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Study of nonlinear PDE with power nonlinearities

Alexander G. Ramm[1,*]

[1] Department of Mathematics, Kansas State University, Manhattan, KS 66506, USA

[*] Correspondence: ramm@ksu.edu

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Abstract: Let $u' + Au = h(u,t) + f(x,t)$ with the initial condition $u(x,0) = u_0(x)$, where $u \in H$, $u' := u_t := \frac{du}{dt}$, and H is a Hilbert space. The nonlinear term satisfies the estimate $\|h(u,t)\| \le a\|u\|^p(1+t)^{-b}$, and the operator A satisfies the coercivity condition $(Au,u) \ge \gamma(t)(u,u)$, where $\gamma(t) = q_0(1+t)^{-q}$. Here, a, p, b, q_0, and q are positive constants. Sufficient conditions are established under which the solution exists and is either bounded or tends to zero as $t \to \infty$.

Keywords: nonlinear PDE problems, boundedness of the solutions, global existence

MSC: 34G20.

1. Introduction

Let

$$u' + Au = h(u,t) + f(x,t), \; u(x,0) = u_0(x), \tag{1}$$

where $u \in H$, $u' := u_t := \frac{du}{dt}$, H is a Hilbert space, $(u,v) = (v,u)$ is an inner product in this space, $\|u\|^2 := (u,u)$, A is a linear operator, $(Au,u) \ge \gamma(t)(u,u)$, where $\gamma(t0 > 0$ is a continuous function, $h(u,t)$ and $f(x,t)$ are continuous functions of their arguments, $t \ge 0$. The assumptions about $\gamma(t)$ are formulated in formula (9) below. We assume that

$$\|h(u,t)\| \le \alpha(t, \|u\|) \le a(t)\|u\|^p, \;\; p \ge 1, \tag{2}$$

and

$$\|f(x,t)\| \le \beta(t), \tag{3}$$

where $\alpha := \alpha(t,s) \ge 0$ is a continuous function of t on $[0,\infty) := \mathbb{R}_+$, non-decreasing as a function of s, locally Lipschitz on $s \in \mathbb{R}_+$, and $\beta := \beta(t)$ is a continuous function on $\mathbb{R}_+$. Let us denote $\|u\| := g(t)$.

We would like to find sufficient conditions on $a(t)$ and $\beta(t)$ for the solution to problem (1) to be bounded as $t \to \infty$ or for this solution to satisfy the condition

$$\lim_{t\to\infty} \|u(x,t)\| = 0. \tag{4}$$

Our basic method is based on the results in [1], [2], [3]. Let us formulate one of these results that we will use.

Consider the following inequality:

$$g'(t) \le -\gamma(t)g(t) + \alpha(t, g(t)) + \beta(t), \;\; t \ge 0, \; g \ge 0, \tag{5}$$

where $\beta = \beta(t)$ and $\alpha = \alpha(t,g)$ satisfy the assumptions mentioned below formula (3), and $\gamma(t)$ is a continuous function on $\mathbb{R}_+$.

Proposition 1. *Suppose that $\mu(t) > 0$ is defined for all $t \in \mathbb{R}_+$ and satisfies the following conditions:*

$$\alpha(t, \frac{1}{\mu(t)}) + \beta(t) \le \frac{1}{\mu(t)}[\gamma(t) - \frac{\mu'(t)}{\mu(t)}], \;\; t > 0, \tag{6}$$

and

$$\mu(0)g(0) < 1. \tag{7}$$

Then any solution $g \geq 0$ to (5) is defined for all $t \in \mathbb{R}_+$ and satisfies the inequality:

$$g(t) \leq \frac{1}{\mu(t)}, \quad t > 0. \tag{8}$$

A proof of Proposition 1 can be found in [3], pp. 104–109.
Assume that:

$$0 \leq \gamma(t) \leq q_0(1+t)^{-q}; \;\; 0 \leq \alpha(t,g) \leq a_0(1+t)^{-a}g^p; \;\; 0 \leq \beta \leq b_0(1+t)^{-b}, \tag{9}$$

where $q_0, q, a_0, a, b_0, b, p$ are positive constants, $p \geq 1$.
Take the inner product of (1) with u and use the following relations:

$$(u', u) = g'g; \;\; -(Au, u) \leq -\gamma(t)g^2; \;\; |(h(t,u), u)| \leq \alpha(t,g)g; \;\; |(f,u)| \leq \beta(t)g. \tag{10}$$

Then one gets inequality (5) for g.
Let us choose

$$\mu(t) = m_0(1+t)^m, \quad m_0, m > 0, \tag{11}$$

where m_0, m are constants.
Let us rewrite (6) and (7) as

$$-\gamma(t)\frac{1}{\mu(t)} + \alpha(t, \frac{1}{\mu(t)}) + \beta(t) \leq \frac{d}{dt}\Big(\frac{1}{\mu(t)}\Big), \quad \mu(0)g(0) < 1. \tag{12}$$

We want to derive from Proposition 1 the following Theorem.

Theorem 1. Assume that m_0, m are chosen so that inequalities (12) hold, where $g \geq 0$ solves (5). Then $g(t)$ exists for all $t > 0$ and inequality (8) holds.

2. Proofs

Let us prove Theorem 1.
To do this, it is sufficient to check that m_0 and m can be chosen as required in Theorem 1 for any fixed $p \geq 1$.
Using estimates (9) one checks that

$$-q_0 m_0^{-1}(1+t)^{-q-m} + a_0(1+t)^{-a}[m_0(1+t)^m]^{-p} + b_0(1+t)^{-b} < -m_0^{-1}m(1+t)^{-m-1}. \tag{13}$$

Let us rewrite this inequality:

$$m_0^{-1}m(1+t)^{-m-1} + a_0(1+t)^{-a}[m_0(1+t)^m]^{-p} + b_0(1+t)^{-b} < q_0 m_0^{-1}(1+t)^{-q-m}. \tag{14}$$

The first inequality (12) is satisfied if

$$a + mp > q + m; \;\; b > q + m; \;\; m + 1 > q + m; \quad \frac{a_0}{m_0^p} + b_0 + \frac{m}{m_0} \leq \frac{q_0}{m_0}. \tag{15}$$

The second inequality (12) is satisfied if

$$m_0 g(0) < 1. \tag{16}$$

Inequalities (15) and (16) are valid for any fixed $p \geq 1$ if, for example, m, q_0, q, b, a_0, b_0 and $g(0)$ are small, and m_0, b, a, are not too small.

There are many choices of these parameters to satisfy the inequalities (15) and (16). For example, one may choose $q = 0,9, p = 1, m = 1, b = 3, m_0 = \frac{1}{2}, a_0 = \frac{1}{10}, a = 1, q_0 = 10, b_0 = \frac{1}{10}$. One can check that the inequalities (15) are satisfied.

Estimate (8) shows that $\lim_{t\to\infty} g(t) = 0$ provided that $\lim_{t\to\infty} \mu(t) = \infty$. If $m > 0$ then $\lim_{t\to\infty} \mu(t) = \infty$.

Since $\mu(t)$ exists for all $t > 0$, the $g(t)$ also exists for all $t > 0$, and so is $u(t)$, the solution to (1). Sufficient conditions for the local solvability of problem (1) are known: if A is an elliptic operator and $h(t, u)$ is locally Lipschitz with respect to u, continuous with respect to t, and $f(x, t)$ is smooth with respect to both arguments, then the solution to (1) exists locally and, by Theorem 1, globally.

Consider now the choice:

$$\mu(t) = m_0 + \frac{m_1}{(1+t)^{m_2}}, \quad m_0, m_1, m_2 > 0, \tag{17}$$

where m_0, m_1, m_2 are constants.

Then

$$0 < c_0 \le \mu(t) \le c_1, \quad \frac{1}{c_1} \le \frac{1}{\mu(t)} \le \frac{1}{c_0}, \tag{18}$$

where $c_0, c_1 > 0$ are constants.

One checks that

$$\frac{d}{dt}\left(\frac{1}{\mu(t)}\right) = -\frac{\mu'}{\mu^2} = m_2 m_1 (1+t)^{-m_2-1}\mu(t)^{-2}, \tag{19}$$

for μ defined in (17).

One chooses parameters $m_2, m_0, m_1, q_0, q, a_0, a, b_0, b$, to satisfy the first inequality (12). The second inequality (12) is satisfied if $g(0)$ is sufficiently small. If these inequalities are satisfied, then estimate (8) shows that $g(t)$ is bounded on $\mathbb{R}_+$. Therefore,

$$\sup_{t\ge 0} \|u(x,t)\| \le c, \tag{20}$$

where $c > 0$ is a constant and $u = u(x, t)$ solves Eq. (1).

3. Conclusion

Sufficient conditions are given for the solution to (1) to exist for all $t > 0$ and to tend to zero as $t \to \infty$, or to be bounded at infinity.

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