

Article

On a generalized Pachpatte inequality through a convexity-based approach

Christophe Chesneau

Department of Mathematics, LMNO, University of Caen-Normandie, 14032 Caen, France;
christophe.chesneau@gmail.com

Received: 20 July 2025; Accepted: 23 September 2025; Published: 16 October 2025.

Abstract: In this article, we extend a key integral inequality established by Pachpatte in 2002 by introducing a new convexity-based approach. Specifically, we incorporate a general convex function to create a flexible framework that can be adapted to various mathematical contexts. The resulting techniques are original and reusable, offering potential for further innovations in related analytical frameworks. We present several examples to illustrate the theory and demonstrate the versatility of the approach.

Keywords: integral inequalities, convexity, convexity inequality, Jensen integral inequality

MSC: 26D15, 33E20.

1. Introduction

Integrals are fundamental tools in mathematics. They are used to define various measures, such as areas, volumes and cumulative quantities, and have applications in diverse fields including physics, engineering, economics, statistics and probability theory. In practice, however, the exact evaluation of integrals can be challenging. Despite the existence of numerous techniques, many integrals cannot be solved in closed form. In such cases, it is essential to establish bounds in order to provide reliable approximations and error margins. These bounds are often derived from sharp integral inequalities, a subject that has been studied extensively for decades. For a more detailed exploration of this topic, the reader is referred to [1–6].

A classical type of integral inequality involves comparing the average value of a function with its values at the endpoints. This yields bounds on the difference between the integral of a function and certain discrete approximations, such as the trapezoidal rule. Pachpatte, in [7], provides a useful example of such an inequality, which is presented in the theorem below.

Theorem 1. [7, First part of Theorem 1] Let $a, b \in \mathbb{R}$ with $b > a$ and $f : [a, b] \mapsto \mathbb{R}$ be a differentiable function on $[a, b]$ with f' continuous on $[a, b]$. Then we have

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{2} [f(a) + f(b)] \right| \leq \frac{1}{2} \int_a^b |f'(t)| dt.$$

This result provides a bound on the error between the average value of f over $[a, b]$ and the arithmetic mean of its endpoint values. This error is controlled by the integral of the absolute value of the derivative of f . Inequalities of this type play a significant role in numerical integration, approximation theory, and the analysis of integral operators.

From a mathematical perspective, convex functions are fundamental entities. The formal definition of a convex function is given below. Let $a, b \in \mathbb{R}$ with $b > a$, and $g : [a, b] \mapsto \mathbb{R}$ be a function. We say that g is convex if and only if, for any $\epsilon \in [0, 1]$ and $x, y \in [a, b]$, the following inequality holds:

$$g[\epsilon x + (1 - \epsilon)y] \leq \epsilon g(x) + (1 - \epsilon)g(y). \quad (1)$$

There are numerous integral inequalities involving convex functions. Classical results, such as the Jensen integral inequality and the Hermite-Hadamard integral inequality, provide powerful tools for bounding

integrals using convexity properties. These inequalities also have significant applications in economics, statistics, information theory and functional analysis. For further details on this topic, the reader is referred to [8–18].

In this article, we propose to extend Theorem 1 by adopting a convexity-based approach. More precisely, we introduce a convex function ϕ and derive a sharp upper bound for the following quantity:

$$\phi \left\{ \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{2} [f(a) + f(b)] \right\}.$$

Theorem 1 corresponds to the particular case where $\phi(x) = |x|$. Our method relies on several key tools from convex analysis, including the standard convexity inequality given in Eq. (1), as well as the Jensen integral inequality. The proofs are detailed and structured in a way that highlights modular arguments, with intermediate results that can be reused in related problems. The theoretical results are complemented by various examples that illustrate how different choices of the function ϕ lead to distinct inequalities. These examples demonstrate the flexibility and generality of the proposed framework.

The rest of the article is organized as follows: §2 contains the main results and proofs. A conclusion is given in §3.

2. Results and proofs

2.1. First result

The result below is an intermediate step in proving our main theorem. A detailed proof is provided immediately after the theorem is stated.

Theorem 2. Let $a, b \in \mathbb{R}$ with $b > a$, $f : [a, b] \mapsto \mathbb{R}$ be a differentiable function on $[a, b]$ such that, for any $t \in [a, b]$, $|f'(t)| \leq 1$, and $\phi : \mathbb{R} \mapsto [0, +\infty)$ be a non-decreasing convex function with $\phi(0) = 0$. Then, for any $x \in [a, b]$, we have

$$\phi \left\{ f(x) - \frac{1}{2} [f(a) + f(b)] \right\} \leq \frac{\phi(b-a)}{2(b-a)} \int_a^b |f'(t)| dt.$$

Proof. Since f is differentiable on $[a, b]$ with f' bounded on $[a, b]$, for any $x \in [a, b]$, we can express f in the following two ways:

$$f(x) = f_1(x) = f(a) + \int_a^x f'(t) dt, \quad f(x) = f_2(x) = f(b) - \int_x^b f'(t) dt.$$

We also have

$$\begin{aligned} f(x) &= \frac{1}{2} [f_1(x) + f_2(x)] \\ &= \frac{1}{2} [f(a) + f(b)] + \frac{1}{2} \int_a^x f'(t) dt + \frac{1}{2} \int_x^b [-f'(t)] dt, \end{aligned}$$

so that

$$f(x) - \frac{1}{2} [f(a) + f(b)] = \frac{1}{2} \int_a^x f'(t) dt + \frac{1}{2} \int_x^b [-f'(t)] dt.$$

Using this expression and the basic convexity inequality applied with the weight $\epsilon = 1/2$, we obtain

$$\begin{aligned} \phi \left\{ f(x) - \frac{1}{2} [f(a) + f(b)] \right\} &= \phi \left\{ \frac{1}{2} \int_a^x f'(t) dt + \frac{1}{2} \int_x^b [-f'(t)] dt \right\} \\ &\leq \frac{1}{2} \phi \left[\int_a^x f'(t) dt \right] + \frac{1}{2} \phi \left\{ \int_x^b [-f'(t)] dt \right\}. \end{aligned} \quad (2)$$

For any $t \in (a, b)$, since $f'(t) \leq |f'(t)|$ and $-f'(t) \leq |f'(t)|$, we have, for any $x \in [a, b]$, $\int_a^x f'(t)dt \leq \int_a^x |f'(t)|dt$ and $\int_x^b [-f'(t)]dt \leq \int_x^b |f'(t)|dt$. These inequalities and the fact that ϕ is non-decreasing yield

$$\frac{1}{2}\phi\left[\int_a^x f'(t)dt\right] + \frac{1}{2}\phi\left\{\int_x^b [-f'(t)]dt\right\} \leq \frac{1}{2}\phi\left[\int_a^x |f'(t)|dt\right] + \frac{1}{2}\phi\left\{\int_x^b |f'(t)|dt\right\}. \quad (3)$$

Suitable decompositions of the two integrands and two applications of the Jensen integral inequality based on the convexity of ϕ give

$$\begin{aligned} \frac{1}{2}\phi\left[\int_a^x |f'(t)|dt\right] + \frac{1}{2}\phi\left\{\int_x^b |f'(t)|dt\right\} &= \frac{1}{2}\phi\left[\frac{1}{x-a}\int_a^x (x-a)|f'(t)|dt\right] + \frac{1}{2}\phi\left\{\frac{1}{b-x}\int_x^b (b-x)|f'(t)|dt\right\} \\ &\leq \frac{1}{2(x-a)}\int_a^x \phi[(x-a)|f'(t)|]dt + \frac{1}{2(b-x)}\int_x^b \phi[(b-x)|f'(t)|]dt. \end{aligned} \quad (4)$$

Since, for any $t \in [a, b]$, $|f'(t)| \leq 1$, and $\phi(0) = 0$, the basic convexity inequality with the weight $\epsilon = |f'(t)|$ ensures that

$$\begin{aligned} \phi[(x-a)|f'(t)|] &= \phi\{(x-a)|f'(t)| + [1-|f'(t)|] \times 0\} \\ &\leq |f'(t)|\phi(x-a) + [1-|f'(t)|]\phi(0) \\ &= |f'(t)|\phi(x-a). \end{aligned}$$

In a similar way, we have

$$\begin{aligned} \phi[(b-x)|f'(t)|] &= \phi\{(b-x)|f'(t)| + [1-|f'(t)|] \times 0\} \\ &\leq |f'(t)|\phi(b-x) + [1-|f'(t)|]\phi(0) \\ &= |f'(t)|\phi(b-x). \end{aligned}$$

These inequalities imply that

$$\begin{aligned} &\frac{1}{2(x-a)}\int_a^x \phi[(x-a)|f'(t)|]dt + \frac{1}{2(b-x)}\int_x^b \phi[(b-x)|f'(t)|]dt \\ &\leq \frac{1}{2(x-a)}\int_a^x |f'(t)|\phi(x-a)dt + \frac{1}{2(b-x)}\int_x^b |f'(t)|\phi(b-x)dt \\ &= \frac{\phi(x-a)}{2(x-a)}\int_a^x |f'(t)|dt + \frac{\phi(b-x)}{2(b-x)}\int_x^b |f'(t)|dt. \end{aligned} \quad (5)$$

Since ϕ is convex with $\phi(0) = 0$, a classical result ensures that the function $\phi(x)/x$ is non-decreasing. Therefore, for any $x \in [a, b]$, we have

$$\frac{\phi(x-a)}{x-a} \leq \frac{\phi(b-a)}{b-a}, \quad \frac{\phi(b-x)}{b-x} \leq \frac{\phi(b-a)}{b-a}.$$

These inequalities and the Chasles integral relation give

$$\begin{aligned} \frac{\phi(x-a)}{2(x-a)}\int_a^x |f'(t)|dt + \frac{\phi(b-x)}{2(b-x)}\int_x^b |f'(t)|dt &\leq \frac{\phi(b-a)}{2(b-a)}\left[\int_a^x |f'(t)|dt + \int_x^b |f'(t)|dt\right] \\ &= \frac{\phi(b-a)}{2(b-a)}\int_a^b |f'(t)|dt. \end{aligned} \quad (6)$$

It follows from Eqs. (2), (3), (4), (5) and (6) that

$$\phi\left\{f(x) - \frac{1}{2}[f(a) + f(b)]\right\} \leq \frac{\phi(b-a)}{2(b-a)}\int_a^b |f'(t)|dt.$$

This concludes the proof. $\square$

The assumption that, for any $t \in [a, b]$, $|f'(t)| \leq 1$, can be replaced with the statement that there exists $M > 0$ such that, for any $t \in [a, b]$, $|f'(t)| \leq M$. It is sufficient to replace f by f/M and $\phi(x)$ by $\phi(Mx)$, and we get

$$\phi \left\{ f(x) - \frac{1}{2}[f(a) + f(b)] \right\} \leq \frac{\phi[M(b-a)]}{2M(b-a)} \int_a^b |f'(t)| dt.$$

Some examples of applications of Theorem 2 are described below.

Example 1. If we take $\phi(x) = |x|$, which satisfies the required assumptions, since $b > a$, then we have

$$\left| f(x) - \frac{1}{2}[f(a) + f(b)] \right| \leq \frac{|b-a|}{2(b-a)} \int_a^b |f'(t)| dt = \frac{1}{2} \int_a^b |f'(t)| dt.$$

In this case, note that the proof does not use the restrictive condition that, for any $t \in [a, b]$, $|f'(t)| \leq 1$.

Example 2. More generally, if we take $\phi(x) = |x|^p$ with $p > 1$, which satisfies the required assumptions, then we have

$$\left| f(x) - \frac{1}{2}[f(a) + f(b)] \right|^p \leq \frac{|b-a|^p}{2(b-a)} \int_a^b |f'(t)| dt = \frac{(b-a)^{p-1}}{2} \int_a^b |f'(t)| dt.$$

Example 3. If we take $\phi(x) = \tan[(\pi/4)x]$, which satisfies the required assumptions, then we have

$$\tan \left[\frac{\pi}{4} \left\{ f(x) - \frac{1}{2}[f(a) + f(b)] \right\} \right] \leq \frac{\tan[(\pi/4)(b-a)]}{2(b-a)} \int_a^b |f'(t)| dt.$$

Example 4. If we take $\phi(x) = \exp(x) - 1$, which satisfies the required assumptions, then we have

$$\exp \left\{ f(x) - \frac{1}{2}[f(a) + f(b)] \right\} - 1 \leq \frac{\exp(b-a) - 1}{2(b-a)} \int_a^b |f'(t)| dt.$$

Many more examples could be given, provided the assumptions of the theorem are met.

2.2. Second result

Our main result, which generalizes [7, First part of Theorem 1], as recalled in Theorem 1, is given below.

Theorem 3. Let $a, b \in \mathbb{R}$ with $b > a$, $f : [a, b] \mapsto \mathbb{R}$ be a differentiable function on $[a, b]$ such that, for any $t \in [a, b]$, $|f'(t)| \leq 1$, and $\phi : \mathbb{R} \mapsto [0, +\infty)$ be a non-decreasing convex function with $\phi(0) = 0$. Then we have

$$\phi \left\{ \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{2}[f(a) + f(b)] \right\} \leq \frac{\phi(b-a)}{2(b-a)} \int_a^b |f'(t)| dt.$$

Proof. Since $\int_a^b dx = b - a$, we can write

$$\phi \left\{ \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{2}[f(a) + f(b)] \right\} = \phi \left[\frac{1}{b-a} \int_a^b h(x) dx \right], \quad (7)$$

where

$$h(x) = f(x) - \frac{1}{2}[f(a) + f(b)].$$

Using the Jensen integral inequality based on the convexity of ϕ , we obtain

$$\phi \left[\frac{1}{b-a} \int_a^b h(x) dx \right] \leq \frac{1}{b-a} \int_a^b \phi[h(x)] dx = \frac{1}{b-a} \int_a^b \phi \left\{ f(x) - \frac{1}{2}[f(a) + f(b)] \right\} dx. \quad (8)$$

Applying Theorem 2, we directly get

$$\phi \left\{ f(x) - \frac{1}{2}[f(a) + f(b)] \right\} \leq \frac{\phi(b-a)}{2(b-a)} \int_a^b |f'(t)| dt. \quad (9)$$

This upper bound does not depend on x . It follows from Equations (7), (8) and (9) that

$$\begin{aligned} \phi \left\{ \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{2}[f(a) + f(b)] \right\} &\leq \frac{1}{b-a} \int_a^b \left[\frac{\phi(b-a)}{2(b-a)} \int_a^b |f'(t)| dt \right] dx \\ &= \frac{\phi(b-a)}{2(b-a)} \int_a^b |f'(t)| dt \left(\int_a^b \frac{1}{b-a} dx \right) \\ &= \frac{\phi(b-a)}{2(b-a)} \int_a^b |f'(t)| dt. \end{aligned}$$

This completes the proof. $\square$

The assumption that, for any $t \in [a, b]$, $|f'(t)| \leq 1$, can be replaced with the statement that there exists $M > 0$ such that, for any $t \in [a, b]$, $|f'(t)| \leq M$. It is sufficient to replace f by f/M and $\phi(x)$ by $\phi(Mx)$, and we get

$$\phi \left\{ \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{2}[f(a) + f(b)] \right\} \leq \frac{\phi[M(b-a)]}{2M(b-a)} \int_a^b |f'(t)| dt.$$

Moreover, Applying Theorem 3 with f^2 instead of f yields

$$\phi \left\{ \frac{1}{b-a} \int_a^b f^2(x) dx - \frac{1}{2}[f^2(a) + f^2(b)] \right\} \leq \frac{\phi(b-a)}{b-a} \int_a^b |f'(t)| |f(t)| dt,$$

which generalizes [7, Second part of Theorem 1] by the presence of ϕ .

Some examples of applications of Theorem 3 are described below.

Example 5. If we take $\phi(x) = |x|$, which satisfies the required assumptions, then we have

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{2}[f(a) + f(b)] \right| \leq \frac{1}{2} \int_a^b |f'(t)| dt.$$

This corresponds to the inequality in Theorem 1.

Example 6. If we take $\phi(x) = |x|^p$ with $p > 1$, which satisfies the required assumptions, then we have

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{2}[f(a) + f(b)] \right|^p \leq \frac{(b-a)^{p-1}}{2} \int_a^b |f'(t)| dt.$$

Example 7. If we take $\phi(x) = \tan[(\pi/4)x]$, which satisfies the required assumptions, then we have

$$\tan \left[\frac{\pi}{4} \left\{ \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{2}[f(a) + f(b)] \right\} \right] \leq \frac{\tan[(\pi/4)(b-a)]}{2(b-a)} \int_a^b |f'(t)| dt.$$

Example 8. If we take $\phi(x) = \exp(x) - 1$, which satisfies the required assumptions, then we have

$$\exp \left\{ \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{2}[f(a) + f(b)] \right\} - 1 \leq \frac{\exp(b-a) - 1}{2(b-a)} \int_a^b |f'(t)| dt.$$

Similar examples could be presented, provided the assumptions of the theorem are met.

3. Conclusion

Integral inequalities play a key role in the analysis of integral functions across a variety of scientific disciplines. By extending the classical result established by Pachpatte in 2002 through a convexity-based framework, we have provided a flexible, unifying approach covering a wide range of bounds. Future work could involve exploring further generalizations involving higher-order derivatives, alternative convex functions, or multi-dimensional extensions in order to better accommodate complex integral operators. Furthermore, applying these generalized inequalities to practical problems in numerical analysis, optimization, and probabilistic modelling offers promising avenues for ongoing research.

References

- [1] Hardy, G. H., Littlewood, J. E., & Pólya, G. (1934). *Inequalities*. Cambridge University Press, Cambridge. MR0046395 (13,727 e), Zbl, 10.
- [2] Bellman, R., & Beckenbach, E. F. (1961). *Inequalities*. Berlin, Germany: Springer-Verlag.
- [3] Walter, W. (2012). *Differential and Integral Inequalities* (Vol. 55). Springer Science & Business Media.
- [4] Bainov, D. D., & Simeonov, P. S. (2013). *Integral Inequalities and Applications* (Vol. 57). Springer Science & Business Media.
- [5] Yang, B. C. (2009). *Hilbert-Type Integral Inequalities*, Bentham Science Publishers, The United Arab Emirates.
- [6] Cvetkovski, Z. (2012). *Inequalities: Theorems, Techniques and Selected Problems*, Springer, Berlin Heidelberg, 2012.
- [7] Pachpatte, B. G. (2002). A note on Ostrowski type inequalities. *Demonstratio Mathematica*, 35(1), 27–30.
- [8] Hadamard, J. (1893). Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. *Journal De Mathématiques Pures Et Appliquées*, 9, 171-215.
- [9] Hermite, C. (1883). Sur deux limites d'une intégrale définie. *Mathesis*, 3(82), 6.
- [10] Jensen, J. L. W. V. (1905). Om konvekse funktioner og uligheder imellem middelværdier. *Nyt Tidsskrift for Matematik*, 16, 49-68.
- [11] Jensen, J. L. W. V. (1906). Sur les fonctions convexes et les inégalités entre les valeurs moyennes. *Acta Mathematica*, 30(1), 175-193.
- [12] Beckenbach, E. F. (1948). Convex functions. *Bulletin of the American Mathematical Society*, 54(5), 439-460.
- [13] Bellman, R. (1961). On the approximation of curves by line segments using dynamic programming. *Communications of the ACM*, 4(6), 284.
- [14] Mitrinovic, D. S. (1970). *Analytic Inequalities*. Springer-Verlag. Berlin and New York.
- [15] Mitrinovic, D. S., Pecaric, J., & Fink, A. M. (2013). *Classical and New Inequalities in Analysis* (Vol. 61). Springer Science & Business Media.
- [16] Roberts, A. W., & Varberg, P. E. (1973). *Convex Functions*, Academic Press.
- [17] Niculescu, C. P. (2000). Convexity according to the geometric mean. *Mathematical Inequalities and Applications*, 3(2), 155-167.
- [18] Hardy, G. H., Littlewood, J. E., & Polya, G. (1952). *Inequalities*. 2nd ed. Cambridge University Press.



© 2025 by the authors; licensee PSRP, Lahore, Pakistan. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC-BY) license (<http://creativecommons.org/licenses/by/4.0/>).