Open Journal of Mathematical Analysis

PSR Press

Article

Positive solutions for fractional boundary value problems with fractional conditions of right-focal type utilizing lower-order problems

Jeffrey W. Lyons

Department of Mathematical Sciences, The Citadel, 171 Moultrie Street, Charleston, SC 29409 USA; jlyons3@citadel.edu

Received: 17 June 2025; Accepted: 24 July 2025; Published: 23 August 2025.

Abstract: In this work, we seek conditions for the existence or nonexistence of solutions for nonlinear Riemann-Liouville fractional boundary value problems of order $\alpha + 2n$, where $\alpha \in (m-1, m]$ with $m \geq 3$ and $m, n \in \mathbb{N}$. The problem's nonlinearity is continuous and also depends on a positive parameter upon which our constraints are established. Our approach involves constructing a Green's function by combining the Green's functions of a lower-order fractional boundary value problem and a right-focal boundary value problem n times. Leveraging the properties of this Green's function, we apply Krasnosel'skii's Fixed Point Theorem to establish our results. Several examples are presented to illustrate the existence and nonexistence regions.

Keywords: positive solutions, nonexistence, convolution, induction, right focal, fractional derivative

MSC: 26A33, 34A08

1. Introduction

Let $m, n \in \mathbb{N}$ with $m \geq 3$. Set $\alpha \in (m-1, m]$ and $\beta \in [1, m-1]$. In this paper, the following Riemann-Liouville fractional boundary value problem is studied

$$D_{0^+}^{\alpha+2n} u(x) + (-1)^n \lambda g(x) f(u) = 0, \quad 0 < x < 1, \tag{1}$$

$$\begin{aligned} u^{(i)}(0) = 0, \quad i = 0, 1, \ldots, m-2, \quad D_{0^+}^{\beta} u(1) = 0, \\ D_{0^+}^{\alpha+2\gamma} u(0) = D_{0^+}^{\alpha+2\gamma+1} u(1) = 0, \quad \gamma = 0, 1, \ldots, n-1. \end{aligned} \tag{2}$$

Of particular note are the second set of boundary conditions which are right-focal inspired. Throughout, we require that $f : [0, \infty) \to [0, \infty)$ be a continuous function and that $g : [0, 1] \to [0, \infty)$ is also a continuous function additionally satisfying $\int_0^1 g(x)\, dx > 0$. Finally, $\lambda > 0$ is a positive parameter upon which we establish our existence and nonexistence of positive solution results for (1), (2). In this work, α is the order of the lower-order fractional boundary value problem, β is the order of the fractional derivative boundary condition evaluated at the right endpoint, and $2n$ is the increased order of the higher-order boundary value problem.

To prove that positive solutions exist, we seek fixed points of the operator

$$Tu(x) = (-1)^n \lambda \int_0^1 G(x, s) g(s) f(u(s))\, ds,$$

where $G(x, s)$ is the Green's function associated with (1), (2). Fixed points of the operator are positive solutions to (1), (2).

The main motivation for this work is a generalization of the paper by Lyons and Neugebauer, [1]. Here, we extend their work by increasing the order of the fractional boundary value problem from a magnitude of $\alpha + 2$ to a magnitude of $\alpha + 2n$. This is done using repeated convolution of the Green's function for a standard right-focal, second-order ordinary boundary value problem with that of a lower-order fractional boundary value problem. As expected, the right-focal work found within [1] is recovered precisely when n is equals to one.

The work of Lyons and Neugebauer was itself a generalization of that done by Graef along with various authors in the early part of the century [2–4]. The initial motivation for their efforts was to prove the existence of positive solutions to beam equations using fixed point theory [5]. If we pick the values in our generalization correctly, we recover those results as a special case.

To construct the Green's function for (1), (2), we use the technique outlined in Eloe and Neugebauer in [6]. This is facilitated by a convolution of the Green's function $G_0(x,s)$ for a lower-order problem with the Green's function of a right-focal boundary value problem. We continue in an iterative process to yield the higher-order Green's function corresponding to (1), (2). Next, we prove that key properties of the lower-order Green's functions are inherited by the higher-order Green's function. Finally, an application of Guo-Krasnosel'skii Fixed Point Theorem is employed to show the existence of positive solutions and a contradiction argument is provided that establishes the nonexistence results. Both types of results are based upon the sizing of λ.

Much research has been done employing fixed point theory to establish the existence of solutions and occassionally nonexistence of solutions to Riemann-Liouville fractional boundary value problems. This study fits into this wide array of research in that vein [1,2,7–16].

Section 2 introduces key definitions related to the RL-fractional derivative and offers directions for further study, along with a statement of the Guo-Krasnosel'skii Fixed Point Theorem. The following sections focus on constructing the Green's function and analyzing its properties. In Sections 5 and 6, we determine parameter intervals for λ that ensure the existence or nonexistence of positive solutions. Lastly, we provide examples to demonstrate the application of our main results.

2. Definitions and theorems

To start, we present the definitions of the Riemann-Liouville fractional integral and derivative. The choice of this type of fractional derivative amongst the many other choices is that the Green's functions and properties therein are well-established for our boundary value problem and it is widely used and adopted.

Definition 1. Let $\nu > 0$. The Riemann-Liouville fractional integral of a function u of order ν, denoted $I_{0^+}^{\nu} u$, is defined as

$$I_{0^+}^{\nu} u(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-s)^{\nu-1} u(s) ds,$$

provided the right-hand side exists.

Definition 2. Let n denote a positive integer and assume $n-1 < \alpha \leq n$. The Riemann-Liouville fractional derivative of order α of the function $u : [0,1] \to \mathbb{R}$, denoted $D_{0^+}^{\alpha} u$, is defined as

$$D_{0^+}^{\alpha} u(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-s)^{n-\alpha-1} u(s) ds = D^n I_{0^+}^{n-\alpha} u(x),$$

provided the right-hand side exists.

For material on fractional calculus and more in depth information about these definitions, we cite [17–20]. Lastly, we present Guo-Krasnosel'skii's Fixed Point Theorem as found in [21,22].

Theorem 1 (Guo-Krasnosel'skii's Fixed Point Theorem)**.** *Let X be a Banach space, and let $\mathcal{P} \subset X$ be a cone. Assume that Ω_1 and Ω_2 are open sets with $0 \in \Omega_1 \subseteq \overline{\Omega}_1 \subset \Omega_2$. Let the operator $T : \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{P}$ be a completely continuous such that either*

1. *for $u \in \mathcal{P} \cap \partial\Omega_1$, $\|Tu\| \geq \|u\|$ and for $u \in \mathcal{P} \cap \partial\Omega_2$, $\|Tu\| \leq \|u\|$; or*
2. *for $u \in \mathcal{P} \cap \partial\Omega_1$, $\|Tu\| \leq \|u\|$ and for $u \in \mathcal{P} \cap \partial\Omega_2$, $\|Tu\| \geq \|u\|$.*

Then, T has a fixed point in the set $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Green's function via convolution

Similar to procedure found in [16], we now build the Green's function for (1), (2). This is accomplished using induction and convolution.

First, the right-focal boundary value problem

$$-u'' = 0, \quad 0 < x < 1, \quad u(0) = 0, \quad u'(1) = 0$$

has Green's function

$$G_{rf}(x,s) = \begin{cases} s, & 0 \le s < x \le 1, \\ x, & 0 \le x < s \le 1. \end{cases}$$

Let $G_0(x,s)$ be the Green's function for

$$-D^{\alpha}_{0^+} u = 0, \quad 0 < x < 1, \quad u^{(i)}(0) = 0,\ i = 0,1,\dots,m-2, \quad D^{\beta}_{0^+} u(1) = 0,$$

which is given by [23]:

$$G_0(x,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} x^{\alpha-1}(1-s)^{\alpha-1-\beta} - (x-s)^{\alpha-1}, & 0 \le s < x \le 1, \\ x^{\alpha-1}(1-s)^{\alpha-1-\beta}, & 0 \le x \le s < 1. \end{cases}$$

Now, define $G_k(x,s)$ for $k = 1,2,\dots,n$ by

$$G_k(x,s) = -\int_0^1 G_{k-1}(x,r) G_{rf}(r,s) dr.$$

Then,

$$G_n(x,s) = -\int_0^1 G_{n-1}(x,r) G_{rf}(r,s) dr,$$

is the Green's function for

$$-D^{\alpha+2n}_{0^+} u(x) = 0, \quad 0 < x < 1,$$

with boundary conditions (2), and $G_{n-1}(x,s)$ is the Green's function for

$$-D^{\alpha+2(n-1)}_{0^+} u(x) = 0, \quad 0 < x < 1,$$

with boundary conditions

$$u^{(i)}(0) = 0,\ i = 0,1,\dots,m-2, \quad D^{\beta}_{0^+} u(1) = 0,$$

$$D^{\alpha+2l}_{0^+} u(0) = D^{\alpha+2l+1}_{0^+} u(1) = 0,\ l = 0,1,\dots,n-2.$$

To see this, first consider $k = 1$ and the linear differential equation

$$D^{\alpha+2}_{0^+} u(x) + h(x) = 0, \quad 0 < x < 1,$$

satisfying boundary conditions

$$u^{(i)}(0) = 0,\ i = 0,1,\dots,m-2, \quad D^{\beta}_{0^+} u(1) = 0,$$

$$D^{\alpha}_{0^+} u(0) = 0, \quad D^{\alpha+1}_{0^+} u(1) = 0.$$

Employ a of variable change

$$v(x) = D^{\alpha+2-2}_{0^+} u(x).$$

Then,

$$D^2 v(x) = D^2 D^{\alpha+2-2}_{0^+} u(x) = D^{\alpha+2}_{0^+} u(x) = -h(x),$$

and since $v(x) = D^{\alpha}_{0^+}u(x)$,

$$v(0) = D^{\alpha}_{0^+}u(0) = 0 \quad \text{and} \quad v'(1) = D^{\alpha+1}_{0^+}u(1) = 0.$$

Thus, v satisfies the right-focal boundary value problem

$$v'' + h(x) = 0, \quad 0 < x < 1,$$

$$v(0) = 0, \quad v'(1) = 0.$$

Also, u now satisfies a lower-order boundary value problem,

$$D^{\alpha}_{0^+}u(x) = v(x), \quad 0 < x < 1,$$

$$u^{(i)}(0) = 0,\ i = 0, 1, \ldots, m-2, \quad D^{\beta}_{0^+}u(1) = 0.$$

Thus,

$$\begin{aligned} u(x) &= \int_0^1 G_0(x,s)(-v(s))ds \\ &= \int_0^1 G_0(x,s)\left(-\int_0^1 G_{rf}(s,r)h(r)ds\right)dr \\ &= \int_0^1 \left(\int_0^1 -G_0(x,s)G_{rf}(s,r)ds\right)h(r)dr, \end{aligned}$$

and

$$u(x) = \int_0^1 G_1(x,s)h(s)ds,$$

where

$$G_1(x,s) = -\int_0^1 G_0(x,r)G_{rf}(r,s)dr.$$

Proceeding inductively, we assume that $k = n-1$ is true and investigate the linear differential equation

$$D^{\alpha+2n}_{0^+}u(x) + k(x) = 0, \quad 0 < x < 1,$$

satisfying boundary conditions (2). Employ a variable change

$$v(x) = D^{\alpha+2(n-1)}_{0^+}u(x).$$

Then, we have that

$$D^2 v(x) = D^{\alpha+2n}_{0^+} = -k(x)$$

and

$$v(0) = D^{\alpha+2(n-1)}_{0^+}u(0) = 0 \quad \text{and} \quad v'(1) = D^{\alpha+2(n-1)+1}_{0^+}v(1) = 0.$$

Now, $v(x)$ satisfies the right-focal boundary value problem

$$v'' + k(x) = 0, \quad 0 < x < 1,$$

$$v(0) = 0, \quad v'(1) = 0$$

while $u(x)$ satisfies the lower-order problem

$$D^{\alpha+2(n-1)}_{0^+}u(x) = v(x), \quad 0 < x < 1,$$

$$u(0) = 0, \quad D^{\beta}_{0^+}u(1) = 0,$$

$$D^{\alpha+2\gamma}_{0^+}u(0) = D^{\alpha+2\gamma+1}_{0^+}u(1) = 0,\ \gamma = 0, 1, \ldots, n-2.$$

Proceeding inductively,

$$\begin{aligned} u(x) &= \int_0^1 G_{n-1}(x,s)(-v(s))ds \\ &= \int_0^1 \left(-\int_0^1 G_{n-1}(x,s)G_{rf}(s,r)ds \right) k(r)dr \\ &= \int_0^1 G_n(x,s)k(s)ds. \end{aligned}$$

Therefore,

$$u(x) = \int_0^1 G_n(x,s)k(s)ds,$$

where

$$G_n(x,s) = -\int_0^1 G_{n-1}(x,r)G_{rf}(r,s)dr.$$

Thus, the unique solution to

$$D_{0^+}^{\alpha+2n}u(x) + k(x) = 0, \quad 0 < x < 1,$$

satisfying boundary conditions (2) is given by

$$u(x) = \int_0^1 G_n(x,s)k(s)ds.$$

4. Properties of the green's function

In this section, we investigate properties of $G_n(x,s)$ that are inherited from $G_0(x,s)$ and $G_{rf}(x,s)$. The first lemma is well-established and presented without proof.

Lemma 1. *For $(x,s) \in [0,1] \times [0,1]$, $G_{rf}(x,s) \in C^{(1)}$ and $G_{rf}(x,s) \geq 0$.*

The following lemma is proved in [1].

Lemma 2.
(1) *If $(x,s) \in [0,1] \times [0,1)$, then $G_0(x,s) \in C^{(1)}$.*
(2) *If $(x,s) \in (0,1) \times (0,1)$, then $G_0(x,s) > 0$ and $\frac{\partial}{\partial x}G_0(x,s) > 0$.*
(3) *If $(x,s) \in [0,1] \times [0,1)$, then $x^{\alpha-1}G_0(1,s) \leq G_0(x,s) \leq G_0(1,s)$.*

The following properties for $G_n(x,s)$ are derived from $G_0(x,s)$ from Lemma 2.

Lemma 3.
1) *If $(x,s) \in [0,1] \times [0,1)$, then $G_n(x,s) \in C^{(1)}$.*
2) *If $(x,s) \in (0,1) \times (0,1)$, then $(-1)^n G_n(x,s) > 0$ and $(-1)^n \frac{\partial}{\partial x}G_n(x,s) > 0$.*
3) *If $(x,s) \in [0,1] \times [0,1)$, then*

$$(-1)^n x^{\alpha-1}G_n(1,s) \leq (-1)^n G_n(x,s) \leq (-1)^n G_n(1,s).$$

Proof. Induction is used to prove each part.
For (1) with $(x,s) \in [0,1] \times [0,1)$, we first consider $k = 1$,

$$G_1(x,s) = -\int_0^1 G_0(x,r)G_{rf}(r,s)ds.$$

By Lemmas 1 and 2, $G_1(x,s) \in C^{(1)}$. Next, assume that $k = n-1$ is true. Then,

$$G_n(x,s) = -\int_0^1 G_{n-1}(x,r)G_{rf}(r,s)ds.$$

By induction along with Lemma 1, $G_n(x,s) \in C^{(1)}$.

For (2) with $(x,s) \in (0,1) \times (0,1)$ and using Lemmas 1 and 2, we first consider $k = 1$,

$$(-1)^1 G_1(x,s) = -\left(-\int_0^1 G_0(x,r)G_{rf}(r,s)dr\right) > 0$$

and

$$(-1)^1 \frac{\partial}{\partial x} G_1(x,s) = -\left(-\int_0^1 \frac{\partial}{\partial x} G_0(x,r)G_{rf}(r,s)dr\right) > 0.$$

Now, proceeding inductively, assume that $k = n-1$ is true. Then, by Lemma 1,

$$\begin{aligned}(-1)^n G_n(x,s) &= (-1)^n \left(-\int_0^1 G_{n-1}(x,r)G_{rf}(r,s)dr\right)\\ &= (-1)^2 \left(\int_0^1 (-1)^{n-1} G_{n-1}(x,r)G_{rf}(r,s)dr\right)\\ &> 0,\end{aligned}$$

and

$$\begin{aligned}(-1)^n \frac{\partial}{\partial x} G_n(x,s) &= (-1)^n \left(-\int_0^1 \frac{\partial}{\partial x} G_{n-1}(x,r)G_{rf}(r,s)dr\right)\\ &= (-1)^2 \left(\int_0^1 (-1)^{n-1} \frac{\partial}{\partial x} G_{n-1}(x,r)G_{rf}(r,s)dr\right)\\ &> 0.\end{aligned}$$

For (3) with $(x,s) \in [0,1] \times [0,1)$ and using Lemma 2 (3), we first consider $k = 1$,

$$\begin{aligned}(-1)^1 x^{\alpha-1} G_1(1,s) &= -x^{\alpha-1} \left(-\int_0^1 G_0(1,r)G_{rf}(r,s)dr\right)\\ &= -\left(\int_0^1 -x^{\alpha-1} G_0(1,r)G_{rf}(r,s)dr\right)\\ &\leq -\left(\int_0^1 -G_0(x,r)G_{rf}(r,s)dr\right)\\ &= -\left(-\int_0^1 G_0(x,r)G_{rf}(r,s)dr\right)\\ &= (-1)^1 G_1(x,s),\end{aligned}$$

and

$$\begin{aligned}(-1)^1 G_1(x,s) &= -\left(-\int_0^1 G_0(x,r)G_{rf}(r,s)dr\right)\\ &= \int_0^1 G_0(x,r)G_{rf}(r,s)dr\\ &\leq \int_0^1 G_0(1,r)G_{rf}(r,s)dr\\ &= -\left(-\int_0^1 G_0(1,r)G_{rf}(r,s)dr\right)\\ &= (-1)^1 G_1(1,s).\end{aligned}$$

Now, proceeding inductively, assume that $k = n-1$ is true. Then,

$$(-1)^n x^{\alpha-1} G_n(1,s) = (-1)^n x^{\alpha-1} \left(-\int_0^1 G_{n-1}(1,r)G_{rf}(r,s)dr\right)$$

$$
\begin{aligned}
&= (-1)^2\left(\int_0^1 (-1)^{n-1} t^{\alpha-1} G_{n-1}(1,r) G_{rf}(r,s)dr\right)\\
&\le (-1)^2\left(\int_0^1 (-1)^{n-1} G_{n-1}(x,r) G_{rf}(r,s)dr\right)\\
&= (-1)^n\left(-\int_0^1 G_{n-1}(x,r) G_{rf}(r,s)dr\right)\\
&= (-1)^n G_n(x,s),
\end{aligned}
$$

and

$$
\begin{aligned}
(-1)^n G_n(x,s) &= (-1)^n\left(-\int_0^1 G_{n-1}(x,r) G_{rf}(r,s)dr\right)\\
&= (-1)^2\left(\int_0^1 (-1)^{n-1} G_{n-1}(x,r) G_{rf}(r,s)dr\right)\\
&\le (-1)^2\left(\int_0^1 (-1)^{n-1} G_{n-1}(1,r) G_{rf}(r,s)dr\right)\\
&= (-1)^n\left(-\int_0^1 G_{n-1}(1,r) G_{rf}(r,s)dr\right)\\
&= (-1)^n G_n(1,s).
\end{aligned}
$$

□

5. Existence of positive solutions

Using our constructed Green's function and its properties, we now demonstrate the existence of positive solutions to (1), (2) by finding bounds for λ. This is done with an application of the Guo-Krasnosel'skii Fixed Point Theorem.

Define the constants

$$
\mathcal{A}_n = \int_0^1 (-1)^n s^{\alpha-1} G_n(1,s) g(s)ds, \quad \mathcal{B}_n = \int_0^1 (-1)^n G_n(1,s) g(s)ds,
$$

$$
\mathcal{F}_0 = \limsup_{u\to 0^+} \frac{f(u)}{u}, \quad f_0 = \liminf_{u\to 0^+} \frac{f(u)}{u},
$$

$$
\mathcal{F}_\infty = \limsup_{u\to \infty} \frac{f(u)}{u}, \quad f_\infty = \liminf_{u\to \infty} \frac{f(u)}{u}.
$$

Let $\mathcal{B} = C[0,1]$ be a Banach space with norm

$$
\|u\| = \max_{x\in[0,1]} |u(x)|.
$$

Define the cone

$$
\begin{aligned}
\mathcal{P} = \{u \in \mathcal{B} : u(0) = 0,\ &u(x) \text{ is nondecreasing, and}\\
&x^{\alpha-1}u(1) \le u(x) \le u(1) \text{ on } [0,1]\Big\}.
\end{aligned}
$$

Define the operator $T : \mathcal{P} \to \mathcal{B}$ by

$$
Tu(x) = (-1)^n \lambda \int_0^1 G_n(x,s) g(s) f(u(s))ds.
$$

Lemma 4. *$T : \mathcal{P} \to \mathcal{P}$ is completely continuous.*

Proof. Set $u \in \mathcal{P}$. Thus,

$$Tu(0) = (-1)^n \lambda \int_0^1 G_n(0,s)g(s)f(u(s))ds = 0.$$

Additionally, for $x \in (0,1)$ and by Lemma 3 (2),

$$\frac{\partial}{\partial x}[Tu(x)] = (-1)^n \lambda \int_0^1 \frac{\partial}{\partial x} G_n(x,s)g(s)f(u(s))ds > 0.$$

This provides that $Tu(x)$ is nondecreasing.

Next, for $x \in [0,1]$ and by Lemma 3,

$$\begin{aligned} x^{\alpha-1}Tu(1) &= x^{\alpha-1}(-1)^n \lambda \int_0^1 G_n(1,s)g(s)f(u(s))ds \\ &\leq (-1)^n \lambda \int_0^1 G_n(x,s)g(s)f(u(s))ds \\ &= Tu(x), \end{aligned}$$

and

$$\begin{aligned} Tu(x) &= (-1)^n \lambda \int_0^1 G_n(x,s)g(s)f(u(s))ds \\ &\leq (-1)^n \lambda \int_0^1 G_n(1,s)g(s)f(u(s))ds \\ &= Tu(1). \end{aligned}$$

Therefore, $Tu \in \mathcal{P}$. T is completely continuous by the Arzeli-Ascoli Theorem. □

Theorem 2. *If*

$$\frac{1}{\mathcal{A}_n f_\infty} < \lambda < \frac{1}{\mathcal{B}_n \mathcal{F}_0},$$

then (1), (2) *has at least one positive solution.*

Proof. Since we have that $\mathcal{F}_0 \lambda \mathcal{B}_n < 1$, $\exists\, \delta > 0$ implying

$$(\mathcal{F}_0 + \delta)\lambda \mathcal{B}_n \leq 1.$$

Also since

$$\mathcal{F}_0 = \limsup_{u \to 0^+} \frac{f(u)}{u},$$

$\exists\, \mathcal{H}_1 > 0$ implying

$$f(u) \leq (\mathcal{F}_0 + \delta)u \quad \text{for} \quad u \in (0, \mathcal{H}_1].$$

Set $\Omega_1 = \{u \in \mathcal{B} : \|u\| < \mathcal{H}_1\}$ and let $u \in \mathcal{P} \cap \partial\Omega_1$. Thus, $\|u\| = \mathcal{H}_1$, and

$$\begin{aligned} |(Tu)(1)| &= (-1)^n \lambda \int_0^1 G_n(1,s)g(s)f(u(s))ds \\ &\leq (-1)^n \lambda \int_0^1 G_n(1,s)g(s)(\mathcal{F}_0 + \delta)u(s)ds \\ &\leq (\mathcal{F}_0 + \delta)u(1)\lambda \int_0^1 (-1)^n G_n(1,s)g(s)ds \\ &\leq (\mathcal{F}_0 + \delta)\|u\|\lambda \mathcal{B}_n \\ &\leq \|u\|. \end{aligned}$$

Since $Tu \in \mathcal{P}$, we have that $\|Tu\| \leq \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_1$.

Since $f_\infty \lambda > \dfrac{1}{\mathcal{A}_n}$, $\exists\ \theta \in (0,1)$ and $\delta > 0$ so that

$$(f_\infty - \delta)\lambda > \left((-1)^n \int_\theta^1 s^{\alpha-1} G_n(1,s) g(s) ds\right)^{-1}.$$

Additionally, as

$$f_\infty = \liminf_{u\to\infty} \frac{f(u)}{u},$$

$\exists\ \mathcal{H}_3 > 0$ implying

$$f(u) \geq (f_\infty - \delta)u \quad \text{for} \quad u \in [\mathcal{H}_3, \infty).$$

Set

$$\mathcal{H}_2 = \max\left\{\frac{\mathcal{H}_3}{\theta^{\alpha-1}}, 2\mathcal{H}_1\right\},$$

and set $\Omega_2 = \{u \in \mathcal{B} : \|u\| < \mathcal{H}_2\}$. Let $u \in \mathcal{P} \cap \partial\Omega_2$. Then, $\|u\| = \mathcal{H}_2$. Notice for $x \in [\theta, 1]$,

$$u(x) \geq x^{\alpha-1} u(1) \geq \theta^{\alpha-1}\mathcal{H}_2 \geq \theta^{\alpha-1}\frac{\mathcal{H}_3}{\theta^{\alpha-1}} = \mathcal{H}_3.$$

Thus,

$$\begin{aligned}
|(Tu)(1)| &\geq (-1)^n \lambda \int_\theta^1 G_n(1,s) g(s) f(u(s)) ds \\
&\geq \lambda \int_\theta^1 (-1)^n G_n(1,s) g(s)(f_\infty - \delta) u(s) ds \\
&\geq \lambda (f_\infty - \delta) u(1) (-1)^n \int_\theta^1 s^{\alpha-1} G_n(1,s) g(s) ds \\
&\geq \|u\|.
\end{aligned}$$

Hence, $\|Tu\| \geq \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_2$. Since $\mathcal{H}_1 < \mathcal{H}_2$, we find $\overline{\Omega}_1 \subset \Omega_2$.
Therefore, by Theorem 1 (1), T has a fixed point $u \in \mathcal{P}$ which is a positive solution of (1), (2). □

Theorem 3. *If*

$$\frac{1}{\mathcal{A}_n f_0} < \lambda < \frac{1}{\mathcal{B}_n \mathcal{F}_\infty},$$

then (1), (2) *has at least one positive solution.*

Proof. Since $f_0 \lambda \mathcal{A}_n > 1$, $\exists\ \delta > 0$ which implies that

$$(f_0 - \delta)\lambda \mathcal{A}_n \geq 1.$$

Additionally, as

$$f_0 = \liminf_{u\to 0^+} \frac{f(u)}{u},$$

$\exists\ \mathcal{H}_1 > 0$ which implies that

$$f(u) \geq (f_0 - \delta)u, \quad x \in (0, \mathcal{H}_1].$$

Set $\Omega_1 = \{u \in \mathcal{B} : \|u\| < \mathcal{H}_1\}$, and let $u \in \mathcal{P} \cap \partial\Omega_1$. Thus, $u(x) \leq \mathcal{H}_1$ for $x \in [0,1]$. Therefore,

$$\begin{aligned}
|(Tu)(1)| &= (-1)^n \lambda \int_0^1 G_n(1,s) g(s) f(u(s)) ds \\
&\geq (-1)^n \lambda \int_0^1 G_n(1,s) g(s)(f_0 - \delta) u(s) ds \\
&\geq \lambda (f_0 - \delta) u(1) \int_0^1 (-1)^n s^{\alpha-1} G_n(1,s) g(s) ds
\end{aligned}$$

$$\begin{aligned} &\geq \lambda(f_0 - \delta)\|u\|\mathcal{A}_n \\ &\geq \|u\|. \end{aligned}$$

Thus, $\|Tu\| \geq \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_1$. Also, since $\mathcal{F}_\infty \mathcal{B}_n \lambda < 1$, $\exists\, \delta \in (0,1)$ implying

$$((\mathcal{F}_\infty + \delta)\mathcal{B}_n + \delta)\lambda \leq 1.$$

Since

$$\mathcal{F}_\infty = \limsup_{u \to \infty} \frac{f(u)}{u},$$

$\exists\, \mathcal{H}_3 > 0$ which implies that

$$f(u) \leq (\mathcal{F}_\infty + \delta)u, \quad u \in [\mathcal{H}_3, \infty).$$

Set

$$M = \max_{u \in [0, \mathcal{H}_3]} f(u).$$

Then, $\exists\, c \in (0,1)$ with

$$(-1)^n \int_0^c G_n(1,s)g(s)ds \leq \frac{\delta}{M}.$$

Let

$$\mathcal{H}_2 = \max\left\{2\mathcal{H}_1, \frac{\mathcal{H}_3}{c^{\alpha-1}}, 1\right\},$$

and set $\Omega_2 = \{u \in \mathcal{B} : \|u\| < \mathcal{H}_2\}$. Let $u \in \mathcal{P} \cap \partial\Omega_2$. Then, $\|u\| = \mathcal{H}_2$ and so,

$$u(1) = \mathcal{H}_2 \geq \frac{\mathcal{H}_3}{c^{\alpha-1}} > \mathcal{H}_3.$$

Now, $u(0) = 0$. The Intermediate Value Theorem tell us that $\exists\, \xi \in (0,1)$ with $u(\xi) = \mathcal{H}_3$. But, for $x \in [c,1]$, we have

$$u(x) \geq x^{\alpha-1}u(1) = x^{\alpha-1}\mathcal{H}_2 \geq c^{\alpha-1}\frac{\mathcal{H}_3}{c^{\alpha-1}} = \mathcal{H}_3.$$

Thus, $\xi \in (0,c]$. Moreover, since $u(x)$ is nondecreasing, we get that

$$0 \leq u(x) \leq \mathcal{H}_3, \quad x \in [0,\xi)$$

and

$$u(x) \geq \mathcal{H}_3, \quad x \in (\xi, 1].$$

Thus,

$$\begin{aligned} |(Tu)(1)| &= (-1)^n \lambda \int_0^1 G_n(1,s)g(s)f(u(s))ds \\ &= \lambda\left((-1)^n \int_0^\xi G_n(1,s)g(s)f(u(s))ds \,+\, (-1)^n \int_\xi^1 G_n(1,s)g(s)f(u(s))ds\right) \\ &\leq \lambda\left(M\int_0^\xi (-1)^n G_n(1,s)g(s)ds \,+\, (-1)^n \int_\xi^1 G_n(1,s)g(s)(\mathcal{F}_\infty + \delta)u(s)ds\right) \\ &\leq \lambda\left(M\frac{\delta}{M} + (\mathcal{F}_\infty + \delta)u(1)\int_\xi^1 (-1)^n G_n(1,s)g(s)ds\right) \\ &\leq \lambda(\delta + (\mathcal{F}_\infty + \delta)\|u\|\,\mathcal{B}_n) \\ &\leq \lambda(\delta\|u\| + (\mathcal{F}_\infty + \delta)\|u\|\,\mathcal{B}_n) \\ &= \lambda\|u\|(\delta + (\mathcal{F}_\infty + \delta)\mathcal{B}_n) \\ &\leq \|u\|. \end{aligned}$$

Thus, $\|Tu\| \leq \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_2$. Notice that since $\mathcal{H}_1 < \mathcal{H}_2$, we have that $\overline{\Omega}_1 \subset \Omega_2$. Therefore, by Theorem 1 (2), T has a fixed point $u \in \mathcal{P}$ which is a positive solution of (1), (2). □

6. Nonexistence of positive solutions

In this section, we seek constraints on λ that would guarantee that no positive solution exists to (1), (2). To that end, we present properties that positive solutions must satisfy in the following lemma.

Lemma 5. *Suppose* $D_{0^+}^{\alpha+2n}u \in C[0,1]$. *If* $(-1)^n(-D_{0^+}^{\alpha+2n}u(x)) \geq 0$ *for all* $x \in [0,1]$ *and* $u(x)$ *satisfies* (2), *then*
1) $u'(x) \geq 0, \quad 0 \leq x \leq 1,$ *and*
2) $x^{\alpha-1}u(1) \leq u(x) \leq u(1), \quad 0 \leq x \leq 1.$

Proof. Let $0 \leq x \leq 1$.
For part (1), we employ Lemma 3 (2) to get

$$\begin{aligned}
u'(x) &= \int_0^1 \frac{\partial}{\partial x} G_n(x,s)(-D_{0^+}^{\alpha+2n}u(s))ds \\
&= \int_0^1 (-1)^n \frac{\partial}{\partial x} G_n(x,s)(-1)^n(-D_{0^+}^{\alpha+2n}u(s))ds \\
&> 0.
\end{aligned}$$

For part (2), we employ Lemma 3 (3) to get

$$\begin{aligned}
x^{\alpha-1}u(1) &= x^{\alpha-1}\int_0^1 G_n(1,s)(-D_{0^+}^{\alpha+2n}u(s))ds \\
&= \int_0^1 (-1)^n x^{\alpha-1} G_n(1,s)(-1)^n(-D_{0^+}^{\alpha+2n}u(s))ds \\
&\leq \int_0^1 (-1)^n G_n(x,s)(-1)^n(-D_{0^+}^{\alpha+2n}u(s))ds \\
&= \int_0^1 G_n(x,s)(-D_{0^+}^{\alpha+2n}u(s))ds \\
&= u(x),
\end{aligned}$$

and

$$\begin{aligned}
u(x) &= \int_0^1 G_n(x,s)(-D_{0^+}^{\alpha+2n}u(s))ds \\
&= \int_0^1 (-1)^n G_n(x,s)(-1)^n(-D_{0^+}^{\alpha+2n}u(s))ds \\
&\leq \int_0^1 (-1)^n G_n(1,s)(-1)^n(-D_{0^+}^{\alpha+2n}u(s))ds \\
&= \int_0^1 G_n(1,s)(-D_{0^+}^{\alpha+2n}u(s))ds \\
&= u(1).
\end{aligned}$$

□

Theorem 4. *If for all* $u \in (0,\infty)$

$$\lambda < \frac{u}{\mathcal{B}_n f(u)},$$

then no positive solution exists to (1), (2).

Proof. For contradiction, assume that $u(x)$ is a positive solution to (1), (2). Then, we have that $(-1)^n(-D_{0^+}^{\alpha+2n}u(x)) = \lambda g(x)f(u(x)) \geq 0$. Therefore, by Lemma 5,

$$\begin{aligned}
u(1) &= (-1)^n\lambda\int_0^1 G_n(1,s)g(s)f(u(s))ds \\
&< (-1)^n(\mathcal{B}_n)^{-1}\int_0^1 G_n(1,s)g(s)u(s)ds
\end{aligned}$$

$$\leq u(1)(\mathcal{B}_n)^{-1}\int_0^1(-1)^nG_n(1,s)g(s)ds$$
$$= u(1),$$

which yields a contradiction. □

Theorem 5. *If for all* $u \in (0,\infty)$
$$\lambda > \frac{u}{\mathcal{A}_n f(u)},$$
then no positive solution exists to (1), (2).

Proof. For contradiction, assume that $u(x)$ is a positive solution to (1), (2). Then, we have that $(-1)^n(-D_{0^+}^{\alpha+2n}u(x)) = \lambda g(x)f(u(x)) \geq 0$. Therefore, by Lemma 5,

$$\begin{aligned} u(1) &= (-1)^n\lambda\int_0^1 G_n(1,s)g(s)f(u(s))ds \\ &> (-1)^n(\mathcal{A}_n)^{-1}\int_0^1 G_n(1,s)g(s)u(s)ds \\ &\geq u(1)(\mathcal{A}_n)^{-1}\int_0^1(-1)^ns^{\alpha-1}G_n(1,s)g(s)ds \\ &= u(1), \end{aligned}$$

which yields a contradiction. □

7. An example

Finally, we approximate bounds on the parameter λ for both the existence and nonexistence of positive solutions for a given example. We use Theorems 2, 4, and 5. Examples that use Theorems 3, 4, and 5 are demonstrated similarly.

Set $n = 2$, $m = 3$, $\alpha = 2.5$, $\beta = 1.5$, and $g(x) = x$. Notice that $g(x) \geq 0$ is continuous for $0 \leq x \leq 1$ and $\int_0^1 g(x)dx > 0$. We find that

$$\begin{aligned} G_0(1,s) &= \frac{1}{\Gamma(2.5)}\begin{cases} 1^{1.5}(1-s)^0 - (1-s)^{1.5}, & 0 \leq s < x \leq 1, \\ 1^{1.5}(1-s)^0, & 0 \leq x \leq s < 1 \end{cases} \\ &= \frac{1-(1-s)^{1.5}}{\Gamma(2.5)} \end{aligned}$$

and calculate

$$\begin{aligned} \mathcal{A}_2 &= \int_0^1(-1)^2s^{1.5}G_2(1,s)(s)ds \\ &= \int_0^1\left[-\int_0^1 G_1(1,r_1)G_{rf}(r_1,s)dr_1\right]s^{2.5}ds \\ &= \int_0^1\left[-\int_0^1\left(\int_0^1 -G_0(1,r_2)G_{rf}(r_2,r_1)dr_2\right)G_{rf}(r_1,s)dr_1\right]s^{2.5}ds \\ &\approx 0.03071, \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}_2 &= \int_0^1(-1)^2G_2(1,s)(s)ds \\ &= \int_0^1\left[-\int_0^1 G_1(1,r_1)G_{rf}(r_1,s)dr_1\right]sds \\ &= \int_0^1\left[-\int_0^1\left(\int_0^1 -G_0(1,r_2)G_{rf}(r_2,r_1)dr_2\right)G_{rf}(r_1,s)dr_1\right]sds \end{aligned}$$

$$\approx 0.04749.$$

By approximating $\mathcal{A}_2$ and $\mathcal{B}_2$ first, applying existence and nonexistence theorems is much simpler as they only have need for the liminfs and limsups of choice of $f(u)$.

Example 1. Here we provide an example using Theorem 2, 4, and 5. Set $f(u) = u\ln(u+1) + 2u$. Notice for $u \geq 0$, $f(u) \geq 0$ is continuous. Thus, our problem is

$$D_{0^+}^{6.5}u(x) + \lambda x(u\ln(u+1) + 2u) = 0, \quad 0 < x < 1, \tag{3}$$

$$\begin{gathered} u(0) = u'(0) = 0, \quad D_{0^+}^{1.5}(1) = 0, \\ D_{0^+}^{2.5}u(0) = D_{0^+}^{3.5}(1) = 0, \quad D_{0^+}^{4.5}(0) = D_{0^+}^{5.5}(1) = 0. \end{gathered} \tag{4}$$

We compute the liminfs and limsups for $f(u)/u = \ln(u+1) + 2$.

$$f_\infty = \liminf_{u\to\infty}(\ln(u+1)+2) = \infty, \qquad \mathcal{F}_0 = \limsup_{u\to 0^+}(\ln(u+1)+2) = 2,$$
$$f_0 = \liminf_{u\to 0^+}(\ln(u+1)+2) = 2, \qquad \mathcal{F}_\infty = \limsup_{u\to\infty}(\ln(u+1)+2) = \infty.$$

Then, we find

$$\frac{1}{\mathcal{A}_2 f_\infty} \approx \frac{1}{0.03031\cdot\infty} = 0,$$

and

$$\frac{1}{\mathcal{B}_2\mathcal{F}_0} \approx \frac{1}{0.04749\cdot 2} \approx 10.52853.$$

Next, for $u \in (0,\infty)$, we investigate

$$\frac{u}{\mathcal{B}_2 f(u)} = \frac{1}{\mathcal{B}_2(\ln(u+1)+2)}.$$

We approximate

$$\inf_{u\in(0,\infty)} \frac{1}{\mathcal{B}_2(\ln(u+1)+2)} = \frac{1}{\mathcal{B}_2}\inf_{u\in(0,\infty)}\frac{1}{\ln(u+1)+2} \approx \frac{1}{0.04749}(0) = 0.$$

Finally, for $u \in (0,\infty)$, we investigate

$$\frac{u}{\mathcal{A}_2 f(u)} = \frac{1}{\mathcal{A}_2(\ln(u+1)+2)}.$$

We approximate

$$\sup_{u\in(0,\infty)} \frac{1}{\mathcal{A}_2(\ln(u+1)+2)} = \frac{1}{\mathcal{A}_2}\sup_{u\in(0,\infty)}\frac{1}{\ln(u+1)+2} \approx \frac{1}{0.030307}\left(\frac{1}{2}\right) \approx 16.49784.$$

Thus, by Theorem 2, if $0 < \lambda < 16.49$, then (3), (4) has at least one positive solution. By Theorem 5, if $\lambda > 16.49$, then (3), (4) does not have a positive solution. We note that in this example Theorem 4 did not yield a meaningful result which was expected as Theorem 2 provides a positive solution for any choice of positive λ.

Remark 1. Lastly, we note that to find a meaning λ range for both nonexistence results and either existence result simultaneously with $g(x) = x$, we could choose a rational function $f(u)$ with a quadratic numerator and linear denominator. Thus, $f(u)/u$ is a rational function with a linear numerator and denominator leading to finite values for each liminf and limsup.

8. Conclusions

We studied Riemann-Liouville fractional differential equations with order $\alpha + 2n$ with $n \in \mathbb{N}$ that includes a parameter λ. The two-point boundary conditions are influenced by standard right-focal conditions. We established the Green's function for the boundary value problem by utilizing a convolution of a lower-order problem and standard right-focal problem by making a change of variables. Then, we inductively defined the Green's function for the higher order problem.

Next, we inductively proved many properties inherited by the Green's function from the lower-order problems. These properties permitted an application of the Guo-Krasnosel'skii Fixed Point Theorem to establish the existence of positive solutions based upon the size of λ. We also established the nonexistence of positive solutions based upon choice of λ via contradiction. Finally, we discussed a specific example and proved existence and nonexistence based on the choice of λ.

Future research may be to use the approach in this work to establish existence and nonexistence of positive solutions for other types of boundary conditions. Another avenue could be considering a singularity at $f(0)$.

The author has no competing interests.

Acknowledgments: The author is deeply indebted to the referees for providing constructive comments and helps in improving the contents of this article.

Conflicts of Interest: "The author declares no conflict of interest."

References

[1] Lyons, J. W., & Neugebauer, J. T. (2018). Two point fractional boundary value problems with a fractional boundary condition. *Fractional Calculus and Applied Analysis, 21*(2), 442-461.

[2] Graef, J. R., Henderson, J., & Yang, B. (2007). Positive solutions of a nonlinear higher order boundary-value problem. *Electronic Journal of Differential Equations (EJDE)[electronic only], 2007*, 1-10.

[3] Graef, J. R., Qian, C., & Yang, B. (2003). A three point boundary value problem for nonlinear fourth order differential equations. *Journal of Mathematical Analysis and Applications, 287*(1), 217-233.

[4] Graef, J. R., & Yang, B. (2006). Positive solutions to a multi-point higher order boundary value problem. *Journal of Mathematical Analysis and Applications, 316*(2), 409-421.

[5] Graef, J. R., Qian, C., & Yang, B. (2003). A three point boundary value problem for nonlinear fourth order differential equations. *Journal of Mathematical Analysis and Applications, 287*(1), 217-233.

[6] Eloe, P. W., & Neugebauer, J. T. (2016). Convolutions and Green's functions for two families of boundary value problems for fractional differential equations. *Electronic Journal of Differential Equations, 2016*, 1–13.

[7] Agarwal, R. P., O'Regan, D., & Staněk, S. (2010). Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations. *Journal of Mathematical Analysis and Applications, 371*(1), 57-68.

[8] Bai, Z., & Lü, H. (2005). Positive solutions for boundary value problem of nonlinear fractional differential equation. *Journal of Mathematical Analysis and Applications, 311*(2), 495-505.

[9] Graef, J. R., & Yang, B. (2014). Positive solutions to a three point fourth order focal boundary value problem. *The Rocky Mountain Journal of Mathematics, 44*(3), 937-951.

[10] Henderson, J., & Luca, R. (2017). Existence of positive solutions for a singular fractional boundary value problem. *Nonlinear Analysis: Modelling and Control, 22*(1), 99-114.

[11] Kaufmann, E. R., & Mboumi, E. (2008). Positive solutions of a boundary value problem for a nonlinear fractional differential equation. *Electronic Journal of Qualitative Theory of Differential Equations, 2008*, 1-11.

[12] Rehman, M. U., Khan, R. A., & Eloe, P. W. (2011). Positive solutions of nonlocal boundary value problem for higher order fractional differential system. *Dynamic Systems and Applications, 20*(2-3), 169–182.

[13] Wang, Y. (2025). A new result regarding positive solutions for semipositone boundary value problems of fractional differential equations. *Fractal and Fractional, 9*(2), 110.

[14] Wang, Y., & Wu, Y. (2020). Positive solutions of fractional differential equation boundary value problems at resonance. *Journal of Applied Analysis and Computation, 10*(6), 2459-2475.

[15] Yang, B. (2012). Upper estimate for positive solutions of the $(p, n-p)$ conjugate boundary value problem. *Journal of Mathematical Analysis and Applications, 390*(2), 535-548.

[16] Neugebauer, J. T., & WINGO, A. G. (2024). Positive solutions for a fractional boundary value problem with Lidstone like boundary conditions. *Kragujevac Journal of Mathematics, 48*(2), 309-322.

[17] Diethelm, K. (2010). *The Analysis of Fractional Differential Equations*, volume 2004 of Lecture Notes in Mathematics.

[18] Kilbas, A. A. (2006). *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204.

[19] Miller, K. S., & Ross, B. (1993). *An Introduction to the Fractional Calculus and Fractional Differential Equations*. A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York.

[20] Podlubny, I. (1999). *Fractional Differential Equations*. Academic Press, Inc., San Diego, CA, 1999.

[21] Guo, D., & Lakshmikantham, V. (1988). *Nonlinear Problems in Abstract Conesi Academic Press*. Inc., Boston, New York.

[22] Krasnosel'skii, M. A. (1964). *Topological Methods in the Theory of Nonlinear Integral Equations*. The Macmillan Co.: New York, NY, USA.

[23] Eloe, P. W., Lyons, J. W., & Neugebauer, J. T. (2015). An ordering on Green's functions for a family of two-point boundary value problems for fractional differential equations. *Communications on Pure and Applied Analysis*, *19*(3-4), 453-462.