!k_3!k_4!}\left[\frac{(\alpha+k_1+k_2-1)!\,(k_3+k_4)!}{(\alpha+n)!}+\frac{(\alpha+k_3+k_4-1)!\,(k_1+k_2)!}{(\alpha+n)!}\right]
\end{aligned}
$$

$$\times\left[\frac{(\beta+k_1+k_3-1)!\,(k_2+k_4)!}{(\alpha+n)!}+\frac{(\beta+k_2+k_4-1)!\,(k_1+k_3)!}{(\alpha+n)!}\right]\times f^{k_1}(a,c)\,f^{k_2}(a,d)\,f^{k_3}(b,c)\,f^{k_4}(b,d)\,. \tag{11}$$

Proof. By the co-ordinated convexity of f, $F(x)=x^n$ and by Jensen integral inequality with multible variables, we get

$$\begin{aligned}
f^n\left(\frac{b+\alpha a}{\alpha+1},\frac{d+\beta c}{\beta+1}\right) &= f^n\left(\frac{\alpha}{(b-a)^\alpha}\int_a^b(b-x)^{\alpha-1}(x)\,dx,\frac{\beta}{(d-c)^\beta}\int_c^d(d-y)^{\beta-1}(y)\,dy\right)\\
&\le \left[\frac{\alpha\beta}{(b-a)^\alpha(d-c)^\beta}\int_a^b\int_c^d(b-x)^{\alpha-1}(d-y)^{\beta-1}f(x,y)\,dydx\right]^n\\
&\le \frac{\alpha\beta}{(b-a)^\alpha(d-c)^\beta}\int_a^b\int_c^d(b-x)^{\alpha-1}(d-y)^{\beta-1}f^n(x,y)\,dydx,\\
f^n\left(\frac{b+\alpha a}{\alpha+1},\frac{\beta d+c}{\beta+1}\right) &= f^n\left(\frac{\alpha}{(b-a)^\alpha}\int_a^b(b-x)^{\alpha-1}(x)\,dx,\frac{\beta}{(d-c)^\beta}\int_c^d(y-c)^{\beta-1}(y)\,dy\right)\\
&\le \left[\frac{\alpha\beta}{(b-a)^\alpha(d-c)^\beta}\int_a^b\int_c^d(b-x)^{\alpha-1}(y-c)^{\beta-1}f(x,y)\,dydx\right]^n\\
&\le \frac{\alpha\beta}{(b-a)^\alpha(d-c)^\beta}\int_a^b\int_c^d(b-x)^{\alpha-1}(y-c)^{\beta-1}f^n(x,y)\,dydx,\\
f^n\left(\frac{\alpha b+a}{\alpha+1},\frac{d+\beta c}{\beta+1}\right) &= f^n\left(\frac{\alpha}{(b-a)^\alpha}\int_a^b(x-a)^{\alpha-1}(x)\,dx,\frac{\beta}{(d-c)^\beta}\int_c^d(d-y)^{\beta-1}(y)\,dy\right)\\
&\le \left[\frac{\alpha\beta}{(b-a)^\alpha(d-c)^\beta}\int_a^b\int_c^d(x-a)^{\alpha-1}(d-y)^{\beta-1}f(x,y)\,dydx\right]^n\\
&\le \frac{\alpha\beta}{(b-a)^\alpha(d-c)^\beta}\int_a^b\int_c^d(x-a)^{\alpha-1}(d-y)^{\beta-1}f^n(x,y)\,dydx,
\end{aligned}$$

and

$$\begin{aligned}
f^n\left(\frac{\alpha b+a}{\alpha+1},\frac{\beta d+c}{\beta+1}\right) &= f^n\left(\frac{\alpha}{(b-a)^\alpha}\int_a^b(x-a)^{\alpha-1}(x)\,dx,\frac{\beta}{(d-c)^\beta}\int_c^d(y-c)^{\beta-1}(y)\,dy\right)\\
&\le \left[\frac{\alpha\beta}{(b-a)^\alpha(d-c)^\beta}\int_a^b\int_c^d(x-a)^{\alpha-1}(y-c)^{\beta-1}f(x,y)\,dydx\right]^n\\
&\le \frac{\alpha\beta}{(b-a)^\alpha(d-c)^\beta}\int_a^b\int_c^d(x-a)^{\alpha-1}(y-c)^{\beta-1}f^n(x,y)\,dydx,
\end{aligned}$$

for all $(x,y)\in\Delta$. Adding both sides of above results, we can write

$$\begin{aligned}
&f^n\left(\frac{b+\alpha a}{\alpha+1},\frac{d+\beta c}{\beta+1}\right)+f^n\left(\frac{b+\alpha a}{\alpha+1},\frac{\beta d+c}{\beta+1}\right)+f^n\left(\frac{\alpha b+a}{\alpha+1},\frac{d+\beta c}{\beta+1}\right)+f^n\left(\frac{\alpha b+a}{\alpha+1},\frac{\beta d+c}{\beta+1}\right)\\
&\le \left[\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta}J^{\alpha,\beta}_{a+,c+}f(b,d)\right]^n+\left[\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta}J^{\alpha,\beta}_{a+,d-}f(b,c)\right]^n\\
&\quad+\left[\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta}J^{\alpha,\beta}_{b-,c+}f(a,b)\right]^n+\left[\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta}J^{\alpha,\beta}_{b-,d-}f(a,c)\right]^n\\
&\le \frac{\alpha\beta}{(b-a)^\alpha(d-c)^\beta}\int_a^b\int_c^d(b-x)^{\alpha-1}(d-y)^{\beta-1}f^n(x,y)\,dydx\\
&\quad+\frac{\alpha\beta}{(b-a)^\alpha(d-c)^\beta}\int_a^b\int_c^d(b-x)^{\alpha-1}(y-c)^{\beta-1}f^n(x,y)\,dydx
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha\beta}{(b-a)^{\alpha}(d-c)^{\beta}}\int_a^b\int_c^d (x-a)^{\alpha-1}(d-y)^{\beta-1} f^n(x,y)\,dydx\\
&+\frac{\alpha\beta}{(b-a)^{\alpha}(d-c)^{\beta}}\int_a^b\int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1} f^n(x,y)\,dydx, \qquad (12)
\end{aligned}
$$

and so the first and the second inequalities of (11) are proved. For the proof of the last inequality in (11), by changing of the variables $x = ta + (1-t)b$, $y = sc + (1-s)d$ and $x = (1-t)a + tb$, $y = (1-s)c + sd$ for $(x,y) \in \Delta$ and $(t,s) \in [0,1]^2$ in (12) respectively, and by co-ordinate convexity of f, we have

$$
\begin{aligned}
&\frac{\alpha\beta}{(b-a)^{\alpha}(d-c)^{\beta}}\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1} f^n(x,y)\,dydx\\
&\quad+\frac{\alpha\beta}{(b-a)^{\alpha}(d-c)^{\beta}}\int_a^b\int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1} f^n(x,y)\,dydx\\
&\quad+\frac{\alpha\beta}{(b-a)^{\alpha}(d-c)^{\beta}}\int_a^b\int_c^d (x-a)^{\alpha-1}(d-y)^{\beta-1} f^n(x,y)\,dydx\\
&\quad+\frac{\alpha\beta}{(b-a)^{\alpha}(d-c)^{\beta}}\int_a^b\int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1} f^n(x,y)\,dydx\\
&\quad=\alpha\beta\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1} f^n(ta+(1-t)b, sc+(1-s)d)\,dsdt\\
&\quad+\alpha\beta\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1} f^n(ta+(1-t)b, (1-s)c+sd)\,dsdt\\
&\quad+\alpha\beta\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1} f^n((1-t)a+tb, sc+(1-s)d)\,dsdt\\
&\quad+\alpha\beta\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1} f^n((1-t)a+tb, (1-s)c+sd)\,dsdt\\
&\quad\le\alpha\beta\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}\left[tsf(a,c)+t(1-s)f(a,d)+(1-t)sf(b,c)+(1-t)(1-s)f(b,d)\right]^n dsdt\\
&\quad+\alpha\beta\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}\left[t(1-s)f(a,c)+tsf(a,d)+(1-t)(1-s)f(b,c)+(1-t)sf(b,d)\right]^n dsdt\\
&\quad+\alpha\beta\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}\left[(1-t)sf(a,c)+(1-t)(1-s)f(a,d)+tsf(b,c)+t(1-s)f(b,d)\right]^n dsdt\\
&\quad+\alpha\beta\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}\left[(1-t)(1-s)f(a,c)+(1-t)sf(a,d)+t(1-s)f(b,c)+tsf(b,d)\right]^n dsdt\\
&\quad=\alpha\beta\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}\sum_{k_1+k_2+k_3+k_4=n}\frac{n!}{k_1!k_2!k_3!k_4!} t^{k_1+k_2}(1-t)^{k_3+k_4}s^{k_1+k_3}(1-s)^{k_2+k_4}\\
&\quad\times f^{k_1}(a,c)f^{k_2}(a,d)f^{k_3}(b,c)f^{k_4}(b,d)\,dsdt\\
&\quad+\alpha\beta\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}\sum_{k_1+k_2+k_3+k_4=n}\frac{n!}{k_1!k_2!k_3!k_4!} t^{k_1+k_2}(1-t)^{k_3+k_4}s^{k_2+k_4}(1-s)^{k_1+k_3}\\
&\quad\times f^{k_1}(a,c)f^{k_2}(a,d)f^{k_3}(b,c)f^{k_4}(b,d)\,dsdt\\
&\quad+\alpha\beta\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}\sum_{k_1+k_2+k_3+k_4=n}\frac{n!}{k_1!k_2!k_3!k_4!} t^{k_3+k_4}(1-t)^{k_1+k_2}s^{k_1+k_3}(1-s)^{k_2+k_4}\\
&\quad\times f^{k_1}(a,c)f^{k_2}(a,d)f^{k_3}(b,c)f^{k_4}(b,d)\,dsdt\\
&\quad+\alpha\beta\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}\sum_{k_1+k_2+k_3+k_4=n}\frac{n!}{k_1!k_2!k_3!k_4!} t^{k_3+k_4}(1-t)^{k_1+k_2}s^{k_2+k_4}(1-s)^{k_1+k_3}\\
&\quad\times f^{k_1}(a,c)f^{k_2}(a,d)f^{k_3}(b,c)f^{k_4}(b,d)\,dsdt\\
&\quad=\alpha\beta\sum_{k_1+k_2+k_3+k_4=n}\frac{n!}{k_1!k_2!k_3!k_4!}\int_0^1\int_0^1\left[t^{\alpha+k_1+k_2-1}(1-t)^{k_3+k_4}+t^{\alpha+k_3+k_4-1}(1-t)^{k_1+k_2}\right]\\
&\quad\times\left[s^{\beta+k_1+k_3-1}(1-s)^{k_2+k_4}+s^{\beta+k_2+k_4-1}(1-s)^{k_1+k_3}\right] f^{k_1}(a,c)f^{k_2}(a,d)f^{k_3}(b,c)f^{k_4}(b,d)\,dsdt
\end{aligned}
$$

$$= \alpha\beta \sum_{k_1+k_2+k_3+k_4=n} \frac{n!}{k_1!k_2!k_3!k_4!} \left[\frac{(\alpha+k_1+k_2-1)!\,(k_3+k_4)!}{(\alpha+n)!} + \frac{(\alpha+k_3+k_4-1)!\,(k_1+k_2)!}{(\alpha+n)!} \right]$$
$$\times \left[\frac{(\beta+k_1+k_3-1)!\,(k_2+k_4)!}{(\alpha+n)!} + \frac{(\beta+k_2+k_4-1)!\,(k_1+k_3)!}{(\alpha+n)!} \right] f^{k_1}(a,c)\, f^{k_2}(a,d)\, f^{k_3}(b,c)\, f^{k_4}(b,d)\,.$$

Using the multinomial theorem, for an expression of the form $(x_1+x_2+...+x_n)^n$ the expansion is given by:

$$(x_1+x_2+...+x_n)^n = \sum_{k_1+k_2+...+k_m=n} \frac{n!}{k_1!k_2!...k_m!} x_1^{k_1} x_2^{k_2} ... x_m^{k_m}.$$

Here $k_1, k_2, \ldots, k_m$ are non-negative integers representing the powers of each term in the expansion, and their sum equals n. Thus, we have the conclusion (11). □

Remark 4. Under the assumptions of Theorem 7, if we take $n = 1$, the inequality (11) reduces to the fractional integral inequality for co-ordinated convex functions as in Theorem 5.

Moreover, the conditions that f is non-negative or that n is even ensure that all terms $f^n(x,y)$ are well-defined and non-negative over the domain Δ, which is necessary for the validity of the inequalities. Hence, Theorem 7 generalizes the previous results to powers of the function while maintaining the structure of the Hermite–Hadamard type inequalities.

Conflicts of Interest: The author declares no competing interests.

References

[1] Dragomir, S. S., & Pearce, C. (2003). Selected topics on Hermite-Hadamard inequalities and applications. *Science Direct Working Paper,* (S1574-0358), 04.

[2] Mitrinovic, D. S., Pecaric, J., & Fink, A. M. (2013). *Classical and New Inequalities in Analysis* (Vol. 61). Springer Science & Business Media.

[3] Kian, M., & Moslehian, M. (2013). Refinements of the operator Jensen-Mercer inequality. *The Electronic Journal of Linear Algebra, 26,* 742-753.

[4] Matković, A., Pečarić, J., & Perić, I. (2006). A variant of Jensen's inequality of Mercer's type for operators with applications. *Linear Algebra and its Applications, 418*(2-3), 551-564.

[5] Peng, S., Wei, W., & Wang, J. (2014). On the Hermite-Hadamard inequalities for convex functions via Hadamard fractional integrals. *Facta Universitatis, Series: Mathematics and Informatics, 29*(1), 55-75.

[6] Sarikaya, M. Z., Set, E., Yaldiz, H., & Başak, N. (2013). Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. *Mathematical and Computer Modelling, 57*(9-10), 2403-2407.

[7] Zabandan, G. (2009). A new refinement of the Hermite-Hadamard inequality for convex functions. *Journal of Inequalities in Pure and Applied Mathematics, 10*(2), 7.

[8] Zabandan, G., & Kılıçman, A. (2012). A new version of Jensen's inequality and related results. *Journal of Inequalities and Applications, 2012*(1), 238.

[9] Wang, J., Li, X., & Zhu, C. (2013). Refinements of Hermite-Hadamard type inequalities involving fractional integrals. *Bulletin of the Belgian Mathematical Society-Simon Stevin, 20*(4), 655-666.

[10] Dragomir, S. S. (2001). On the Hadamard's inequlality for convex functions on the co-ordinates in a rectangle from the plane. *Taiwanese Journal of Mathematics,* 775-788.

[11] Sarıkaya, M. Z. (2014). On the Hermite–Hadamard-type inequalities for co-ordinated convex function via fractional integrals. *Integral Transforms and Special Functions, 25*(2), 134-147.

[12] Fleming, W. (1977). *Functions of Several Variables*. Spring-Verlang, New-York, Heidelberg, Berlin, 1977.