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On Abel summability and Littlewood Paley functions for Hermite polynomials expansions

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Abstract: We introduce Littlewood Paley functions defined in terms of a reparameterization of the Ornstein-Uhlenbeck semigroup obtaining that these operators are bounded in L^p, $1 < p < \infty$, with respect to the unidimensional gaussian measure, by means of singular integrals theory. In addition, we study the Abel summability of the Fourier Hermite expansions considering their pointwise convergence and their convergence in the L^p sense, obtaining a version of Tauber's theorem.

Keywords: Hermite expansions, Gaussian measure, singular integrals, Littlewood Paley functions, Tauber's theorem

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1. Preliminaries

In this section, we begin by introducing some definitions and notations necessary for our subsequent development. Let us consider the unidimensional Gaussian measure $\gamma(x) = \frac{e^{-x^2}}{\sqrt{\pi}}$, with $x \in \mathbb{R}$ and as usual $\|f\|_{p,\gamma}$ denotes the norm $\left(\int_{\mathbb{R}} |f(x)|^p \gamma(dx)\right)^{1/p}$ of an element $f \in L^p(\gamma)$, for $1 \leq p < \infty$. By A_p, $C_{p,x}$, etc. we mean constants, not necessarily always the same, depending exclusively on the parameters shown as subscripts.

The normalized Hermite polynomials of order $k \in \mathbb{N} \cup \{0\}$ is defined by

$$h_k(x) = \frac{1}{(2^k k!)^{1/2}} (-1)^k e^{x^2} \frac{d^k}{dx^k}(e^{-x^2}). \tag{1}$$

Given a function $f \in L^1(\gamma)$ its k-Fourier-Hermite coefficient and the Fourier Hermite expansion are defined respectively by

$$c_k^f = \int_{\mathbb{R}} f(x) h_k(x) \gamma(dx) \text{ and } \sum_{k=0}^{\infty} c_k^f h_k(x).$$

It is well known that Hermite polynomials satisfy the following identity known as Mehler′s formula, (see [1]),

$$\sum_{k=0}^{\infty} h_k(x) h_k(y) r^n = \frac{e^{-\frac{r^2(x^2+y^2)-2rxy}{1-r^2}}}{\sqrt{1-r^2}}, \ 0 \leq r < 1.$$

Moreover, the following estimate is true for each $x \in \mathbb{R}$, (see [1, (8.22.8)] and [2]),

$$|h_k(x)| \leq e^{x^2} \frac{\sqrt{k!}}{2^{k/2}\Gamma(\frac{k}{2}+1)} \tag{2}$$

and with respect to the norm $L^p(\gamma)$, $1 < p < \infty$, asymptotic estimates are obtained (see [3, Theorem 2.1])

$$\|h_n\|_{p,\gamma} = \begin{cases} \frac{c_1(p)}{n^{1/4}} \left(1 + O\left(\frac{1}{n}\right)\right) & \text{if } 0 < p < 2, \\ \frac{c_2(p)}{n^{1/4}} (p-1)^{n/2} \left(1 + O\left(\frac{1}{n}\right)\right) & \text{if } 2 < p < \infty, \end{cases} \tag{3}$$

where

$$c_1(p) = \left(\frac{2}{\pi}\right)^{1/4} \mu_p \left(\frac{2}{2-p}\right)^{1/2p},$$
$$c_2(p) = \left(\frac{2}{\pi}\right)^{1/4} \left(\frac{p-1}{2(p-2)}\right)^{1/2p},$$
$$\mu_p = \left(\int_0^1 |\sin(\pi x)|^p dx\right)^{1/p},$$

and recall that with this normalization, from (1), then $\|h_n\|_{2,\gamma} = 1$. Also, the normalized Hermite polynomials constitute an orthonormal system in $L^2(\gamma)$.

For $n \in \mathbb{N} \cup \{0\}$ the partials sums of the Fourier Hermite expansions $S_n f(x)$ and the arithmetic mean $C_n f(x)$ are defined as,

$$S_n f(x) = \sum_{k=0}^{n} c_k^f h_k(x),$$
$$C_n f(x) = \frac{1}{n+1} \sum_{k=0}^{n} S_k f(x) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) c_k^f h_k(x),$$

and it has been proved in [4,5] that $\lim_{n\to\infty} \|S_n f - f\|_{p,\gamma} = 0$ if and only if $p = 2$.

Following [6] and [2] let us consider the Abel means associated with the Fourier Hermite expansions defined by

$$A_r f(x) = \sum_{k=0}^{\infty} r^k c_k^f h_k(x), \quad 0 \le r < 1,$$

thus, Mehler′s formula allows us to obtain the following integral representation

$$A_r f(x) = \int_{\mathbb{R}} \frac{e^{-\frac{r^2(x^2+y^2)-2rxy}{1-r^2}}}{\sqrt{1-r^2}} f(y)\gamma(dy) = \int_{\mathbb{R}} \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{\sqrt{\pi(1-r^2)}} f(y)dy.$$

$A_r f(x)$ exists for every $f \in L^1(\gamma)$ whether or not it has Hermite expansion. Note that by means of the change of parameter $r = e^{-t}$ then $A_r = T_{-log(r)}$, where $\{T_t\}_{t=0}^{\infty}$ is the Ornstein-Uhlenbeck semigroup. Furthermore, $\{A_r\}_{r=0}^{1}$, is a family of strongly continuous linear operators and the maximal function $A^* f(x) = \sup_{0\le r<1} |A_r f(x)| \in L^p(\gamma)$, that is to say,

$$\|A^* f\|_{p,\gamma} \le C_p \|f\|_{p,\gamma},\ 1 < p \le \infty. \tag{4}$$

Also, $\lim_{r\to 1^-} \|A_r f - f\|_{p,\gamma} = 0$, as well as, $\lim_{r\to 1^-} A_r f(x) = f(x)$ almost everywhere, for $1 \le p \le \infty$ (see [2, Theorem 2]) .

On the other hand, if $f \in L^2(\gamma)$ then $\sum_{k=0}^{\infty} r^k c_k^f h_k(x)$ converges absolutely to $A_r(f)(x)$ almost everywhere, but for every $1 \le p < 2$, there exist a function $f \in L^p(\gamma)$ and $r < 1$, such that, $\sum_{k=0}^{\infty} r^k c_k^f h_k(x)$ diverges for every x. (see [2, Lemma 2]).

Now following [7] given a function $f \in L^1(\gamma)$, $m \ge 0$, we consider the integrals operators

$$Q_{r,m} f(x) = \int_{\mathbb{R}} \frac{|y-rx|^m}{(1-r^2)^{(m+1)/2}} e^{-\frac{|y-rx|^2}{1-r^2}} f(y)dy, \tag{5}$$
$$Q_m^* f(x) = \sup_{0\le r<1} |Q_{r,m} f(x)|,$$

and for $0 \le r < 1$, $1 < p < \infty$, it was obtained that these operators are $L^p(\gamma)$ continuous, (see [7, Theorem 3]), therefore

$$\|Q_{r,m} f\|_{p,\gamma} \le C_p \|f\|_{p,\gamma}, \text{ and } \|Q_m^* f\|_{p,\gamma} \le C_p \|f\|_{p,\gamma}. \tag{6}$$

Also, let us consider the operators defined as

$$L_m f(x) = \int_{|x-y|>1\wedge\frac{1}{|x|}} \int_0^1 \varphi(r) \frac{|y-rx|^m}{(1-r^2)^{(m+3)/2}} e^{-\frac{|y-rx|^2}{1-r^2}} dr f(y) dy, \tag{7}$$

where φ is a bounded function on $[0,1]$. Therefore,

$$\|L_m f\|_{p,\gamma} \leq C_p \|f\|_{p,\gamma},\ 1 < p < \infty, \tag{8}$$

(see [7, Theorem 6]).

Finally, for $f \in L^p(\gamma)$ let k be a Calderón-Zygmund kernel. We consider the integral operator K defined as

$$Kf(x) = \int_{|x-y|<1\wedge\frac{1}{|x|}} k(x-y) f(y) dy, \tag{9}$$

and we have that K is an $L^p(\gamma)$-continuous operator (see [7, Theorem 5]),

$$\|Kf\|_{p,\gamma} \leq C_p \|f\|_{p,\gamma},\ 1 < p < \infty. \tag{10}$$

In particular, these operators $Q_{r,m}$, L_m and K, as well as their $L^p(\gamma)$-continuity will be key in the development of our argument.

2. The results

2.1. Littlewood Paley function

Inspired by the classic case, (see [6, chapter XIV]), we define a Littlewood Paley g function as

$$gf(x) = \left(\int_0^1 (1-r) |\partial_r A_r f(x)|^2 dr \right)^{1/2},$$

and we obtain the $L^p(\gamma)$ continuity of the g function, for $1 < p < \infty$. For this purpose, we need the following technical result (see [8, Lemma 3, chapter V] for a similar version). Formally,

Lemma 1. *Let us denote* $u(s) = \frac{|y-\sqrt{1-s}x|^2}{s}$ *where* $s \in (0,1)$. *Then, for all* $m \in \mathbb{R}$, *if* $|x-y| < 1 \wedge \frac{1}{|x|}$, *there exist a constant* $C > 0$, *such that,*

$$\int_0^1 \frac{u^m(s)}{s^{3/2}} e^{-u(s)} \frac{ds}{\sqrt{1-s}} \leq \frac{C}{|x-y|}. \tag{11}$$

Similarly,

$$\int_0^1 \frac{u^{1/2}(s)}{s} e^{-u(s)} \frac{ds}{\sqrt{1-s}} \leq \frac{C}{|x-y|}. \tag{12}$$

Proof. If $|x-y| < 1 \wedge \frac{1}{|x|}$, then $|x-y||x| < 1$ and therefore,

$$\begin{aligned} |y-\sqrt{1-s}x|^2 = |(y-x)+(1-\sqrt{1-s})x|^2 &\geq ||y-x|-(1-\sqrt{1-s})|x||^2 \\ &\geq |y-x|^2 - 2|x-y||x|(1-\sqrt{1-s}) \\ &= |y-x|^2 - 2|x-y||x| \frac{s}{(1+\sqrt{1-s})} \\ &\geq |y-x|^2 - 2s. \end{aligned}$$

Then, we consider $\delta \in (0,1)$ and since $u^m(s) e^{-(1-\delta)u(s)}$ is uniformly bounded in $(0,1)$, for all m, we just need to estimate the integral

$$\int_0^1 \frac{e^{-\delta u(s)}}{s^{3/2}} \frac{ds}{\sqrt{1-s}}.$$

Now, let us denote by $a = a(x,y) = |x|^2 + |y|^2$ and $b = b(x,y) = 2xy$, so $u(s) \geq \frac{|x-y|^2}{s} - 2 = \frac{a-b}{s} - 2$ and $e^{-\delta u(s)} \leq e^{-2\delta} e^{-\delta \frac{(a-b)}{s}}$. Considering the changes of variables, first, $w = \frac{a-b}{s}$ and then, $\theta = w - (a-b)$ we can express,

$$\begin{aligned} \int_0^1 \frac{e^{-\delta u(s)}}{s^{3/2}} \frac{ds}{\sqrt{1-s}} &\leq \frac{e^{-2\delta}}{\sqrt{a-b}} \int_{a-b}^{\infty} \frac{e^{-\delta w} dw}{\sqrt{w-(a-b)}} \\ &\leq \frac{e^{-2\delta} e^{-\delta(a-b)}}{\sqrt{a-b}} \int_0^{\infty} e^{-\delta\theta} \theta^{-1/2} d\theta \\ &\leq \frac{e^{-2\delta}\Gamma(1/2)}{\sqrt{\delta}\sqrt{a-b}}. \end{aligned}$$

Then, (11) follows from recalling that $a - b = |x-y|^2$. Finally, by an absolutely similar argument we obtain (12). □

Theorem 1. *If $1 < p < \infty$ and $f \in L^p(\gamma)$, then there exist a constant $C_p > 0$ such that*

$$\|gf\|_{p,\gamma} \leq C_p \|f\|_{p,\gamma}.$$

Proof. We first prove the theorem in case $p = 2$. Let $f \in L^2(\gamma)$, thus if f has Fourier Hermite expansion $\sum_{k=0}^{\infty} c_k^f h_k(x)$, then $A_r f(x)$ and $\partial_r A_r f(x)$ have expansions

$$\sum_{k=0}^{\infty} r^k c_k^f h_k(x) \text{ and } \sum_{k=1}^{\infty} k r^{k-1} c_k^f h_k(x),$$

respectively. So, by orthonormality of Hermite polynomials

$$\int_{\mathbb{R}} |\partial_r A_r f(x)|^2 \gamma(dx) = \sum_{k=1}^{\infty} k^2 r^{2k-2} (c_k^f)^2,$$

and therefore, Tonelli′s theorem allows us to obtain that

$$\begin{aligned} \|gf\|_{2,\gamma}^2 &= \int_{\mathbb{R}} \int_0^1 (1-r) |\partial_r A_r f(x)|^2 dr \gamma(dx) \\ &= \sum_{k=1}^{\infty} k^2 (c_k^f)^2 \left(\int_0^1 (1-r) r^{2k-2} dr \right) \\ &= \sum_{k=1}^{\infty} \frac{k^2 (c_k^f)^2}{2k(2k-1)}. \end{aligned}$$

Consequently, $\|gf\|_{2,\gamma} \leq \frac{1}{\sqrt{2}} \|f\|_{2,\gamma}$.

Now, let us considering $1 < p \leq 2$ and $f \in L^p(\gamma)$. Suppose that the inequality is true for some p and verify it for $1 < k < p$. First, we observe that if $f \in L^p(\gamma)$ then $f \in L^k(\gamma)$. For each $0 < r < 1$, let us denote by $u(r,x) = A_r f(x)$ and $h(r,x) = u^{k/p}(r,x)$, thus

$$\|h(r,.)\|_{p,\gamma}^p = \|u(r,.)\|_{k,\gamma}^k \leq C_k \|f\|_{k,\gamma}^k. \tag{13}$$

Therefore,

$$\begin{aligned} g^2 f(x) &= \int_0^1 (1-r) |\partial_r u(r,x)|^2 dr \\ &= \int_0^1 (1-r) | \left(\frac{p}{k}\right) h^{(p/k)-1}(r,x) \partial_r h(r,x)|^2 dr \end{aligned}$$

$$\leq \left(\frac{p}{k}\right)^2 (A^*f)(x)^{2(p-k)/p} \left(\int_0^1 (1-r)|\partial_r h(r,x)|^2 dr\right),$$

since

$$h^{2(p-k)/k}(r,x) = u^{2(p-k)/p}(r,x) \leq (A^*f(x))^{2(p-k)/p},$$

which implies that,

$$gf(x) \leq \left(\frac{p}{k}\right)(A^*f(x))^{(p-k)/p} gh(x).$$

Then,

$$\int_{\mathbb{R}} g^k f(x)\gamma(dx) \leq \left(\frac{p}{k}\right)^k \int_{\mathbb{R}} (A^*f(x))^{k(p-k)/p} g^k h(x)\gamma(dx),$$

and Hölder inequality, with exponents $p/(p-k)$ and p/k, allows us to obtain

$$\|gf\|_{k,\gamma}^k \leq \left(\frac{p}{k}\right)^k \|A^*f\|_{k,\gamma}^{k(p-k)/p} \|gh\|_{p,\gamma}^k.$$

But by hypothesis and from (13) we get that $\|gh\|_{p,\gamma}^k \leq C_{p,k}\|f\|_{p,\gamma}^{k^2/p}$. Thus, (4) allows us to conclude that,

$$\|gf\|_{k,\gamma} \leq C_k \|f\|_{k,\gamma}, \text{ if } 1 < k < p,$$

and therefore, $\|gf\|_{p,\gamma} \leq C_p\|f\|_{p,\gamma}$ for all $1 < p \leq 2$.

Now, we consider $p \geq 4$ and let q be such that $\frac{1}{q} + \frac{2}{p} = 1$. So, $q = p/(p-2)$ and $1 \leq q \leq 2$. Let ϕ be a testing function. We assume $\phi \geq 0$, with support on $|x| \leq 1$, such that, $\|\phi\|_{2,\gamma} = 1$. Then,

$$\|gf\|_{p,\gamma}^2 = \|g^2 f\|_{p/2,\gamma} = \sup_{\{\|\phi\|_{q,\gamma} \leq 1\}} \int_{\mathbb{R}} g^2 f(x)\phi(x)\gamma(dx).$$

But,

$$\begin{aligned}\int_{\mathbb{R}} g^2 f(x)\phi(x)\gamma(dx) &= \int_{\mathbb{R}}\int_0^1 (1-r)(\partial_r A_r f)^2(x) dr\phi(x)\gamma(dx)\\ &= \int_{\mathbb{R}}\int_0^1 (1-r)\left(\int_{\mathbb{R}} \partial_r \left(\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{\sqrt{\pi(1-r^2)}}\right) f(y)dy\right)^2 dr\phi(x)\gamma(dx),\end{aligned}$$

and since

$$\partial_r\left(\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{\sqrt{1-r^2}}\right) = \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{\sqrt{1-r^2}}\left(\frac{2x|y-rx|}{1-r^2} - \frac{2r|y-rx|^2}{(1-r^2)^2} + \frac{r}{1-r^2}\right),$$

noting that

$$\int_{\mathbb{R}} \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{\sqrt{\pi(1-r^2)}} dy = 1,$$

then by Jensen′s integral inequality we obtain that

$$\int_{\mathbb{R}}\int_0^1 (1-r)(\partial_r A_r f)^2(x) dr\phi(x)\gamma(dx) \leq \int_{\mathbb{R}}\int_0^1 (1-r)\int_{\mathbb{R}} \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{\sqrt{\pi(1-r^2)}}\left(\frac{2x|y-rx|}{1-r^2} - \frac{2r|y-rx|^2}{(1-r^2)^2} + \frac{r}{1-r^2}\right)^2 f^2(y) dy dr\phi(x)\gamma(dx) = I.$$

Now,

$$\left(\frac{2x|y-rx|}{1-r^2}-\frac{2r|y-rx|^2}{(1-r^2)^2}+\frac{r}{1-r^2}\right)^2$$
$$=\frac{4x^2|y-rx|^2}{(1-r^2)^2}-\frac{8xr|y-rx|^3}{(1-r^2)^3}+\frac{4r^2|y-rx|^4}{(1-r^2)^4}+\frac{4xr|y-rx|}{(1-r^2)^2}-\frac{4r^2|y-rx|^2}{(1-r^2)^3}+\frac{r^2}{(1-r^2)^2},$$

and since $1-r\leq 1-r^2$ if $r\in[0,1)$, then we obtain explicity

$$\begin{aligned} I &\leq \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}\int_0^1\int_{\mathbb{R}} e^{-\frac{|y-rx|^2}{1-r^2}}\left(\frac{4x^2|y-rx|^2}{(1-r^2)^{3/2}}+\frac{8xr|y-rx|^3}{(1-r^2)^{5/2}}+\frac{4r^2|y-rx|^4}{(1-r^2)^{7/2}}\right.\\ &\quad\left.+\frac{4xr|y-rx|}{(1-r^2)^{3/2}}+\frac{4r^2|y-rx|^2}{(1-r^2)^{5/2}}+\frac{r^2}{(1-r^2)^{3/2}}\right)f^2(y)dydr\phi(x)\gamma(dx)\\ &=\sum_{j=1}^{6}\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}\int_0^1\int_{\mathbb{R}} e^{-\frac{|y-rx|^2}{1-r^2}}\Delta_j(r,x,y)f^2(y)dydr\phi(x)\gamma(dx), \end{aligned}$$

where $\Delta_j(r,x,y)$ represents each fraction of the sum. Now, if we denote by

$$W_j=\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}\int_0^1\int_{\mathbb{R}} e^{-\frac{|y-rx|^2}{1-r^2}}\Delta_j(r,x,y)f^2(y)dydr\phi(x)\gamma(dx),$$

then we have to estimate each integral W_j, for each $j=1,..,6$.

To estimate W_1 note that from Eq. (5), we can express

$$\int_{\mathbb{R}} e^{-\frac{|y-rx|^2}{1-r^2}}\frac{|y-rx|^2}{(1-r^2)^{3/2}}f^2(y)dy=Q_{r,2}(f^2)(x).$$

Thus, applying Hölder′s inequality and (6) we get

$$\begin{aligned} W_1\leq\frac{4}{\sqrt{\pi}}\int_{\mathbb{R}}x^2Q_2^*(f^2)(x)\phi(x)\gamma(dx) &\leq \frac{4}{\sqrt{\pi}}\|Q_2^*(f^2)\|_{p/2,\gamma}\left(\int_{\mathbb{R}}|x|^{2q}|\phi(x)|^q\gamma(dx)\right)^{1/q}\\ &\leq C_p\|f^2\|_{p/2,\gamma}\left(\int_{|x|\leq 1}|\phi(x)|^q\gamma(dx)\right)^{1/q}\\ &\leq C_p\|f\|_{p,\gamma}^2. \end{aligned}$$

Now we estimate W_2. First, we express

$$\mathbb{R}=\left\{y:|x-y|\leq 1\wedge\frac{1}{|x|}\right\}\cup\left\{y:|x-y|>1\wedge\frac{1}{|x|}\right\}=R_1\cup R_2,$$

then using (7) and Tonelli′s theorem we write

$$\begin{aligned} W_2 &\leq \int_{\mathbb{R}}\int_{R_1}\int_0^1\frac{8|x||y-rx|^3}{\sqrt{\pi}(1-r^2)^3}e^{-\frac{|y-rx|^2}{1-r^2}}f^2(y)dydr\phi(x)\gamma(dx)+\int_{\mathbb{R}}|x|L_3(f^2)(x)\phi(x)\gamma(dx)\\ &= T_1+T_2, \end{aligned}$$

where $\varphi(r)=8r(1-r^2)^{1/2}$. Again, by means of Hölder′s inequality and (8) we obtain

$$\begin{aligned} T_2=\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}|x|L_3(f^2)(x)\phi(x)\gamma(dx) &\leq \frac{1}{\sqrt{\pi}}\|L_3(f^2)\|_{p/2,\gamma}\left(\int_{\mathbb{R}}|x|^q|\phi(x)|^q\gamma(dx)\right)^{1/q}\\ &\leq C_p\|f^2\|_{p/2,\gamma}\left(\int_{|x|\leq 1}|\phi(x)|^q\gamma(dx)\right)^{1/q}\\ &\leq C_p\|f\|_{p,\gamma}^2. \end{aligned}$$

On the other hand, if $y \in R_1$, then under the change of variable $s = 1 - r^2$, we have that (11) of the Lemma 1, allows us to conclude that

$$\begin{aligned}\int_0^1 \frac{|y-rx|^3}{(1-r^2)^3} e^{-\frac{|y-rx|^2}{1-r^2}} dr &= \int_0^1 \frac{|y-\sqrt{1-s}x|^3}{s^3} e^{-\frac{|y-\sqrt{1-s}x|^2}{s}} \frac{ds}{\sqrt{1-s}} \\ &= \int_0^1 \frac{u^{3/2}(s)}{s^{3/2}} e^{-u(s)} \frac{ds}{\sqrt{1-s}} \\ &\leq \frac{C}{|x-y|}.\end{aligned}$$

By using (9), Hölder's inequality and (10), where $k(x-y) = |x-y|^{-1}$ we obtain that

$$\begin{aligned}T_1 \leq C \int_{\mathbb{R}} |x| K(f^2)(x)\phi(x)\gamma(dx) &\leq \|K(f^2)\|_{p/2,\gamma} \left(\int_{\mathbb{R}} |x|^q |\phi(x)|^q \gamma(dx) \right)^{1/q} \\ &\leq C_p \|f\|^2_{p,\gamma}\end{aligned}$$

and in conclusion, $W_2 \leq C_p \|f\|^2_{p,\gamma}$.

The estimation of W_3 follows a similar argument to the previous case. Thus, again considering $\mathbb{R} = R_1 \cup R_2$, we express

$$\begin{aligned}W_3 &\leq \int_{\mathbb{R}} \int_{R_1} \int_0^1 \frac{4r^2 |y-rx|^4}{\sqrt{\pi}(1-r^2)^{7/2}} e^{-\frac{|y-rx|^2}{1-r^2}} f^2(y)dydr\phi(x)\gamma(dx) + \int_{\mathbb{R}} L_4(f^2)(x)\phi(x)\gamma(dx) \\ &= T_1 + T_2,\end{aligned}$$

where $\varphi(r) = 4r^2$ is a bounded function. Again, if $y \in R_1$ we apply the lemma 1 by observing that

$$\begin{aligned}\int_0^1 \frac{|y-rx|^4}{(1-r^2)^{7/2}} e^{-\frac{|y-rx|^2}{1-r^2}} dr &= \int_0^1 \frac{|y-\sqrt{1-s}x|^4}{s^{7/2}} e^{-\frac{|y-\sqrt{1-s}x|^2}{s}} \frac{ds}{\sqrt{1-s}} \\ &= \int_0^1 \frac{u^2(s)}{s^{3/2}} e^{-u(s)} \frac{ds}{\sqrt{1-s}} \\ &\leq \frac{C}{|x-y|}.\end{aligned}$$

This way, Hölder's inequality allows us to obtain

$$T_1 \leq C \int_{\mathbb{R}} K(f^2)(x)\phi(x)\gamma(dx) \leq C\|K(f^2)\|_{p/2,\gamma} \|\phi\|_{q,\gamma} \leq C_p \|f\|^2_{p,\gamma},$$

and

$$T_2 = \int_{\mathbb{R}} L_4(f^2)(x)\phi(x)\gamma(dx) \leq C\|L_4(f^2)\|_{p/2,\gamma} \|\phi\|_{q,\gamma} \leq C_p \|f\|^2_{p,\gamma},$$

therefore, $W_3 \leq C_p \|f\|^2_{p,\gamma}$.

Again, the estimation of W_4 follows a similar argument developed for W_2 and W_3. But in this case, if $y \in R_2$, we consider the operator

$$L_1(f^2)(x) = \int_{R_2} \int_0^1 \varphi(r) \frac{|y-rx|}{(1-r^2)^2} e^{-\frac{|y-rx|^2}{1-r^2}} dr f^2(y)dy,$$

where $\varphi(r) = 4r(1-r^2)^{1/2}$. If $y \in R_1$, then by using (12) of the lemma 1, we note that

$$\begin{aligned}\int_0^1 \frac{|y-rx|}{(1-r^2)^{3/2}} e^{-\frac{|y-rx|^2}{1-r^2}} dr &= \int_0^1 \frac{|y-\sqrt{1-s}x|}{s^{3/2}} e^{-\frac{|y-\sqrt{1-s}x|^2}{s}} \frac{ds}{\sqrt{1-s}} \\ &= \int_0^1 \frac{u^{1/2}(s)}{s} e^{-u(s)} \frac{ds}{\sqrt{1-s}} \leq \frac{C}{|x-y|},\end{aligned}$$

and the rest of the argument follows similarly to the previous cases, obtaining that $W_4 \le C_p\|f\|^2_{p,\gamma}$.

Now, the estimate of W_5 is as follows. Once more, we express $\mathbb{R} = R_1 \cup R_2$. If $y \in R_2$ then we consider

$$L_2(f^2)(x) = \int_{R_2}\int_0^1 \varphi(r)\frac{|y-rx|^2}{(1-r^2)^{5/2}}e^{-\frac{|y-rx|^2}{1-r^2}}dr f^2(y)dy,$$

where $\varphi(r) = 4r^2$. But if $y \in R_1$, then from (11) we get

$$\begin{aligned}\int_0^1 \frac{|y-rx|^2}{(1-r^2)^{5/2}}e^{-\frac{|y-rx|^2}{1-r^2}}dr &= \int_0^1 \frac{|y-\sqrt{1-s}x|^2}{s^{5/2}}e^{-\frac{|y-\sqrt{1-s}x|^2}{s}}\frac{ds}{\sqrt{1-s}}\\ &= \int_0^1 \frac{u(s)}{s^{3/2}}e^{-u(s)}\frac{ds}{\sqrt{1-s}}\\ &\le \frac{C}{|x-y|},\end{aligned}$$

and similarly we conclude that $W_5 \le C_p\|f\|^2_{p,\gamma}$.

Finally, we need to estimate W_6. To do this, if $y \in R_2$ let us consider the operator

$$L_0(f^2)(x) = \int_{R_2}\int_0^1 \varphi(r)\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{3/2}}dr f^2(y)dy,$$

where $\varphi(r) = r^2$ and if $y \in R_1$ let us consider, from (11) the estimation

$$\begin{aligned}\int_0^1 \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{3/2}}dr &= \int_0^1 \frac{|y-\sqrt{1-s}x|^0}{s^{3/2}}e^{-\frac{|y-\sqrt{1-s}x|^2}{s}}\frac{ds}{\sqrt{1-s}}\\ &= \int_0^1 \frac{u^0(s)}{s^{3/2}}e^{-u(s)}\frac{ds}{\sqrt{1-s}}\\ &\le \frac{C}{|x-y|},\end{aligned}$$

and thus, $W_6 \le C_p\|f\|^2_{p,\gamma}$.

In summary, we have obtained that

$$\|gf\|^2_{p,\gamma} \le I \le \sum_{j=1}^{6} W_j \le C_p\|f\|^2_{p,\gamma},$$

and therefore

$$\|gf\|_{p,\gamma} \le C_p\|f\|_{p,\gamma}, \quad \text{for} \quad p \ge 4.$$

For a general function in $L^p(\gamma)$, we need only approximate in norm by a sequence of indefinitely differentiable functions with compact support.

Finally, if $2 < p < 4$ the results follows by Marcinkiewicz interpolation theorem and this completes the proof of this theorem. □

2.2. Auxiliary functions related to g

Following [6] we introduce various auxiliary functions related to g-function, defined in the previous section. Thus, the auxiliary functions we introduce are

$$\sigma_1 f(x) = \left(\sum_{k=0}^{\infty} |S_k f(x) - C_k f(x)|^2\right)^{1/2},$$

$$\sigma_2 f(x) = \left(\sum_{k=1}^{\infty} \frac{|S_k f(x) - C_k f(x)|^2}{k}\right)^{1/2},$$

$$\sigma f(x) = \left(\sum_{k=0}^{\infty} |\Delta_k f(x)|^2 \right)^{1/2},$$

where $\Delta_k f(x) = S_{2^k} f(x) - S_{2^{k-1}} f(x)$ and $k = 0, 1, 2, \ldots$.

Then comparisons are made between these functions and the Littlewood Paley operator. Thus, we first have the following result

Theorem 2. *Suppose that $f \in L^2(\gamma)$, then $\sigma_2 f < \infty$ almost everywhere in $\mathbb{R}$.*

Proof. We have that $S_k f(x) - C_k f(x) = \sum_{j=0}^{k} \frac{j}{k+1} c_j^f h_j(x)$. Then by orthonormality of Hermite polynomials,

$$\int_{\mathbb{R}} |S_k f(x) - C_k f(x)|^2 \gamma(dx) = \sum_{j=0}^{k} \frac{j^2 (c_j^f)^2}{(k+1)^2},$$

and taking the sum with respect to k,

$$\int_{\mathbb{R}} \sum_{k=1}^{\infty} \frac{|S_k f(x) - C_k f(x)|^2}{k} \gamma(dx) = \sum_{k=1}^{\infty} \sum_{j=0}^{k} \frac{j^2 (c_j^f)^2}{k(k+1)^2}.$$

But,

$$\sum_{k=1}^{\infty} \sum_{j=0}^{k} \frac{j^2 (c_j^f)^2}{k(k+1)^2} \leq \sum_{j=1}^{\infty} j^2 (c_j^f)^2 \left(\sum_{k=j}^{\infty} \frac{1}{k^3} \right) \leq \frac{1}{2} \sum_{j=1}^{\infty} (c_j^f)^2,$$

therefore,

$$\int_{\mathbb{R}} \sum_{k=1}^{\infty} \frac{|S_k f(x) - C_k f(x)|^2}{k} \gamma(dx) \leq \frac{1}{2} \sum_{j=1}^{\infty} (c_j^f)^2 < \infty,$$

which implies that $\sigma_2 f < \infty$ almost everywhere. □

Corollary 1. *Suppose $f \in L^2(\gamma)$, such that, $c_j^f = O(j^{3/2})$, $\forall j$. Then, $\sigma_1 f < \infty$ almost everywhere in $\mathbb{R}$.*

Proof. Similar to the previous result, we obtain that

$$\begin{aligned} \int_{\mathbb{R}} \sum_{k=0}^{\infty} |S_k f(x) - C_k f(x)|^2 \gamma(dx) &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{j^2 (c_j^f)^2}{(k+1)^2} \\ &\leq \sum_{j=0}^{\infty} j^2 (c_j^f)^2 \left(\sum_{k=j}^{\infty} \frac{1}{k^2} \right) \\ &\leq \frac{1}{2} \sum_{j=1}^{\infty} j (c_j^f)^2 \leq \frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty, \end{aligned}$$

since $c_j^f = O(j^{3/2})$. □

Now we establish the following result similar to the classical case (see [6])

Theorem 3. *For each $x \in \mathbb{R}$, there exist a constant $C > 0$, such that,*

$$gf(x) \leq C \sigma_2 f(x).$$

Proof. We have $(k+1)(S_kf(x)-C_kf(x))=\sum\limits_{j=1}^{k} jc_j^f h_j(x)$ and since

$$\sum_{k=1}^{\infty} kc_k^f r^{k-1}h_k(x)=(1-r)\sum_{k=1}^{\infty}\sum_{j=1}^{k} jc_j^f h_j(x)r^{k-1},$$

then,

$$\partial_r A_r f(x)=(1-r)\sum_{k=1}^{\infty}(k+1)(S_kf(x)-C_kf(x))r^{k-1}.$$

If $r_n=1-\frac{1}{n}$ we can express

$$\begin{aligned} g^2f(x) &= \sum_{n=1}^{\infty}\int_{r_n}^{r_n+1}(1-r)|\partial_r A_r f(x)|^2 dr \\ &\leq \sum_{n=1}^{\infty}\int_{r_n}^{r_n+1}(1-r)\left((1-r_n)\sum_{k=1}^{\infty}(k+1)|S_kf(x)-C_kf(x)|r_{n+1}^{k-1}\right)^2 dr \\ &\leq \sum_{n=1}^{\infty}\frac{(1-r_n)^2}{n^3}\left(\sum_{k=1}^{\infty}(k+1)|S_kf(x)-C_kf(x)|r_{n+1}^{k-1}\right)^2 \\ &\leq \sum_{n=1}^{\infty}\frac{1}{n^5}\left(\sum_{k=1}^{\infty}(k+1)|S_kf(x)-C_kf(x)|r_{n+1}^{k-1}\right)^2. \end{aligned}$$

Now, as for each $n\in\mathbb{N}$ we have that

$$\begin{aligned} &\left(\sum_{k=1}^{\infty}(k+1)|S_kf(x)-C_kf(x)|r_{n+1}^{k-1}\right)^2 \\ &\leq 2\left(\sum_{k=1}^{n}(k+1)|S_kf(x)-C_kf(x)|r_{n+1}^{k-1}\right)^2+2\left(\sum_{k=n+1}^{\infty}(k+1)|S_kf(x)-C_kf(x)|r_{n+1}^{k-1}\right)^2, \end{aligned}$$

then

$$g^2f(x)\leq P(x)+Q(x),$$

where,

$$P(x)=2\sum_{n=1}^{\infty}\frac{1}{n^5}\left(\sum_{k=1}^{n}(k+1)|S_kf(x)-C_kf(x)|r_{n+1}^{k-1}\right)^2,$$

and

$$Q(x)=2\sum_{n=1}^{\infty}\frac{1}{n^5}\left(\sum_{k=n+1}^{\infty}(k+1)|S_kf(x)-C_kf(x)|r_{n+1}^{k-1}\right)^2.$$

Now by using Cauchy-Schwarz inequality

$$\begin{aligned} P(x) &\leq 2\sum_{n=1}^{\infty}\frac{1}{n^5}\left(\sum_{k=1}^{n}|S_kf(x)-C_kf(x)|^2\right)\left(\sum_{k=1}^{n}(k+1)^2\right) \\ &\leq 2C\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sum_{k=1}^{n}|S_kf(x)-C_kf(x)|^2\right) \\ &= 2C\sum_{k=1}^{\infty}|S_kf(x)-C_kf(x)|^2\left(\sum_{n=k}^{\infty}\frac{1}{n^2}\right) \\ &\leq 2C\sum_{k=1}^{\infty}\frac{|S_kf(x)-C_kf(x)|^2}{k}. \end{aligned}$$

On the other hand, again by means of the Cauchy Schwarz inequality we obtain

$$Q(x) \leq 2\sum_{n=1}^{\infty}\frac{1}{n^5}\left(\sum_{k=n+1}^{\infty}\frac{|S_kf(x)-C_kf(x)|^2}{k^2}\right)\left(\sum_{k=n+1}^{\infty}k^2(k+1)^2r_{n+1}^{2k-2}\right).$$

But,

$$\sum_{k=n+1}^{\infty}k^2(k+1)^2r_{n+1}^{2k-2} \leq \sum_{k=0}^{\infty}k^2(k+1)^2r_{n+1}^{2k-2} \leq \frac{C}{(1-r_{n+1})^5},$$

since $0 < r_{n+1} < 1$. Therefore,

$$\begin{aligned} Q(x) &\leq 2C\sum_{n=1}^{\infty}\frac{1}{n^5}\left(\sum_{k=n+1}^{\infty}\frac{|S_kf(x)-C_kf(x)|^2}{k^2}\right)\frac{1}{(1-r_{n+1}^2)^5} \\ &= 2C\sum_{n=1}^{\infty}\left(\frac{n+1}{n}\right)^5\left(\sum_{k=n+1}^{\infty}\frac{|S_kf(x)-C_kf(x)|^2}{k^2}\right) \\ &\leq 2^6C\sum_{n=1}^{\infty}\sum_{k=n+1}^{\infty}\frac{|S_kf(x)-C_kf(x)|^2}{k^2} \\ &= A\sum_{k=1}^{\infty}\sum_{n=1}^{k}\frac{|S_kf(x)-C_kf(x)|^2}{k^2} \\ &= A\sum_{k=1}^{\infty}\frac{|S_kf(x)-C_kf(x)|^2}{k}, \end{aligned}$$

and the result follows. □

Now, we present a version of Tauber's theorem. To do so, the following results about the asymptotic behavior of the Fourier-Hermite coefficients is necessary.

Lemma 2. *Suppose $f \in L^1(\gamma)$. If $c_k^f = O(e^{\alpha k})$ with $\alpha < -1/2$, $\forall k \in \mathbb{N}$, then*

$$\lim_{k\to\infty} k|c_k^f||h_k(x)| = 0,$$

for each $x \in \mathbb{R}$.

Proof. From (2) and the identity $\sqrt{2\pi}\Gamma(2k) = 2^{2k-1}\Gamma(k)\Gamma(k+\frac{1}{2})$ we obtain

$$k|h_k(x)| \leq \frac{e^{x^2}}{2\sqrt{2\pi}}\frac{\Gamma(k+1)(k!)^{1/2}2^{3k/2}}{\Gamma(2k)}.$$

But $\Gamma(k+1) = k!$ and $\Gamma(2k) = (2k-1)!$ thus,

$$k|h_k(x)| \leq \frac{e^{x^2}}{2\sqrt{2\pi}}\frac{(k!)^{3/2}2^{3k/2}}{(2k-1)!}.$$

Then, by means of Stirling's formula, $k! \cong \sqrt{2\pi k}k^ke^{-k}$ and denoting $A_x = \frac{e^{x^2}}{2e\sqrt[4]{2\pi}}$ we have that,

$$\begin{aligned} \lim_{k\to\infty} k|h_k(x)||c_k^f| &\leq \frac{e^{x^2}}{2\sqrt{2\pi}}\lim_{k\to\infty}|c_k^f|\frac{(\sqrt{2\pi k}k^ke^{-k})^{3/2}2^{3k/2}}{\sqrt{2\pi(2k-1)}(2k-1)^{2k-1}e^{-(2k-1)}} \\ &= A_x\lim_{k\to\infty}|c_k^f|k^{5/2}e^{k/2}\left(\frac{k}{2k-1}\right)^{1/2}\left(\frac{2k}{2k-1}\right)^{2k-1}(2k)^{-k/2} \\ &\leq A_x\lim_{k\to\infty}k^{5/2}e^{(\alpha+(1/2))k}\left(\frac{k}{2k-1}\right)^{1/2}\left(\frac{2k}{2k-1}\right)^{2k-1}(2k)^{-k/2} \\ &= 0, \end{aligned}$$

since by hypothesis, we have $|c_k^f| \leq Me^{\alpha k}$ for some constant $M > 0$ and also, $\alpha < -1/2$. $\square$

Then, under the hypotheses of previous Lemma we affirm that

$$\lim_{k\to\infty} \frac{1}{n} \sum_{k=1}^{n} k|c_k^f||h_k(x)| = 0, \text{ for each } x \in \mathbb{R} \text{ and } n \in \mathbb{N}. \tag{14}$$

In fact, it is enough to observe that (14) are arithmetic means $\tilde{C}_n f(x)$ of the partials sums $\tilde{S}_k f(x)$ defined as $\tilde{S}_k f(x) = k|c_k^f||h_k(x)|, k = 1, \cdots, n$.

In an analogous way, we obtain the following asymptotic behavior with respect to the $L^p(\gamma)$-norm of the Fourier Hermite coefficients.

Lemma 3. *Suppose that $f \in L^p(\gamma)$ and let $\sum_{n=1}^{\infty} c_n^f h_n$ be the Fourier-Hermite expansion of f.*

i) *If $1 < p < 2$ and $c_k^f = O(k^\alpha)$ with $\alpha < -3/4$, $\forall k \in \mathbb{N}$, then*

$$\lim_{k\to\infty} k|c_k^f|\|h_k\|_{p,\gamma} = 0.$$

ii) *If $2 < p < \infty$ and $c_k^f = O((p-1)^\beta k^\alpha)$ with $\alpha < -3/4$ and $\beta < -1/2$, $\forall k \in \mathbb{N}$, then*

$$\lim_{k\to\infty} k|c_k^f|\|h_k\|_{p,\gamma} = 0.$$

Proof. i) We consider $1 < p < 2$. Then, using (3) we immediately obtain that $\|h_k\|_{p,\gamma} \leq M_p k^{-1/4}$ for some constant $M_p > 0$ that depends only on p. Therefore,

$$\lim_{k\to\infty} k|c_k^f|\|h_k\|_{p,\gamma} \leq M_p \lim_{k\to\infty} k^{\alpha+(3/4)} = 0.$$

ii) Similarly, if $2 < p < \infty$ from (3) we get again that

$$\lim_{k\to\infty} k|c_k^f|\|h_k\|_{p,\gamma} \leq M_p \lim_{k\to\infty} k^{\alpha+(3/4)}(p-1)^{\beta+(1/2)} = 0,$$

and the result of the lemma follows. $\square$

In this way, we are ready to establish the following theorems.

Theorem 4 (Tauber). *Suppose that $f \in L^1(\gamma)$. If f has the Hermite expansion, $\sum_{n=1}^{\infty} c_n^f h_n$, such that, $c_0^f = 0$ and $c_k^f = O(e^{\alpha k})$ with $\alpha < -1/2$, then*

$$lim_{k\to\infty} S_k f(x) = f(x),$$

for each $x \in \mathbb{R}$.

Proof. First, we recall that $\lim_{r\to 1^-} A_r f(x) = f(x)$, if and only if, $\lim_{k\to\infty} A_{r_k} f(x) = f(x)$, $\forall (r_k)_{k=1}^{\infty}$, such that, $\lim_{k\to\infty} r_k = 1^-$. Fix $x \in \mathbb{R}$, let $\epsilon > 0$ and we set $r = 1 - \frac{1}{k}$. Then, there exists $N_1 \in \mathbb{N}$ such that,

$$|A_{r_k} f(x) - f(x)| < \epsilon/3, \ \forall k \geq N_1.$$

Now, from Lemma 2 there exists $N_2 \in \mathbb{N}$, such that,

$$k|c_k^f||h_k(x)| < \epsilon/3, \ \forall k \geq N_2$$

and finally, from (14) there exists $N_3 \in \mathbb{N}$, such that,

$$\frac{1}{n} \sum_{k=1}^{n} k|c_k^f||h_k(x)| < \epsilon/3 \ \forall k \geq N_3 \text{ and each } n \in \mathbb{N}.$$

Considering $N_0 = \max(N_1, N_2, N_3)$ and $k \geq N_0$ we obtain,

$$\begin{aligned} |S_k f(x) - f(x)| &\leq |S_k f(x) - A_{r_k} f(x)| + |A_{r_k} f(x) - f(x)| \\ &\leq \sum_{j=0}^{k} (1 - r^j)|c_j^f||h_j(x)| + \sum_{j=k+1}^{\infty} r^j |c_j^f||h_j(x)| + |A_{r_k} f(x) - f(x)| \\ &\leq (1 - r) \sum_{j=0}^{k} j|c_j^f||h_j(x)| + \sum_{j=k+1}^{\infty} r^j \frac{j|c_j^f||h_j(x)|}{j} + |A_{r_k} f(x) - f(x)| \\ &\leq \frac{1}{k} \sum_{j=0}^{k} j|c_j^f||h_j(x)| + \frac{\epsilon}{3k} \sum_{j=0}^{\infty} r^j + |A_{r_k} f(x) - f(x)| \\ &< \epsilon, \end{aligned}$$

and the result follows. □

Similarly, we obtain the following theorem.

Theorem 5. *Let $f \in L^p(\gamma)$ where $1 < p < \infty$. Suppose that f has the Hermite expansion, $\sum_{n=1}^{\infty} c_n^f h_n$, such that, $c_0^f = 0$. Then,*

i) *If $1 < p < 2$ and $c_k^f = O(k^\alpha)$ with $\alpha < -3/4$, then we have that $lim_{k\to\infty} \|S_k f - f\|_{p,\gamma} = 0$.*

ii) *If $2 < p < \infty$ and $c_k^f = O\left((p-1)^{\beta k} k^\alpha\right)$ where $\alpha < -3/4$ and $\beta < -1/2$, then $lim_{k\to\infty} \|S_k f - f\|_{p,\gamma} = 0$.*

iii) *If $p = 2$ and $c_k^f = o(1/k)$, then $lim_{k\to\infty} \|S_k f - f\|_{p,\gamma} = 0$.*

Proof. i) The proof of this item follows a similar argument to that developed in the previous theorem. Thus, given $\epsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that, $\|A_{r_k} f - f\| < \epsilon/3$, $\forall k \geq N_1$. On the other hand, Lemma 3 allows us to conclude that there is $N_2 \in \mathbb{N}$, such that,

$$k|c_k^f|\|h_k\|_{p,\gamma} < \epsilon/3, \quad \forall k \geq N_2,$$

and therefore, for some $N_3 \in \mathbb{N}$ we get

$$\frac{1}{n} \sum_{k=1}^{n} k|c_k^f|\|h_k\|_{p,\gamma} < \epsilon/3 \; \forall k \geq N_3.$$

Then, defining $N_0 = \max(N_1, N_2, N_3)$ we have

$$\begin{aligned} \|S_k f - f\|_{p,\gamma} &\leq \frac{1}{k} \sum_{j=0}^{k} j|c_j^f|\|h_j\|_{p,\gamma} + \frac{\epsilon}{3k} \sum_{j=0}^{\infty} r^j + \|A_{r_k} f - f\|_{p,\gamma} \\ &< \epsilon. \end{aligned}$$

In a similar way, item ii) is demonstrated. The case $p = 2$ is deduced from the fact that $\|h_k\|_{2,\gamma} = 1$, so it is enough to consider Fourier-Hermite coefficients c_k^f, such that, $\lim_{k\to\infty} k|c_k^f| = 0$, as in the classical case. □

Therefore, we observe that for certain functions f, such that, they satisfy the hypotheses of the Theorem 4, the behaviour of the kth partial sum of $\sum_{n=1}^{\infty} c_n^f h_n(x)$ is similar to the behavior of $A_r(f)(x) = \sum_{n=1}^{\infty} r^n c_n^f h_n(x)$, for $r = r_k = 1 - \frac{1}{k}$. In this way we obtain the following result.

Theorem 6. *Under the hypotheses of Theorem 4 we obtain,*

$$\sigma f(x) \leq C_x g f(x),$$

for each $x \in \mathbb{R}$.

Proof. First note that from Theorem 4, considering $r = r_k = 1 - \frac{1}{2^k}$ and by means of Cauchy-Schwarz inequality we get

$$\begin{aligned}
\sigma^2 f(x) = \sum_{k=0}^{\infty} |S_{2^k} f(x) - S_{2^{k-1}} f(x)|^2 &\approx C_x \sum_{k=0}^{\infty} |A_{r_k} f(x) - A_{r_{k-1}} f(x)|^2 \\
&\leq C_x \sum_{k=0}^{\infty} \left(\int_{r_{k-1}}^{r_k} |\partial_s A_s f(x)| ds \right)^2 \\
&\leq C_x \sum_{k=0}^{\infty} (r_k - r_{k-1}) \int_{r_{k-1}}^{r_k} |\partial_s A_s f(x)|^2 ds \\
&= C_x \sum_{k=0}^{\infty} \int_{r_{k-1}}^{r_k} (1 - r_k) |\partial_s A_s f(x)|^2 ds \\
&\leq C_x \int_0^{\infty} (1 - s) |\partial_s A_s f(x)|^2 ds = g^2 f(x),
\end{aligned}$$

and the result of the Theorem follows. □

2.3. Theorems about L^p norms, $1 < p < \infty$

Then we start this section giving the following Lemmas (see [6, Lemma 2.10, Lemma 2.22] for similar versions) and we present it with details for the sake of completeness.

Lemma 4. *Suppose that G is a linear subespace of functions $\phi \in L^p(\gamma)$, $1 < p < \infty$, and $\psi = T\phi$ an additive operation defined for $\phi \in G$, such that,*

i) *If ϕ is real-valued so is ψ.*

ii) $\|\psi\|_{p,\gamma} \leq M \|\phi\|_{p,\gamma}$ *with M independent of ϕ.*

Let $\phi_1, \phi_2, ...$ be a set of functions in G and $\psi_j = T\phi_j$. Then,

$$\|\Psi\|_{p,\gamma} \leq M \|\Phi\|_{p,\gamma},$$

where $\Phi = \left(\sum_j |\phi_j|^2 \right)^{1/2}$ *and* $\Psi = \left(\sum_j |\psi_j|^2 \right)^{1/2}$.

Proof. Let $\alpha_1, \alpha_2, .., \alpha_n$ be a fixed set of direction angles in $\mathbb{R}^n$, so $(\cos(\alpha_1), ..., \cos(\alpha_n)) \in \Sigma$, where Σ is the n-dimensional unit sphere. If we denote $\phi = \sum_j \phi_j \cos(\alpha_j)$ and $\psi = \sum_j \psi_j \cos(\alpha_j)$, then $\psi = T\phi$ and by ii) we obtain

$$\int_{\mathbb{R}} \left| \sum_j \psi_j \cos(\alpha_j) \right|^p \gamma(dx) \leq M^p \int_{\mathbb{R}} \left| \sum_j \phi_j \cos(\alpha_j) \right|^p \gamma(dx). \tag{15}$$

Now, we observe that

$$\left| \sum_j \psi_j \cos(\alpha_j) \right| = |\Psi(x)| |\cos(\delta)| \text{ and } \left| \sum_j \phi_j \cos(\alpha_j) \right| = |\Phi(x)| |\cos(\beta)|,$$

where δ, β denoting the angles between the vectors $(\psi_j)_j$ and $(\cos(\alpha_j))_j$ and $(\phi_j)_j$ and $(\cos(\alpha_j))_j$ respectively. Therefore, integrate (15) over Σ, Fubini's Theorem allows us to write

$$\int_{\mathbb{R}} |\Psi(x)|^p \left(\int_{\Sigma} |\cos(\delta)|^p d\sigma \right) \gamma(dx) \leq M^p \int_{\mathbb{R}} |\Phi(x)|^p \left(\int_{\Sigma} |\cos(\beta)|^p d\sigma \right) \gamma(dx).$$

But, we observe that

$$\int_{\Sigma} |\cos(\delta)|^p d\sigma = \int_{\Sigma} |\cos(\beta)|^p d\sigma,$$

and these integrals are independent of x. Then, we deduce that

$$\|\Psi\|_{p,\gamma} \leq M\|\Phi\|_{p,\gamma},$$

and the result follows. □

Lemma 5. *Let $1 < p < \infty$, $0 \leq r_j < 1$ for $j = 1, 2...M$ and let I_j denote any subinterval of $(r_j, 1)$. Then under the hypotheses of Theorem 5,*

$$\int_{\mathbb{R}} \left(\sum_{j=1}^{M} |S_j(A_{r_j}f)(x)|^2 \right)^{p/2} \gamma(dx) \leq A_p \int_{\mathbb{R}} \left(\sum_{j=1}^{M} \frac{1}{|I_j|} \int_{I_j} |A_r f(x)|^2 dr \right)^{p/2} \gamma(dx).$$

Proof. From Theorem 5 with $r_j = 1 - \frac{1}{j}$ and (4), we have that

$$\|S_j(A_{r_j}f)\|_{p,\gamma} \leq A_p\|A_{r_j}f\|_{p,\gamma} \leq A_p\|f\|_{p,\gamma},$$

$\forall j = 1, ..., M$. Now, considering $\phi_j = A_{r_j}f$ and $\psi_j = S_j(A_{r_j}f)$ by Lemma 4 we can obtain that

$$\int_{\mathbb{R}} \left(\sum_{j=1}^{M} |S_j(A_{r_j})f(x)|^2 \right)^{p/2} \gamma(dx) \leq A_p \int_{\mathbb{R}} \left(\sum_{j=1}^{M} |f(x)|^2 \right)^{p/2} \gamma(dx).$$

Now, we suppose that I_j no contains the point $r = 1$ and we consider $\rho_j \in (r_j, 1)$, $j = 1, .., M$. Thus, $r_j = R_j\rho_j$, with $0 < R_j < 1$ and therefore, $A_{r_j}f(x) = A_{R_j\rho_j}f(x) = A_{R_j}(A_{\rho_j}f)(x)$. Again,

$$\|S_j(A_{r_j}f)\|_{p,\gamma} = \|S_j(A_{R_j}(A_{\rho_j}f))\|_{p,\gamma} \leq A_p\|A_{\rho_j}f\|_{p,\gamma},$$

and from Lemma 4, we deduce that

$$\int_{\mathbb{R}} \left(\sum_{j=1}^{M} |S_j(A_{r_j})f(x)|^2 \right)^{p/2} \gamma(dx) \leq A_p \int_{\mathbb{R}} \left(\sum_{j=1}^{M} |A_{\rho_j}f(x)|^2 \right)^{p/2} \gamma(dx), \tag{16}$$

where $\rho_j > r_j$ for each $j = 1, .., M$.

Now, we split each interval I_j into m equal parts, so $I_j = \bigcup_{i=1}^{m} I_j^{(i)}$ and denote by $\rho_j^{(i)}$ the left-hand ends of $I_j^{(i)}$. Observing that

$$|S_j(A_{r_j})f(x)|^2 = \sum_{i=1}^{m} m^{-1}|S_j(A_{r_j})f(x)|^2,$$

and if we simultaneously replace the term $|A_{\rho_j}f(x)|^2$ by $\sum_{i=1}^{m} m^{-1}|A_{\rho_j^{(i)}}f(x)|^2$ in (16) we have that

$$\int_{\mathbb{R}} \left(\sum_{j=1}^{M} \sum_{i=1}^{m} m^{-1}|S_j(A_{r_j})f(x)|^2 \right)^{p/2} \gamma(dx) \leq A_p \int_{\mathbb{R}} \left(\sum_{j=1}^{M} \sum_{i=1}^{m} m^{-1}|A_{\rho_j^{(i)}}f(x)|^2 \right)^{p/2} \gamma(dx),$$

and therefore,

$$\int_{\mathbb{R}} \left(\sum_{j=1}^{M} |S_j(A_{r_j}f)(x)|^2 \right)^{p/2} \gamma(dx) \leq A_p \int_{\mathbb{R}} \left(\sum_{j=1}^{M} \sum_{i=1}^{m} \frac{1}{|I_j^{(i)}|} |A_{\rho_j^{(i)}}f(x)|^2 \frac{|I_j^{(i)}|}{m} \right)^{p/2} \gamma(dx).$$

Then, making $m \to \infty$ we conclude the result of the Lemma. □

Now we are ready to prove the next theorem.

Theorem 7. *Under the hypotheses of Theorem 5, then there exist a constant* $A_p > 0$ *such that*

$$\|\sigma_2 f\|_{p,\gamma} \leq A_p \|gf\|_{p,\gamma}.$$

Proof. Let us start this proof denoting by

$$S_N(f,\rho)(x) = \sum_{j=0}^{N} \rho^j c_j^f h_j(x), \; S'_N(f,\rho)(x) = \sum_{j=0}^{N} j\rho^{j-1} c_j^f h_j(x),$$

$$\omega_N(f)(x) = \sum_{j=0}^{N} j c_j^f h_j(x), \text{ and } \omega_N(f,\rho)(x) = \sum_{j=0}^{N} j\rho^j c_j^f h_j(x),$$

where $0 \leq \rho < 1$, for each $x \in \mathbb{R}$. Then, recalling Abel's formula

$$\sum_{k=0}^{N} u_k v_k = \sum_{k=0}^{N-1} U_k (v_k - v_{k+1}) + U_N v_n,$$

where $U_K = u_0 + ... + u_k$, if we consider $v_k = \rho^{-k-1}$ and $U_K = \omega(f,\rho)(x)$ we obtain that

$$\sum_{k=0}^{N} (\omega_k(f,\rho)(x) - \omega_{k-1}(f,\rho)(x))\rho^{-k-1} = \sum_{k=0}^{N-1} \omega_k(f,\rho)(x)(\rho^{-k-1} - \rho^{-k-2}) + \omega_N(f,\rho)(x)\rho^{-N-1}. \tag{17}$$

Now,

$$\sum_{k=0}^{N} (\omega_k(f,\rho)(x) - \omega_{k-1}(f,\rho)(x))\rho^{-k-1} = \rho^{-1}\omega_N f(x), \tag{18}$$

and on the other hand,

$$\begin{aligned} &\sum_{k=0}^{N-1} \omega_k(f,\rho)(x)(\rho^{-k-1} - \rho^{-k-2}) + \omega_N(f,\rho)(x)\rho^{-N-1} \\ &\quad = \sum_{k=0}^{N-1} \omega_k(f,\rho)(x)\rho^{-k-2}(\rho - 1) + \omega_N(f,\rho)(x)\rho^{-N-1}. \end{aligned} \tag{19}$$

Then, replacing (18) and (19) in (17), we get

$$\omega_N f(x) = \rho^{-N}\omega_N(f,\rho)(x) - (1-\rho)\sum_{k=0}^{N-1} \omega_k(f,\rho)(x)\rho^{-k-1},$$

and therefore,

$$|\omega_N f(x)|^2 \leq 2\left[\rho^{-2N}|\omega_N(f,\rho)(x)|^2 + \left((1-\rho)\sum_{k=0}^{N-1} |\omega_k(f,\rho)(x)|\rho^{-k-1}\right)^2\right].$$

Now, Jessen's inequality allows us to express

$$\left((1-\rho)\sum_{k=0}^{N-1} |\omega_k(f,\rho)(x)|\rho^{-k-1}\right)^2 \leq (1-\rho)(\rho^{-N} - 1)\sum_{k=0}^{N-1} \rho^{-k-1}|\omega_k(f,\rho)(x)|^2,$$

and we obtain that

$$|\omega_N f(x)|^2 \leq 2\left[\rho^{-2N}|\omega_N(f,\rho)(x)|^2 + \frac{(1-\rho)}{\rho^N}\sum_{k=0}^{N-1} \rho^{-k-1}|\omega_k(f,\rho)(x)|^2\right].$$

Let $\rho=\rho_N=1-\frac{1}{N+1}$ and $I_N=(\rho_N,\rho_{N+1})$. Then from the definition of the function σ_2,

$$\begin{aligned}\|\sigma_2\|_{p,\gamma}^p=&\int_{\mathbb{R}}\left(\sum_{N=1}^{\infty}\frac{|\omega_N f(x)|^2}{N(N+1)^2}\right)^{p/2}\gamma(dx)\\ \leq&2^{p/2}\int_{\mathbb{R}}\left(\sum_{N=1}^{\infty}\left[(\rho_N)^{-2N}\frac{|\omega_N(f,\rho_N)(x)|^2}{N(N+1)^2}+\frac{(1-\rho_N)}{(\rho_N)^N}\sum_{k=0}^{N-1}(\rho_N)^{-k-1}\frac{|\omega_k(f,\rho_N)(x)|^2}{N(N+1)^2}\right]\right)^{p/2}\gamma(dx)\\ \leq&2^{p/2}\int_{\mathbb{R}}\left(\sum_{N=1}^{\infty}\left[\frac{|\omega_N(f,\rho_N)(x)|^2}{(\rho_N)^{2N}N^3}+\frac{(1-\rho_N)}{(\rho_N)^N}\sum_{k=0}^{N-1}(\rho_N)^{-k-1}\frac{|\omega_k(f,\rho_N)(x)|^2}{N^3}\right]\right)^{p/2}\gamma(dx)\\ \leq&A_p\int_{\mathbb{R}}\left(\sum_{N=1}^{\infty}\left[\frac{|\omega_N(f,\rho_N)(x)|^2}{N^3}+\frac{1}{N^3(N+1)}\sum_{k=0}^{N-1}|\omega_k(f,\rho_N)(x)|^2\right]\right)^{p/2}\gamma(dx),\end{aligned}$$

since $1-\rho_N=\frac{1}{N+1}$, $(\rho_N)^{-2N}<e^2$ and we denote $A_p=2^{p/2}e^p$.

But by definition, $\omega_k(f,\rho)(x)=\rho S_k'(f,\rho)(x)$, $k=0,..,N$, $\forall\rho\in[0,1)$ and thus,

$$\|\sigma_2\|_{p\gamma}^p\leq A_p\int_{\mathbb{R}}\left(\sum_{N=1}^{\infty}\left[\frac{|S_N'(f,\rho_N)(x)|^2}{N^3}+\frac{1}{N^3(N+1)}\sum_{k=0}^{N-1}|S_k'(f,\rho_N)(x)|^2\right]\right)^{p/2}\gamma(dx).$$

In this way, applying the Lemma 5 we obtain that

$$\begin{aligned}\|\sigma_2\|_{p,\gamma}\leq&A_p\int_{\mathbb{R}}\left(\sum_{N=1}^{\infty}\frac{1}{N^3|I_N|}\int_{I_N}|\partial_\rho A_\rho f(x)|^2d\rho\right.\\ &\left.+\frac{1}{N^4}\left[\sum_{k=0}^{N-1}\frac{1}{|I_N|}\int_{I_N}|\partial_\rho A_\rho f(x)|^2d\rho\right]\right)^{p/2}\gamma(dx),\end{aligned}$$

since $S_N'(f,\rho)(x)$ are the partial sums of the Fourier Hermite expansion of $\partial_\rho A_\rho f(x)$. Therefore, observing that $|I_N|=\frac{1}{(N+1)(N+2)}$ we obtain,

$$\|\sigma_2\|_{p\gamma}^p\leq A_p\int_{\mathbb{R}}\left(\sum_{N=1}^{\infty}\frac{2(N+1)(N+2)}{N^3}\int_{I_N}|\partial_\rho A_\rho f(x)|^2d\rho\right)^{p/2}\gamma(dx).$$

But, $1=(N+1)(1-\rho)\leq(N+2)(1-\rho)$ then

$$\begin{aligned}\|\sigma_2\|_{p,\gamma}^p&\leq A_p\int_{\mathbb{R}}\left(\sum_{N=1}^{\infty}\left(\frac{N+2}{N}\right)^3\int_{I_N}(1-\rho)|\partial_\rho A_\rho f(x)|^2d\rho\right)^{p/2}\gamma(dx)\\ &\leq 3^{p/2}A_p\int_{\mathbb{R}}\left(\sum_{N=1}^{\infty}\int_{I_N}(1-\rho)|\partial_\rho A_\rho f(x)|^2d\rho\right)^{p/2}\gamma(dx)\\ &\leq A_p\int_{\mathbb{R}}\left(\int_0^1(1-\rho)|\partial_\rho A_\rho f(x)|^2d\rho\right)^{p/2}\gamma(dx)\\ &= A_p\|\,gf\|_{p,\gamma}^p,\end{aligned}$$

and we conclude the result. □

Finally, it is appropriate to comment on the need for the hypothesis, $c_0^f=0$, established in some previous results. Indeed, in [5] H. Pollard demonstrated, considering the function $f(x)=e^{cx^2}$, for $1\leq p\leq 2$ and $1/2<c<1/p$, that $\lim_{n\to\infty}\|S_nf-f\|_{p,\gamma}\neq 0$. But we observe that for this function the condition $c_0^f=0$ is not fulfilled. Thus, by requiring the condition $c_0^f=0$ we avoid this type of cases in our results.

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