

Article

Renormalization of unbounded stretched graphons

Ali Shojaei-Fard

PhD Independent Scholar, 1461863596 Marzdaran Blvd., Tehran, Iran; shojaeifa@yahoo.com

Received: 29 August 2025; Accepted: 12 October 2025; Published: 16 October 2025.

Abstract: The paper considers real valued stretched graphons defined on the Lebesgue measure space $([0, \infty), m)$. The topological space of these graph functions is equipped with the core Hopf algebra to assign renormalized values to unbounded stretched graphons.

Keywords: unbounded stretched graphons, topological core Hopf algebra, Hopf–Birkhoff factorization, deformation of star products

MSC: Primary: 16T05, 05C63. Secondary: 17B63.

1. Introduction

On the one hand, the graphon theory in infinite combinatorics studies convergence properties of sequences of dense or sparse weighted finite graphs or networks in terms of assigning suitable graphon models and random graphs to their graph limits [1–5]. These graphon models are modifiable in terms of changing the ground σ -finite measure spaces or applying rescaling techniques [6–9]. The graphon theory and its extensions have provided interesting mathematical tools to deal with various problems in applied and functional analysis [5,7,8,10,11], (theoretical) computer science [5,6,9,12,13] and mathematical physics [11–15]. On the other hand, the renormalization theory in mathematical physics is a well-known platform for the extraction of finite values from divergent Feynman integrals / diagrams and divergent perturbative series of higher loop order Feynman diagrams which contribute to Green’s functions in quantum field theories [13,16–18]. The method of the Bogoliubov–Parasiuk–Hepp–Zimmermann (BPHZ), formulated on the basis of the (Bogoliubov–)Zimmermann’s forest formula, performs perturbative renormalization of Feynman integrals / diagrams in terms of an inductive step by step removal of sub-divergences [19]. The interrelation between analytic and combinatorial nature of the perturbative renormalization theory can be considered by the formalism of the renormalization Hopf algebra [18,20–23]. The Connes–Kreimer–Marcolli theory describes the BPHZ perturbative renormalization on the basis of the Riemann–Hilbert problem / correspondence and the theory of motives [17,18,20,22,24]. The method of (stretched) Feynman graphons formulated a new topological extension of the renormalization Hopf algebraic formalism. This setting recovers Feynman graph limits for the study of the convergence properties of sequences of higher loop order Feynman diagrams which contribute to divergent perturbative series in the structure of Green’s functions or solutions of their fixed point equations [11,14,15,25]. The method of (stretched) Feynman graphons enables us to apply fundamental tools of theoretical computer science such as random graphs, homomorphism densities, Halting problem and Kolmogorov complexity for the extraction of meaningful information beyond perturbation theory [12,13,25].

The present work applies this background to address some new interconnections between the graphon theory and the renormalization theory where a new practical setting is introduced and developed to assign renormalized values to unbounded stretched graphons defined on any σ -finite measure space.

1.1. Basics of the graphon theory

For the Lebesgue measure space $([0, 1], m)$, any $[0, 1]$ -valued symmetric Lebesgue measurable function W defined on $[0, 1] \times [0, 1]$ is called *graphon*. For any invertible Lebesgue measure preserving transformation ρ on

$[0, 1]$, such as $u \mapsto nu \pmod{1}$, $n \geq 1$, the new function $W^\rho(x, y) := W(\rho(x), \rho(y))$ is called *labeled graphon*. The set $\mathcal{W}^{([0,1],m)}([0, 1])$ of graphons is equipped with a semi-norm, called *cut-norm*, given by

$$\|W\|_{cut} := \sup_{A,B \subseteq [0,1]} \left| \int_{A \times B} W(x,y) dx dy \right|, \tag{1}$$

which can be extended to a pseudo-metric, called *cut-distance*, given by

$$d_{cut}(W_1, W_2) = \inf_{\rho_1, \rho_2} \sup_{A,B \subseteq [0,1]} \left| \int_{A \times B} \left(W_1^{\rho_1}(x,y) - W_2^{\rho_2}(x,y) \right) dx dy \right|, \tag{2}$$

such that A, B are Lebesgue measurable subsets in $[0, 1]$ and ρ_1, ρ_2 are (invertible) Lebesgue measure preserving transformations on $[0, 1]$. Graphons W_1, W_2 are called *weakly isomorphic* (i.e. $W_1 \approx W_2$) iff there exist Lebesgue measure preserving transformations τ_1, τ_2 on $[0, 1]$ such that $W_1^{\tau_1} = W_2^{\tau_2}$ almost everywhere with respect to the Lebesgue measure. Because of

$$W_1 \approx W_2 \Leftrightarrow d_{cut}(W_1, W_2) = 0, \tag{3}$$

up to the weakly isomorphism, the quotient space $\mathcal{W}_{\approx}^{([0,1],m)}([0, 1])$ is a metric space. Elements of this quotient space are called *unlabeled graphons* [4,5].

Lemma 1. *The set of finite weighted graphs embeds in $\mathcal{W}_{\approx}^{([0,1],m)}([0, 1])$.*

Proof. Consider any finite graph G with the vertex set $V(G)$, such that $|V(G)| = n$ and each vertex v_i is weighted by some $\alpha_i \in \mathbb{R}$, and the edge set $E(G)$ such that any edge $v_i v_j$ is weighted by $\beta_{ij} \in \mathbb{R}$. For any partition $\sigma := (I_1, \dots, I_n)$ of $[0, 1]$ with $m(I_k) = \frac{|\alpha_k|}{\sum_{i=1}^n |\alpha_i|}$, the graph function

$$W_G^\sigma(x, y) = \begin{cases} \beta_{ij} - \lfloor \beta_{ij} \rfloor, & (x,y) \in I_i \times I_j \\ 0, & \text{otherwise} \end{cases}, \tag{4}$$

is a labeled graphon associated to G . While relabeling for the graph is given by permutation of its vertices, any measure preserving bijection ρ on $[0, 1]$ defines a relabeling $W^{\rho(\sigma)}$ associated to G such that $\rho(\sigma) := (\rho(I_1), \dots, \rho(I_n))$ is a partition of $[0, 1]$. Up to the weakly isomorphism, the unique class

$$W_G := [W_G^\sigma]_{\approx} = \left\{ W^{\rho(\sigma)} : \rho \text{ Lebesgue measure preserving transformation} \right\}, \tag{5}$$

is called the *canonical unlabeled graphon* associated to G . \square

The interrelation between the theory random graphs and the graphon theory is started by extracting Erdos–Renyi random graph models $\mathbb{G}(n, \pi, W)$, $n \geq 1$, from graphons W defined on any probability measure space (Ω, π) . For the vertex set $V_n = \{v_1, \dots, v_n\}$ of $\mathbb{G}(n, \pi, W)$, each vertex v_i assigns a type variable $x_{v_i} \in \Omega$ distributed by π independent of each other such that with the probability $W(x_{v_i}, x_{v_j})$, there exists an edge $v_i v_j$ in $\mathbb{G}(n, \pi, W)$ independent of each other. The random graphs $\mathbb{G}(n, \pi, W)$ and stochastic block model have the same distribution. Thanks to Sampling Lemma, for the probability measure space $([0, 1], m)$, with the probability at least $1 - 2 \exp(-n/(2 \log n))$, we get $d_{cut}(\mathbb{G}(n, m, W), W) \leq \frac{22}{\sqrt{\log n}}$. For weakly isomorphic graphons W_1, W_2 , the random graphs $\mathbb{G}(n, \pi, W_1)$ and $\mathbb{G}(n, \pi, W_2)$ have the same distribution for any $n \geq 1$. It shows that for any sequence $\{W_k\}_{k \geq 1}$ of graphons on (Ω, π) which converges to W with respect to the metric (2), the sequence $\{\mathbb{G}(n, \pi, W_k)\}_{k \geq 1}$ of random graphs converges in distribution to the random graph $\mathbb{G}(n, \pi, W)$ for any $n \geq 1$. In relation to unbounded graphons on the probability measure space $([0, 1], m)$, for any graphon W and $n \geq 1$, there exists a Lebesgue measurable symmetric step function $U_n : \Omega \times \Omega \rightarrow \mathbb{R}$ with n steps such that $\|W - U_n\|_{cut} \leq \frac{2}{\sqrt{\log n}}$ [1–5,9].

The space $\mathcal{W}_{\approx}^{([0,1],m)}([0, 1])$ is a complete compact Hausdorff separable metric space which topologically completes the space of finite weighted graphs. Any sequence $\{G_n\}_{n \geq 1}$ of finite dense graphs with increasing

vertex sets has a subsequence $\{G_{n_i}\}_{i \geq 1}$ which converges to a graph limit with respect to the metric $d_{cut}(G_{n_i}, G_{n_j}) := d_{cut}(W_{G_{n_i}}, W_{G_{n_j}})$. The graph limit of this subsequence is represented by a non-trivial unique unlabeled graphon class such that

$$\lim_{i \rightarrow \infty} G_{n_i} = X \Leftrightarrow \lim_{i \rightarrow \infty} W_{G_{n_i}} = W_X. \tag{6}$$

The graph limit of any Cauchy sequence $\{H_n\}_{n \geq 1}$ of finite sparse graphs with increasing vertex sets is represented by the zero graphon. It is possible to assign another non-trivial graph limit in terms of (i) using rescaled techniques such as $W_{H_n} \mapsto \frac{1}{\|W_{H_n}\|_{cut}} W_{H_n}$ or $W_{H_n} \mapsto \frac{1}{\|W_{H_n}\|_p} W_{H_n}$ with respect to L^p norms for $p \geq 1$, or (ii) replacing $([0, 1], m)$ with other suitable σ -finite measure space $\mathfrak{U} := (\Omega \subseteq [0, \infty), \mu)$. The generalizations of graphons, defined on arbitrary σ -finite measure spaces, are called *stretched graphons*. They are real valued symmetric μ -measurable functions defined on $\Omega \times \Omega$. For the case of mathematical physics, they are called stretched Feynman graphons. Up to the weakly isomorphism, the quotient space $\mathcal{W}_{\approx}^{(\Omega \subseteq [0, \infty), \mu)}(\mathbb{R})$ of bounded and unbounded stretched graphons on Ω is a complete Hausdorff metric space [1,2,6,9,10,14,15].

For the rest of this paper, we work on weakly isomorphic classes of stretched graphons $[\cdot]_{\approx}$ in $\mathcal{W}_{\approx}^{\mathfrak{U}}(\mathbb{R})$, namely real valued unlabeled stretched graphons. We simplify the presentation by using W instead of $[W]_{\approx}$ and calling them stretched graphons.

For $p \geq 1$, W is called L^p -stretched graphon if

$$\|W\|_{p,cut} := \inf_{\rho} \sup_{A,B \subseteq \Omega} \left(\int_{A \times B} |W^{\rho}(x,y)|^p d\mu(x)d\mu(y) \right)^{1/p} < \infty, \tag{7}$$

such that A, B are μ -measurable subsets in Ω and ρ is any μ -measure preserving transformation on Ω . We have $\|W\|_{cut} \leq \|W\|_{1,cut} \leq \|W\|_{2,cut} \leq \|W\|_{\infty}$. Its corresponding pseudo-metric is given by

$$d_{p,cut}(W_1, W_2) = \|W_1 - W_2\|_{p,cut}, \tag{8}$$

which is a metric up to the weakly isomorphism. It can be checked that

$$d_{1,cut}(W_1, W_2) = 0 \Leftrightarrow d_{cut}(W_1, W_2) = 0. \tag{9}$$

It means that convergence in $d_{1,cut}$ implies convergence in d_{cut} , however, in general, it might fail for $p > 1$ where stretched graphons could be unbounded. For $p > 1$ and $c > 0$, a sequence $\{G_n\}_{n \geq 1}$ of finite weighted graphs is called a c -upper L^p regular sequence if for any $\epsilon > 0$ there exists some n_{ϵ} such that for any $n \geq n_{\epsilon}$, G_n is $(c + o(1), o(1))$ -upper L^p regular. It means that for any $n \geq n_{\epsilon}$, $\frac{G_n}{\|G_n\|_1}$ has L^p norm at most c after averaging over any partition of vertices of G_n into blocks of at least $\epsilon|V(G_n)|$ in size. The L^p upper regularity implies unboundedness. For any c -upper L^p regular sequence $\{G_n\}_{n \geq 1}$, there exists a graph limit X represented by a L^p -stretched graphon W_X with $\|W_X\|_p \leq c$ such that, [7-9];

$$\liminf_{n \rightarrow \infty} d_{cut} \left(\frac{G_n}{\|G_n\|_1}, X \right) = 0 \Leftrightarrow \lim_{n \rightarrow \infty} d_{cut} \left(\frac{W_{G_n}}{\|W_{G_n}\|_1}, W_X \right) = 0. \tag{10}$$

1.2. From combinatorial representation of the renormalization Hopf algebra to (stretched) graphons

Feynman diagrams are a certain class of finite weighted decorated oriented graphs with nested loops without self-loop which encode interactions of particles in a quantum field theory Φ . The edge set $\Gamma^{[1]} = \Gamma_{int}^{[1]} \sqcup \Gamma_{ext}^{[1]}$ of any Feynman diagram Γ is divided in two types, namely internal edges which have beginning and ending vertices and external edges which have only beginning or ending vertices. A Feynman diagram is called 1PI if it remains connected after removing a single internal edge. A subgraph γ in Γ is a graph such that $\gamma_{int}^{[1]} \subseteq \Gamma_{int}^{[1]}$, $\gamma^{[0]} \subseteq \Gamma^{[0]}$ and for each vertex $v \in \gamma^{[0]}$, $res_{\gamma}(v) = res_{\Gamma}(v)$ [16,17].

Feynman integrals are iterated integrals on the momentum parameter which varies between zero and infinity. Sub-divergences in Feynman integrals are represented by nested loops in Feynman diagrams and the (Bogoliubov-)Zimmermann's forest formula for the step by step removal of nested loops is encapsulated by Kreimer's renormalization coproduct on the graded \mathbb{K} -vector space $\bigoplus_{n=0}^{\infty} H^{(n)}$ such that for each $n \geq 1$, $H^{(n)}$ is

the \mathbb{K} -vector space generated by 1PI Feynman diagrams with the loop number n or products of 1PI Feynman diagrams with the overall loop number n . For $n = 0$, $H^{(0)} = \mathbb{K}$. The Kreimer’s renormalization coproduct and its corresponding antipode are given by

$$\Delta_{FG}(\Gamma) = \mathbb{I} \otimes \Gamma + \Gamma \otimes \mathbb{I} + \sum_{\gamma} \gamma \otimes \Gamma/\gamma, S_{FG}(\Gamma) = -\Gamma - \sum_{\gamma} S(\gamma)\Gamma/\gamma, S(\mathbb{I}) = \mathbb{I}, \tag{11}$$

such that \mathbb{I} is the symbol for the class of zero loop graphs and empty graph, and the sum is over all disjoint unions of those non-trivial 1PI subgraphs of Γ that their corresponding Feynman sub-integrals have some sub-divergences. The graph Γ/γ is obtained by shrinking all internal edges of γ into a new vertex v_{γ} in Γ such that external edges of γ are attached to v_{γ} . The resulting Hopf algebra, presented by $H_{FG}(\Phi)$ and called *renormalization Hopf algebra* encodes the BPHZ renormalization on the basis of the Riemann–Hilbert problem [17,18,20,22,24].

Rooted trees are basic tools for the construction of the combinatorial version of the renormalization Hopf algebra. A *rooted tree* is a simple simply connected oriented graph with a particular vertex, called the root, which has only outgoing edges while every other vertex has exactly one incoming edge. There exists at least a path from the root to any leaf. A rooted tree embedded in the plane is called planar, and otherwise it is called non-planar. Up to the isomorphism of trees, the renormalization Hopf algebra is combinatorially reconstructed on the graded polynomial algebra $\bigoplus_{n=0}^{\infty} A_{rt}^{(n)}$ of non-planar rooted trees with concatenation as its multiplication. For each $n \geq 1$, $A_{rt}^{(n)}$ is the \mathbb{K} -vector space generated by non-planar rooted trees with the vertex number n or forests of non-planar rooted trees with the overall vertex number n . For $n = 0$, $A_{rt}^{(0)} = \mathbb{K}$. The reformulation of the coproduct and antipode (11) on this graded polynomial algebra is given in terms of the notion of *admissible cut*. An admissible cut on a rooted tree t is a subset C of its edge set $E(t)$ such that along any path from the root to any of its leaf, there exists at most one element of C in the path. We get

$$\Delta_{ren}(t) = \mathbb{I} \otimes t + t \otimes \mathbb{I} + \sum_C R^C(t) \otimes P^C(t), S(t) = -t - \sum_C S(R^C(t))P^C(t), S(\mathbb{I}) = \mathbb{I}, \tag{12}$$

such that \mathbb{I} is the empty tree and the sum is over all non-trivial admissible cuts of t where C divides t into two parts. The subtree $R^C(t)$ which contains the root of t and a forest $P^C(t)$ of the remaining subtrees of t . The resulting combinatorial Hopf algebra, presented by H_{CK} and called *Connes–Kreimer Hopf algebra*, is graded connected unital counital free commutative non-cocommutative Hopf algebra [26,27].

Lemma 2. *The renormalization Hopf algebra has a universal combinatorial representation [18,22,26,27].*

Proof. On the one hand, any Feynman diagram Γ has a rooted tree (or forest) representation t_{Γ} such that the loop number of Γ identifies the number of vertices of t_{Γ} . Each vertex v_{γ} in t_{Γ} is decorated by a (1PI) primitive Feynman subdiagram γ of Γ . For 1PI primitive Feynman subdiagrams γ_1, γ_2 of Γ , if $\gamma_1 \subseteq \gamma_2$, then there exists an edge from the vertex v_{γ_1} to the vertex v_{γ_2} in t_{Γ} . If $\gamma_1 \not\subseteq \gamma_2$ and $\gamma_2 \not\subseteq \gamma_1$, then there is no edge between vertices v_{γ_1} and v_{γ_2} . If Γ has overlapping loops, then t_{Γ} is a linear combination of decorated non-planar rooted trees. Therefore the renormalization Hopf algebra $H_{FG}(\Phi)$ embeds, by an injective Hopf algebra homomorphism, in a decorated version of the Connes–Kreimer Hopf algebra presented by $H_{CK}(\Phi)$ such that vertices of rooted trees are decorated by (1PI) primitive Feynman (sub)diagrams.

On the other hand, let B^+ be a linear operator on H_{CK} which maps each forest $t_1 \dots t_n$ to a new non-planar rooted tree t by adding a new root \bullet together with n new edges which connect \bullet to the roots of t_1, \dots, t_n . For any $t = B^+(t_1 \dots t_n)$, we get $\Delta_{ren}(t) = (id \otimes B^+)\Delta_{ren}(t_1 \dots t_n) + t \otimes \mathbb{I}$. Thanks to the Hochschild cohomology of commutative Hopf algebras, consider a category of pairs (H, L) of a commutative (graded) Hopf algebra and a linear operator $L : H \rightarrow H$ which satisfies the equation

$$\Delta_H L(x) = (id \otimes L)\Delta_H(x) + L(x) \otimes \mathbb{I}. \tag{13}$$

Morphisms of this category are Hopf algebra homomorphisms $f : H_1 \rightarrow H_2$ such that $L_2 \circ f = f \circ L_1$. The object (H_{CK}, B^+) is the universal object. It means that for any object (H, L) , there exists a unique Hopf algebra homomorphism $f_H : H_{CK} \rightarrow H$ such that $L \circ f_H = f_H \circ B^+$.

The injective Hopf algebra homomorphism $\Gamma \mapsto t_\Gamma$ together with the unique Hopf algebra homomorphism $f_{H_{FG}(\Phi)} : H_{CK} \rightarrow H_{FG}(\Phi)$ determine the quotient Hopf algebra $H_{CK}(\Phi)/I_\Phi$ for some Hopf ideal I_Φ as the universal representation of $H_{FG}(\Phi)$. \square

Lemma 3. *The Connes–Kreimer Hopf algebra of non-planar rooted trees topologically embeds in the metric space $\mathcal{W}_{\approx}^{([0,1),m)}(\mathbb{R})$.*

Proof. Thanks to Lemma 1, the adjacency matrix of any $t \in H_{CK}$ determines its unique canonical unlabeled graphon $W_t \in \mathcal{W}_{\approx}^{([0,1),m)}(\mathbb{R})$. For a fixed $p \geq 1$, we apply (7) to define a distance between non-planar rooted trees given by

$$d_{p,cut}(s, t) = d_{cut} \left(\frac{W_s}{\|W_s\|_{p,cut}}, \frac{W_t}{\|W_t\|_{p,cut}} \right). \tag{14}$$

The graph limits of all Cauchy sequences in H_{CK} are represented by some elements in $\mathcal{W}_{\approx}^{([0,1),m)}(\mathbb{R})$. Consider any recursive equation $X = \mathbb{I} + \sum_{n=1}^{\infty} c^n B_\bullet^+(X^{n+1})$ with $c > 1$ in H_{CK} such that $X = \sum_{n=0}^{\infty} c^n X_n$ is given by the recursive relations

$$X_n = \sum_{j=1}^n B_\bullet^+ \left(\sum_{k_1+\dots+k_{j+1}=n-j, k_i \geq 0} X_{k_1} \dots X_{k_{j+1}} \right), \quad X_0 = \mathbb{I}. \tag{15}$$

It can be presented by a stretched graphon $W_X \in \mathcal{W}_{\approx}^{([0,1),m)}(\mathbb{R})$ such that

$$\lim_{m \rightarrow \infty} Y_m = X \Leftrightarrow \lim_{m \rightarrow \infty} \frac{W_{Y_m}}{\|W_{Y_m}\|_{p,cut}} = W_X, \tag{16}$$

where $Y_m = \sum_{k=1}^m c^k X_k$. The stretched graphon $W_{Y_m} \in \mathcal{W}_{\approx}^{([0,1),m)}(\mathbb{R})$ is determined by stretched graphons W_{X_j} defined on subintervals $I_j \subset [0, 1)$ with $m(I_j) = \frac{c^j}{\sum_{k=1}^m c^k}$.

For any $x \in H_{CK} \otimes H_{CK}$,

$$\|x\|_{p,cross} = \inf \left\{ \sum_{i=1}^n \|W_{s'_i}\|_{p,cut} \|W_{s''_i}\|_{p,cut}, \quad x = \sum_{i=1}^n s'_i \otimes s''_i \right\}, \tag{17}$$

defines a norm structure such that graph limits of Cauchy sequences in $H_{CK} \otimes H_{CK}$ are represented by some elements in $\mathcal{W}_{\approx}^{([0,1),m)}(\mathbb{R})$. It leads us to describe Δ_{ren} as a linear bounded operator between completed normed spaces which should be continuous. Therefore for any Cauchy sequence $\{t_n\}_{n \geq 1}$ in H_{CK} which converges to the graph limit X , $\Delta_{ren}(X)$ and $S_{ren}(X)$ are defined as graph limits of the sequences $\{\Delta_{ren}(t_n)\}_{n \geq 1}$ and $\{S_{ren}(t_n)\}_{n \geq 1}$ such that

$$\lim_{n \rightarrow \infty} \frac{W_{\Delta_{ren}(t_n)}}{\|W_{\Delta_{ren}(t_n)}\|_{p,cut}} = W_{\Delta_{ren}(X)}, \quad \lim_{n \rightarrow \infty} \frac{W_{S_{ren}(t_n)}}{\|W_{S_{ren}(t_n)}\|_{p,cut}} = W_{S_{ren}(X)}. \tag{18}$$

\square

1.3. Achievements

Unbounded stretched graphons are useful tools to develop the graphon theory for the study of the convergence properties of sequences of sparse graphs in applied mathematics and mathematical physics [3,5–15,25]. The original achievement of this research is to introduce and develop a new concept of "renormalization" for the study of unbounded stretched graphons defined on the Lebesgue measure space $\mathcal{U} = ([0, \infty), m)$. Thanks to the formalism of the Hopf algebraic renormalization, this research provides a new platform to associate meaningful renormalized values to unbounded stretched graphons which contribute as the infinite direct sums of (L^p) -stretched graphons or limits of Cauchy sequences of (L^p) -stretched graphons in $\mathcal{W}_{\approx}^{\mathcal{U}}(\mathbb{R})$ with respect to the metrics (2) and (8).

- The new operation of "direct sum" is introduced for (L^p) -stretched graphons in $\mathcal{W}_{\approx}^{\mathbb{M}}(\mathbb{R})$ to generate a class of unbounded stretched graphons. See Definition 1, Lemmas 4, 5.

- New topological core Hopf algebras $\mathcal{H}^{\mathbb{M},cut}$ and $\mathcal{H}_p^{\mathbb{M},cut}$ are formulated on the spaces of (L^p) -stretched graphons to address some new properties of unbounded stretched graphons. See Theorem 1 and Corollaries 1, 2, 3, 4.

- The BPHZ renormalization is formulated on $\mathcal{H}^{\mathbb{M},cut}$ underlying the Hopf–Birkhoff factorization to associate renormalized values to unbounded stretched graphons. See Theorem 2.

- Thanks to the Rota–Baxter property of (A_{dr}, R_{ms}) , the noncommutative associative convolution algebra $C_{\lambda} := (L(\mathcal{H}^{\mathbb{M},cut}, A_{dr}), \circ_{\lambda})$ is deformed to introduce a new family of Poisson structures. The symplectic geometry background of these Poisson structures are applied to show an analytic continuation for the map $(U, z) \mapsto \phi_{+}^z(U)$ in any infinitesimal punctured disk in the complex plane around $z = 0$. See Lemma 6 and Corollaries 6 and 7.

2. Topological core Hopf algebra of stretched graphons

Unbounded stretched graphons can be generated by an infinite family of bounded (L^p) -stretched graphons in $\mathcal{W}^{\mathbb{M}}(\mathbb{R})$ for any $p \geq 1$.

Definition 1. • For any (L^p) -stretched graphon $W \in \mathcal{W}^{\mathbb{M}}(\mathbb{R})$ and $c \in \mathbb{R}$, the stretched graphon $cW \in \mathcal{W}^{\mathbb{M}}(\mathbb{R})$ is called a rescaled version of W defined on a subinterval $I \subset \Omega$ with $m(I) = |c|$.

- For any family $\{U_n\}_{n \geq 1}$ of (L^p) -stretched graphons in $\mathcal{W}^{\mathbb{M}}(\mathbb{R})$ together with real values $\{c_n\}_{n \geq 1}$, define a new stretched graphon $U : \bigsqcup_{n=1}^{\infty} I_n \times \bigsqcup_{n=1}^{\infty} I_n \subseteq \Omega \times \Omega \rightarrow \mathbb{R}$ in $\mathcal{W}^{\mathbb{M}}(\mathbb{R})$ such that (i) for $r \neq s$, $I_r \cap I_s = \emptyset$, (ii) for any $r \geq 1$, the stretched graphon $c_r U_r$ is a rescaled version of U_r defined on a subinterval I_r with $\mu(I_r) = |c_r|$, and (iii) for any $r \geq 1$,

$$U|_{I_r \times I_r} = \begin{cases} U_r(x,y), & c_r \geq 0 \\ -U_r(x,y), & c_r < 0. \end{cases} \tag{19}$$

The unbounded stretched graphon U is called a *direct sum* of the rescaled versions of U_r of weights c_r and it is presented by $U := c_1 U_1 + \dots + c_n U_n + \dots$

Lemma 4. The pair $(\mathcal{W}^{\mathbb{M}}(\mathbb{R}), +)$ is a commutative group.

Proof. The zero graphon $W_{\mathbb{I}}$ is its unit. For any stretched graphon W defined on $I \times I \subseteq [0, \infty) \times [0, \infty)$, consider the equation $W + W' = W_{\mathbb{I}}$ such that $W + W' : I \sqcup I' \times I \sqcup I' \rightarrow \mathbb{R}$ with $I \cap I' = \emptyset$, $m(I) = m(I') = 1$. We apply Lebesgue measure preserving transformations $x \mapsto x - \alpha$ and $x \mapsto nx \pmod{1}$ to project I' onto I or vice versa to replace the direct sum with its corresponding pointwise summation. Therefore $W + W'$, which is weakly isomorphic to $W_{\mathbb{I}}$, is also weakly isomorphic to the pointwise summation $W(x, y) + W'(x, y)$ on $I \times I$ or on $I' \times I'$. It means that $W(x, y) + W'(x, y) = 0$ (i.e. $W' = -W$) Lebesgue almost everywhere on $I \times I$ or on $I' \times I'$. □

Lemma 5. The cut-distance metric (2) and its L^p -extension (8) are well-defined for (infinite) direct sums of stretched graphons.

Proof. For direct sums U, V defined on $\bigsqcup_{n=1}^{\infty} I_n \times \bigsqcup_{n=1}^{\infty} I_n$ and $\bigsqcup_{n=1}^{\infty} J_n \times \bigsqcup_{n=1}^{\infty} J_n$, and also $K := \bigsqcup_{n=1}^{\infty} I_n \cap \bigsqcup_{n=1}^{\infty} J_n$, we have the following situations.

- If $(x, y) \in K \times K$, then $(U - V)(x, y) = U(x, y) - V(x, y)$. (i.e. pointwise summation)
- If $(x, y) \in \bigsqcup_{n=1}^{\infty} I_n \setminus K \times \bigsqcup_{n=1}^{\infty} I_n \setminus K$, then $(U - V)(x, y) = U(x, y)$,
- If $(x, y) \in \bigsqcup_{n=1}^{\infty} J_n \setminus K \times \bigsqcup_{n=1}^{\infty} J_n \setminus K$, then $(U - V)(x, y) = V(x, y)$.

Therefore

$$\begin{aligned}
 d_{p,cut}(U, V) &= \left\| U + (-V) \right\|_{p,cut} \\
 &= \left\| U|_{\bigsqcup_{n=1}^{\infty} I_n \setminus K \times \bigsqcup_{n=1}^{\infty} I_n \setminus K} \right\|_{p,cut} + \left\| -V|_{\bigsqcup_{n=1}^{\infty} J_n \setminus K \times \bigsqcup_{n=1}^{\infty} J_n \setminus K} \right\|_{p,cut} + \left\| (U - V)|_{K \times K} \right\|_{p,cut}. \quad (20)
 \end{aligned}$$

□

The core Hopf algebra is introduced as an extension of the renormalization Hopf algebra where the coproduct (11) is replaced by the new one which involves almost all subgraphs in any Feynman diagram. It is the \mathbb{K} -vector space generated by connected 1PI Feynman diagrams which is graded by the loop number. The disjoint union is its commutative multiplication where the empty graph \mathbb{I} , as the equivalence class of all graphs with the zero loop number, is the unit of this multiplication. Its coproduct is given by

$$\Delta_{core}(\Gamma) = \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \sum_{\gamma^{core}} \gamma^{core} \otimes \Gamma / \gamma^{core}, \quad (21)$$

such that the sum is over all non-trivial subgraphs $\gamma^{core} = \bigsqcup_i \gamma_i$ where γ_i are 1PI subgraphs and Γ / γ^{core} is a new graph inside Γ which completes γ^{core} . It is obtained by shrinking any γ_i to a new vertex v_{γ_i} in Γ where at the end of shrinking all γ_i , any 2-valent vertex is replaced by an edge. The renormalization Hopf algebra can be described as a quotient of the core Hopf algebra [28].

Theorem 1. *The topological space $\mathcal{W}_{\approx}^{\mathbb{M}}(\mathbb{R})$ with respect to the metrics (2) or (8) can be equipped with the core Hopf algebra structure.*

Proof. We apply §1.1 and Lemmas 4, 5 to present the proof for the metric (2). The structure of the proof for the metric (8) is the same.

Define $\mathcal{H}^{\mathbb{M}}$ as the free commutative algebra generated by stretched graphons $\{W_G, W_X : G, X\} \subset \mathcal{W}_{\approx}^{\mathbb{M}}(\mathbb{R})$ associated to finite graphs G without self-loops and graph limits X of Cauchy sequences of finite graphs with respect to the metric (2). We have $\mathcal{H}^{\mathbb{M}} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\mathbb{M},(n)}$ such that for each $n \geq 1$, $\mathcal{H}^{\mathbb{M},(n)}$ is the \mathbb{K} -vector space generated by W_G with $|G| = n$ with respect to the direct sum given by Definition 1. We have $\mathcal{H}^{(0)} = \mathcal{H}^{(1)} = \mathbb{K}$, and for any graph limit X , $W_X \in \mathcal{H}^{(\infty)}$. For weakly isomorphic graphs G_1, G_2 with the vertex numbers larger than one, their corresponding stretched graphons $W_{G_1}, W_{G_2} \in \mathcal{W}_{\approx}^{\mathbb{M}}(\mathbb{R})$ are weakly isomorphic which means that there exists some $n \geq 2$ such that $W_{G_1}, W_{G_2} \in \mathcal{H}^{\mathbb{M},(n)}$.

The core coproduct (21) can be lifted onto $\mathcal{H}^{\mathbb{M}}$ by

$$\Delta(W_G) = \sum_{H^{core}} W_{H^{core}} \otimes W_{G/H^{core}}, \quad S(W_G) = -W_G - \sum_{W_{H^{core}}} S(W_{H^{core}})W_{G/H^{core}}, \quad S(W_{\mathbb{I}}) = W_{\mathbb{I}}, \quad (22)$$

such that, up to the isomorphism of graphs and weakly isomorphism of stretched graphons, the sum is over all non-trivial subgraphs $H^{core} = \bigsqcup_i H_i$ where H_i are connected subgraphs in G and G/H^{core} is defined by shrinking any H_i to a new vertex v_{H_i} in G where at the end of shrinking all H_i , any 2-valent vertex is replaced by an edge. In addition, for the graph limit X of a Cauchy sequence $\{G_n\}_n$ of finite graphs and the corresponding stretched graphon W_X as the limit of the sequence $\left\{ \frac{1}{\|W_{G_n}\|_{cut}} W_{G_n} \right\}_{n \geq 1}$, the stretched graphons $\Delta(W_X)$ in $\mathcal{H}^{\mathbb{M}} \otimes \mathcal{H}^{\mathbb{M}}$ and $S(W_X)$ in $\mathcal{H}^{\mathbb{M}}$ are defined as the limits of the sequences

$$\left\{ \frac{1}{\|W_{G_n}\|_{cut}} \Delta(W_{G_n}) \right\}_{n \geq 1}, \quad \left\{ \frac{1}{\|W_{G_n}\|_{cut}} S(W_{G_n}) \right\}_{n \geq 1}. \quad (23)$$

We have

$$\Delta(W_X) = \sum_{W_Z} W_Z \otimes W_{X/Z}, \quad (24)$$

such that the sum is over all stretched graphons $W_Z \in \mathcal{W}_{\approx}^{\mathbb{M}}(\mathbb{R})$ such that $Dom(W_Z) \subseteq Dom(W_X)$ and W_Z is weakly isomorphic to $W_X|_{Dom(W_Z)}$. The stretched graphon $W_{X/Z}$ is defined on $Dom(W_X)$ which is zero on

$Dom(W_Z)$ and is weakly isomorphic to W_X on $Dom(W_X) \setminus Dom(W_Z)$. Furthermore, stretched graphons $W_{\Delta(X)}$ and $W_{S(X)}$ are defined as the limits of the sequences $\left\{ \frac{1}{\|W_{G_n}\|_{cut}} W_{\Delta_{core}(G_n)} \right\}_{n \geq 1}$ and $\left\{ \frac{1}{\|W_{G_n}\|_{cut}} W_{S_{core}(G_n)} \right\}_{n \geq 1}$ such that $\mathcal{H}^{\mathbb{U}} \otimes \mathcal{H}^{\mathbb{U}}$ is equipped with the cross norm

$$\|x\|_{cross} = inf \left\{ \sum_{i=1}^n \|W_{G'_i}\|_{cut} \|W_{G''_i}\|_{cut}, x = \sum_{i=1}^n W_{G'_i} \otimes W_{G''_i} \right\}. \tag{25}$$

Let W be a stretched graphon in $\mathcal{W}^{\mathbb{U}}(\mathbb{R})$ which is not the representation of any graph limit. A suitable invertible Lebesgue measure preserving transformations on $[0, \infty)$ projects W onto its corresponding stretched graphon $\tilde{W} \in \mathcal{W}^{([0,1],m)}(\mathbb{R})$ [9]. Thanks to [4,5], for any $n \geq 1$, there exists a Lebesgue measurable symmetric step function $\tilde{U}_n \in \mathcal{W}^{([0,1],m)}(\mathbb{R})$ such that $\|\tilde{W} - \tilde{U}_n\|_{cut} \leq \frac{2}{\sqrt{\log n}}$. It means that with the probability at least $1 - 2exp(-n/(2\log n))$, we get $d_{cut}(\mathbb{G}(n, m, \tilde{U}_n), \tilde{W}) \leq \frac{22}{\sqrt{\log n}}$ where $\mathbb{G}(n, m, \tilde{U}_n)$ is the random graph with the vertex set $\{x_1, \dots, x_n\} \subset [0, 1]$, uniformly selected, such that with the probability $\|\tilde{U}_n\|_1^{-1} \tilde{U}_n(x_i, x_j)$ there exists an edge between x_i and x_j in $\mathbb{G}(n, m, \tilde{U}_n)$. When n tends to infinity, the sequence $\{\mathbb{G}(n, m, \tilde{U}_n)\}_{n \geq 1}$ of random graphs converges in distribution to \tilde{W} . This supports the fact that the sequence $\{\tilde{U}_n\}_{n \geq 1}$ converges to \tilde{W} , when n tends to infinity, with respect to the cut-distance metric (2). The inverse of the fixed Lebesgue measure transformation can be applied to lift \tilde{U}_n onto its original version U_n in $\mathcal{W}^{\mathbb{U}}(\mathbb{R})$ in a way that the sequence $\{U_n\}$ converges to W with respect to the cut-distance metric (2). It is now possible to define $\Delta(W)$ and $S(W)$ as the graph limits of the sequences $\{\Delta(U_n)\}_{n \geq 1}$ and $\{S(U_n)\}_{n \geq 1}$ with respect to the cut-distance metric (2).

For a given infinite direct sum $U = c_1U_1 + \dots + c_rU_r + \dots$ of stretched graphon in $\mathcal{W}^{\mathbb{U}}(\mathbb{R})$, and each U_j , consider its corresponding stretched graphon \tilde{U}_j defined on $[0, 1) \times [0, 1)$. There exists a sequence $\{\tilde{P}_{r_j}\}_{r \geq 1}$ of Lebesgue measurable step functions with the property $\|\tilde{U}_j - \tilde{P}_{r_j}\|_{cut} \leq \frac{2}{\sqrt{\log r_j}}$ such that the corresponding sequence $\{\mathbb{G}(r_j, m, \tilde{P}_{r_j})\}_{r \geq 1}$ of random graphs converges in distribution to \tilde{U}_j . Therefore the sequence $\{U^{(m)}\}_{m \geq 1}$ of the partial direct sums $U^{(m)} := c_1U_1 + \dots + c_mU_m : \bigsqcup_{n=1}^m I_n \times \bigsqcup_{n=1}^m I_n \subseteq \Omega \times \Omega \rightarrow \mathbb{R}$ can be replaced by the sequence $\{\tilde{U}^{(m)}\}_{m \geq 1}$ such that $\tilde{U}^{(m)} := c_1\tilde{U}_1 + \dots + c_m\tilde{U}_m : \bigsqcup_{n=1}^m \tilde{I}_n \times \bigsqcup_{n=1}^m \tilde{I}_n \subseteq [0, 1) \times [0, 1) \rightarrow \mathbb{R}$ with the property that there exists a sequence $\left\{ \mathbb{G}(r_1 + \dots + r_m, m, \tilde{P}_{r_1} + \dots + \tilde{P}_{r_m}) \right\}_{r \geq 1}$ of random graphs which converges in distribution to $\tilde{U}^{(m)}$. This technique shows that $\Delta(U)$ and $S(U)$ are definable as the graph limits of the sequences

$$\left\{ \frac{1}{\|U^{(m)}\|_{cut}} \Delta(U^{(m)}) \right\}_{m \geq 1}, \left\{ \frac{1}{\|U^{(m)}\|_{cut}} S(U^{(m)}) \right\}_{m \geq 1}, \tag{26}$$

with respect to the cut-distance metric (2). The resulting topological Hopf algebra on $\mathcal{W}^{\mathbb{U}}(\mathbb{R})$, presented by $\mathcal{H}^{\mathbb{U},cut}$, is called *topological core Hopf algebra of stretched graphons*.

We show the compatibility of the axioms of the Hopf algebra with infinite direct sums. For the counit $\varepsilon : \mathcal{H}^{\mathbb{U},cut} \rightarrow \mathbb{K}$, which sends $W_{\mathbb{I}}$ to 1 and other non-trivial stretched graphons W_G to zero, and the identity operator id on $\mathcal{H}^{\mathbb{U},cut}$, new stretched graphons $(\varepsilon \otimes id)\Delta(U)$ and $(id \otimes \varepsilon)\Delta(U)$ are defined as the limits of the sequences

$$\left\{ \frac{1}{\|U^{(m)}\|_{cut}} (\varepsilon \otimes id)\Delta(U^{(m)}) \right\}_{m \geq 1}, \left\{ \frac{1}{\|U^{(m)}\|_{cut}} (id \otimes \varepsilon)\Delta(U^{(m)}) \right\}_{m \geq 1}, \tag{27}$$

with respect to the cut-distance metric (2). For $j \geq 1$, we have

$$\begin{aligned} (\varepsilon \otimes id)\Delta(U_j) &= (\varepsilon \otimes id)\Delta(\lim_{r \rightarrow \infty} \tilde{P}_{r_j}) \\ &= \lim_{r \rightarrow \infty} (\varepsilon \otimes id)\Delta(\tilde{P}_{r_j}) = \lim_{r \rightarrow \infty} \sum_{W_j} \varepsilon(W_j) \otimes \tilde{Q}_{J,r_j} = 1 \otimes \tilde{Q}_{\emptyset,r_j} = \tilde{P}_{r_j}, \end{aligned} \tag{28}$$

$$\begin{aligned} (id \otimes \varepsilon)\Delta(U_j) &= (id \otimes \varepsilon)\Delta(\lim_{r \rightarrow \infty} \tilde{P}_{r_j}) \\ &= \lim_{r \rightarrow \infty} (id \otimes \varepsilon)\Delta(\tilde{P}_{r_j}) = \lim_{r \rightarrow \infty} \sum_{W_j} W_j \otimes \varepsilon(\tilde{Q}_{J,r_j}) = \tilde{P}_{r_j} \otimes 1 = \tilde{P}_{r_j}, \end{aligned} \tag{29}$$

with

$$\Delta(\tilde{P}_{r_j}) = \sum_{W_j} W_j \otimes \tilde{Q}_{J,r_j}, \tag{30}$$

such that W_j is the Lebesgue measurable symmetric step function defined on $J \times J$ with $J = \sqcup_a J_a \subseteq \text{Dom}(\tilde{P}_{r_j})$, for some subintervals J_a , such that \tilde{Q}_{J,r_j} is a Lebesgue measurable symmetric step function generated by removing the interval J from the domain of \tilde{P}_{r_j} . In addition, for the multiplication $\langle \rangle$ of the Hopf algebra, new stretched graphons $\langle id \otimes S \rangle \Delta(U)$ and $\langle S \otimes id \rangle \Delta(U)$ are defined as the limits of the sequences

$$\left\{ \frac{1}{\|U^{(m)}\|_{cut}} \langle id \otimes S \rangle \Delta(U^{(m)}) \right\}_{m \geq 1}, \left\{ \frac{1}{\|U^{(m)}\|_{cut}} \langle S \otimes id \rangle \Delta(U^{(m)}) \right\}_{m \geq 1}, \tag{31}$$

with respect to the cut-distance metric (2). For $j \geq 1$, we have

$$\begin{aligned} \langle id \otimes S \rangle \Delta(U_j) &= \langle id \otimes S \rangle \Delta(\lim_{r \rightarrow \infty} \tilde{P}_{r_j}) = \lim_{r \rightarrow \infty} \langle id \otimes S \rangle \Delta(\tilde{P}_{r_j}) \\ &= \lim_{r \rightarrow \infty} \sum_{W_j} W_j S(\tilde{Q}_{J,r_j}) = \lim_{r \rightarrow \infty} \sum_{W_j} S(W_j) \tilde{Q}_{J,r_j} = \langle S \otimes id \rangle \Delta(U_j). \end{aligned} \tag{32}$$

The tensor space $\mathcal{H}^{\mathbb{U}} \otimes \mathcal{H}^{\mathbb{U}} \otimes \mathcal{H}^{\mathbb{U}}$ is equipped with the cross norm

$$\|y\|_{cross} = inf \left\{ \sum_{i=1}^n \|W_{G'_i}\|_{cut} \|W_{G''_i}\|_{cut} \|W_{G'''_i}\|_{cut}, y = \sum_{i=1}^n W_{G'_i} \otimes W_{G''_i} \otimes W_{G'''_i} \right\}. \tag{33}$$

New stretched graphons $(\Delta \otimes id)\Delta(U)$ and $(id \otimes \Delta)\Delta(U)$ are defined as the limits of the sequences

$$\left\{ \frac{1}{\|U^{(m)}\|_{cut}} (\Delta \otimes id)\Delta(U^{(m)}) \right\}_{m \geq 1}, \left\{ \frac{1}{\|U^{(m)}\|_{cut}} (id \otimes \Delta)\Delta(U^{(m)}) \right\}_{m \geq 1}, \tag{34}$$

with respect to the cut-distance metric (2). For $j \geq 1$, the order does not matter in keeping the recursively removal of Lebesgue measurable symmetric step functions defined on $J \times J$ with disjoint unions $J = \sqcup_a J_a$ of sub-intervals chosen from the domain of \tilde{P}_{r_j} . Therefore both sides of the coassociativity equation

$$\begin{aligned} (\Delta \otimes id)\Delta(\lim_{r \rightarrow \infty} \tilde{P}_{r_j}) &= \lim_{r \rightarrow \infty} (\Delta \otimes id)\Delta(\tilde{P}_{r_j}) = \lim_{r \rightarrow \infty} \sum_{W_j} \Delta(W_j) \otimes \tilde{Q}_{J,r_j} \\ &= \lim_{r \rightarrow \infty} \sum_{W'_j} W'_j \otimes \Delta(\tilde{Q}_{J,r_j}) = \lim_{r \rightarrow \infty} (id \otimes \Delta)\Delta(\tilde{P}_{r_j}), \end{aligned} \tag{35}$$

enumerate the same family of decompositions which are sorted differently. \square

Corollary 1. Let $U = c_1 U_1 + \dots + c_r U_r + \dots$ be any (infinite) direct sum of bounded stretched graphon in $\mathcal{W}^{\mathbb{U}}(\mathbb{R})$ with the property that the series $\sum_{n=0}^{\infty} |c_n| \leq \infty$ converges to a finite value or to infinity. Then U can be described as the graph limit of a sequence of random graphs.

Proof. Consider the sequence $\{U^{(m)}\}_{m \geq 1}$ of its corresponding partial direct sums. Thanks to [9], we apply Lebesgue measure preserving transformations on $[0, \infty)$ to project $U^{(m)}$ to its corresponding stretched graphon $\tilde{U}^{(m)} : \sqcup_{k=1}^m J_k \times \sqcup_{k=1}^m J_k \subseteq [0, 1) \times [0, 1) \rightarrow \mathbb{R}$ with respect to the stretched graphons $\tilde{U}_k : J_k \times J_k \rightarrow \mathbb{R}$ for each $1 \leq k \leq m$ such that (i) for $\sum_{k=1}^m |c_k| < 1$, $m(J_k) = \frac{|c_k|}{\sum_{k=1}^m |c_k|}$, (ii) for $\sum_{k=1}^m |c_k| > 1$, $m(J_k) = \frac{1/|c_k|}{\sum_{k=1}^m (1/|c_k|)}$, and (iii) for

$$\sum_{k=1}^m |c_k| = 1, m(J_k) = \frac{2^{-val(U_k)}}{\sum_{k=1}^m 2^{-val(U_k)}} \text{ with } val(U_k) := \text{Max} \{n \geq 1 : U_k \in \bigoplus_{l \geq n} \mathcal{H}^{\mathbb{U},(l)}\}.$$

For each $m \geq 1$, define a random graph R_m with the vertex set $A_m := \{x_1, \dots, x_{m(m+1)/2}\}$ uniformly chosen from nodes in $\sqcup_{n=1}^m J_n$ such that with the probability $\|\tilde{U}^{(m)}\|_1^{-1} \tilde{U}^{(m)}(x_i, x_j)$ there exists an edge between x_i and x_j in R_m . When m tends to infinity, the sequence $\{R_m\}_{m \geq 1}$ converges in distribution to the stretched graphon $\tilde{U} : \sqcup_{k=1}^{\infty} J_k \times \sqcup_{k=1}^{\infty} J_k \subseteq [0, 1) \times [0, 1) \rightarrow \mathbb{R}$ which is weakly isomorphic to U . \square

Homomorphism densities of (stretched) graphons are discussed in [4,5,25] and here we address an extension to infinite direct sums.

Corollary 2. *Let $U = c_1U_1 + \dots + c_nU_n + \dots$ be an infinite direct sum of bounded stretched graphons in $\mathcal{W}^{\mathfrak{U}}(\mathbb{R})$ with $\|U_n\|_{cut} \geq 1$. The probability distribution of any finite graph G in U is well-defined.*

Proof. Consider the sequence $\{U^{(m)}\}_{m \geq 1}$ of partial direct sums of U . The probability distribution of any finite graph G in U is given in terms of the probability distribution of G in the partial direct sums $U^{(m)}$ given by integrating over all possible choices of $x_1, \dots, x_{|V(G)|} \in A_m$ with respect to $U^{(m)}$ such that A_m is determined in the proof of Corollary 1. In other words,

$$\mathbb{P}_{U^{(m)}}(G) = \int_{(\bigsqcup_{k=1}^m I_k)^{|V(G)|}} \prod_{e_{ij} \in E(G)} \frac{U^{(m)}(x_i, x_j)}{\|U^{(m)}\|_{cut}} \prod_{e_{ij} \notin E(G)} \left(1 - \frac{U^{(m)}(x_i, x_j)}{\|U^{(m)}\|_{cut}}\right) dx_1 \dots dx_{|V(G)|} \tag{36}$$

$$\Rightarrow \mathbb{P}_U(G) = \lim_{m \rightarrow \infty} \mathbb{P}_{U^{(m)}}(G). \tag{37}$$

□

Corollary 3. *For a fixed $p \geq 1$, define $\mathcal{H}_p^{\mathfrak{U}} = \bigoplus_{n=0}^{\infty} \mathcal{H}_p^{\mathfrak{U}(n)}$ as the graded free commutative algebra generated by stretched graphons $\{W_G, W_Y : G, Y\} \subset \mathcal{W}_{\approx}^{\mathfrak{U}}(\mathbb{R})$ associated to finite graphs G without self-loops and graph limits Y of Cauchy sequences of finite graphs with respect to the metric (8). $\mathcal{H}_p^{\mathfrak{U}}$ is equipped with the core Hopf algebra.*

Proof. The proof is similar to the proof of Theorem 1 for all L^p -stretched graphons which represent some graph limits. The resulting topological Hopf algebra, presented by $\mathcal{H}_p^{\mathfrak{U}, cut}$, is called L^p -topological core Hopf algebra of stretched graphons. □

Corollary 4. *For any $p \geq 1$, $\mathcal{H}_p^{\mathfrak{U}, cut}$ embeds in $\mathcal{H}^{\mathfrak{U}, cut}$. In addition, topological Hopf algebra of renormalization and H_{CK}^{cut} embed in $\mathcal{H}_p^{\mathfrak{U}, cut}$.*

Proof. The core coproduct is definable for any stretched graphon in $\mathcal{W}_{\approx}^{\mathfrak{U}}(\mathbb{R})$ with respect to the cut-distance metric (2), but the core coproduct of a L^p -stretched graphon which does not represent a graph limit might fail to be defined. Therefore for any $p \geq 1$, $\mathcal{H}_p^{\mathfrak{U}, cut}$ embed in $\mathcal{H}^{\mathfrak{U}, cut}$. The rest of the cases are the result of Lemma 3, Theorem 1 and Corollary 3. □

3. Renormalized stretched graphons

The Connes–Kreimer theory describes the BPHZ renormalization in terms of the Riemann–Hilbert problem underlying the Hopf–Birkhoff factorization on the complex Lie group of characters of the renormalization Hopf algebra [20,24]. A topological enrichment of the renormalization Hopf algebra has been formulated to extend the BPHZ renormalization to deal with divergent perturbative series of solutions of fixed point equations of (1PI) Green’s functions in terms of direct sums of stretched Feynman graphons [12,14,25]. Here we are going to explain a new extension of the BPHZ renormalization for the extraction of finite values from unbounded stretched graphons. Since the space of stretched Feynman graphons embeds in $\mathcal{W}_{\approx}^{\mathfrak{U}}(\mathbb{R})$, this new extension, which is independent of physical theories, has the universal property.

3.1. Method of the Hopf–Birkhoff factorization

Thanks to Milnor–Moore theorem [17], the graded dual of $\mathcal{H}^{\mathfrak{U}, cut}$ determines the complex Lie group $\mathbb{G}^{\mathfrak{U}}(\mathbb{C}) = Hom(\mathcal{H}^{\mathfrak{U}, cut}, \mathbb{C})$ of characters of $\mathcal{H}^{\mathfrak{U}, cut}$. The product $*_{core}$ of this particular Lie group, determined by the core coproduct (24), is given by

$$\psi_1 *_{core} \psi_2(W_X) = \psi_1 \otimes \psi_2(\Delta(W_X)) = \sum_{W_Z} \psi_1(W_Z) \psi_2(W_{X/Z}), \tag{38}$$

such that $\psi^{*core^{-1}} = \psi \circ S$ for any $\psi, \psi_1, \psi_2 \in \mathbb{G}^{\mathbb{U}}(\mathbb{C})$. The Lie structure on $\mathbb{G}^{\mathbb{U}}(\mathbb{C})$ is given by the topology of the pointwise convergence. In other words, the sequence $\{\phi_m\}_{m \geq 1}$ in $\mathbb{G}^{\mathbb{U}}(\mathbb{C})$ converges to character ϕ when n tends to infinity iff for any stretched graphon W , $\phi_m(W)$ tends to $\phi(W)$.

Definition 2. Let A_{dr} be the unital commutative algebra of Laurent series with finite pole parts, and $R_{ms} : A_{dr} \rightarrow A_{dr}$ be the linear map which sends any series $\sum_{n \geq -m}^{\infty} a_n z^n$ onto its pole parts $\mapsto T \sum_{n \geq -m}^{\infty} a_n z^n := \sum_{n \geq -m}^{-1} a_n z^n$.

The pair (A_{dr}, R_{ms}) , which has the Rota–Baxter property, is applied to regularize the dimension of divergent integrals to replace them with some Laurent series. The map R_{ms} , called minimal subtraction, is a projector which keeps the finite part by removing the polar part $\sum_{n \geq -m}^{\infty} a_n z^n - T \sum_{n \geq -m}^{\infty} a_n z^n$.

The Atkinson’s theorem shows that A_{dr} has the subdirect Birkhoff factorization $(R_{ms}(A_{dr}), -\tilde{R}_{ms}(A_{dr})) \subset A_{dr} \times A_{dr}$ with $\tilde{R}_{ms} = id_{A_{dr}} - R_{ms}$ such that each $x \in A_{dr}$ has a unique factorization $x = R_{ms}(x) + \tilde{R}_{ms}(x)$. This factorization can be extended to the Lie algebra of infinitesimal characters

$$\mathcal{L}(\mathcal{H}^{\mathbb{U},cut}, A_{dr}) := \left\{ f : \mathcal{H}^{\mathbb{U},cut} \rightarrow A_{dr} : \text{linear}, f(W_1 W_2) = f(W_1)\varepsilon(W_2) + \varepsilon(W_1)f(W_2) \right\}, \tag{39}$$

such that ε is the counit in $\mathcal{H}^{\mathbb{U},cut}$. The factorization $f = \mathcal{R}_{ms}(f) + \tilde{\mathcal{R}}_{ms}(f)$ is called Hopf–Birkhoff factorization such that

$$\mathcal{R}_{ms} : L(\mathcal{H}^{\mathbb{U},cut}, A_{dr}) \rightarrow L(\mathcal{H}^{\mathbb{U},cut}, A_{dr}), \psi \mapsto R_{ms} \circ \psi. \tag{40}$$

[21,26,27].

Remark 1. For any stretched graphon W , its corresponding infinitesimal character $f_W : \mathcal{H}^{\mathbb{U},cut} \rightarrow \mathbb{C}$ is given by $f_W(U) = \delta_{W,U}$. The collection $\{f_W : W \in \mathcal{H}^{\mathbb{U}}\}$ generates the Lie algebra $\mathcal{L}(\mathcal{H}^{\mathbb{U},cut}, A_{dr})$ with respect to the commutator $[f_W, f_U] = f_W *_{core} f_U - f_U *_{core} f_W$.

Theorem 2. Unbounded stretched graphons in $\mathcal{W}_{\approx}^{\mathbb{U}}(\mathbb{R})$ are renormalizable.

Proof. We apply Theorem 1. Let $W_X \in \mathcal{W}_{\approx}^{\mathbb{U}}(\mathbb{R})$ be any unbounded stretched graphon representation of a graph limit X with $\|W_X\|_{cut} = \infty$.

Step I. Regularized character and its meromorphic extension.

The existence of meromorphic continuation to the complex plane for generalized functions with respect to a parameter was shown in [29] and applied for dimensional regularization parameter in [17,30]. Here we update this technique for our setting. Consider any n variable polynomial $Q(u)$ with $u = (u_1, \dots, u_n)$, and a parameter $f(z, n) \in \mathbb{C}$ which depends on z, n . Define the module of elements $p(u, f(z, n))Q(u)^{-f(z, n)/2-k}$ over the ring of differential operators $P(u, \partial, f(z, n))$ where $p(u, f(z, n))$ are polynomials in u with rational functions of $f(z, n)$ as their coefficients. Since there exist some P and k which contribute to the equation $Q(u)^{-f(z, n)/2-k-1} = P(u, \partial, f(z, n))Q(u)^{-f(z, n)/2-k}$, this module is finitely generated in terms of the filtrations

$$\mathcal{M}_m := \left\{ p(u, f(z, n))Q(u)^{-f(z, n)/2-m} : \text{deg}(p(u, f(z, n))) \leq m(\text{deg}(Q(u)) + 1) \right\}, \tag{41}$$

such that $\dim(\mathcal{M}_m) \sim (\text{deg}(Q(u)) + 1)^n m^n$. Therefore we can find polynomials T and T_j which contribute to the equations $TQ(u) + \sum_j T_j \partial_j Q(u) = 1$. This leads us to define a new polynomial differential operator $L(f(z, n))$ in n variables given by

$$L(f(z, n)) = -\frac{f(z, n)}{2}T + \sum_j T_j \partial_j, \tag{42}$$

and a new polynomial $q(f(z, n)) = -\frac{f(z, n)}{2}$ which satisfy in the equation

$$L(f(z, n))Q(u)^{-f(z, n)/2} = q(f(z, n))Q(u)^{-f(z, n)/2-1}. \tag{43}$$

Consider certain regularized characters ϕ^z in $\mathbb{G}^{\mathfrak{U}}(A_{dr})$ with respect to the parameter $z \in \partial\Delta^*$ which varies in the boundary of an infinitesimal punctured disk Δ^* around zero in the complex plane. They are homomorphisms which assign a Laurent series to any generator W_X of $\mathcal{H}^{\mathfrak{U},cut}$. This Laurent series is the result of defining a new function $\phi^z(W_X)$, which has a meromorphic continuation to the complex plane, such as the one defined in terms of replacing the integrator $dx dy$ in (2) by $d^{1-z}x d^{1-z}y$. Thanks to Lemmas 1.7 and 1.8 in [17], these new functions could have general forms such as

$$\phi^z(W_X) = \int_{u=(u_1, \dots, u_n) \in [0, \infty)^n} g(z, u) Q_{W_X}(u)^{f(z, n)} \prod_{j=1}^n du_j, \quad f(z, n) \in \mathbb{C}, \tag{44}$$

such that $Q_{W_X}(u)$ is a polynomial in n variables built in terms of the Schwinger parametrization

$$\int \frac{dx dy}{W_X(x, y)^n} = \frac{1}{\Gamma(n)} \int dx dy \int_{[0, \infty)} du u^{n-1} e^{-u W_X(x, y)}, \tag{45}$$

and $g(z, u)$ has the Taylor expansion $\sum_{m \in \mathbb{Z}} g_m(u) z^m$ at $z = 0$. Firstly, by applying the integration-by-parts together with (42) and (43), we get

$$\int_{[0, \infty)^n} g(z, u) Q_{W_X}(u)^{f(z, n)+2} \prod_{j=1}^n du_j = q(f(z, n))^{-1} \int_{[0, \infty)^n} L(f(z, n))^* g(z, u) Q_{W_X}(u)^{f(z, n)} \prod_{j=1}^n du_j + B(f(z, n)), \tag{46}$$

such that $L(f(z, n))^*$ is the adjoint operator of $L(f(z, n))$ and $B(f(z, n))$ is a meromorphic boundary term. Secondly, the coefficients $g_m(u)$ of the Taylor expansion of $g(z, u)$ are of the form $Q_{W_X}(u)^{-|m|} h_m$ for some smooth functions h_m of rapid decay defined for $u_j \in [0, \infty)$ whose derivations are also of rapid decay. Therefore, $\phi^z(W_X)$ can be extended meromorphically to the whole complex plane.

Step II. Renormalization program.

Consider the infinite dimensional complex Lie group $\mathbb{G}^{\mathfrak{U}}(\mathbb{C}) = Hom(\mathcal{H}^{\mathfrak{U},cut}, \mathbb{C})$ of characters of the topological core Hopf algebra of stretched graphons. It is the projective limit of finite dimensional Lie groups modeled by suitable Lie subgroups of $GL_n(\mathbb{C})$. Therefore the topology of $\mathbb{G}^{\mathfrak{U}}(\mathbb{C})$, which is inherited from the inverse system of simpler Lie groups, is the topology of pointwise convergence. This topology is valid for the space of algebra homomorphisms from $\mathcal{H}^{\mathfrak{U},cut}$ to any commutative locally convex algebra such as $\mathbb{C}[z^{-1}]$ and $z^{-1}\mathbb{C}[z^{-1}]$. Since $\mathcal{H}^{\mathfrak{U},cut}$ is connected and graded, $\mathbb{G}^{\mathfrak{U}}(\mathbb{C})$ becomes a Baker–Campbell–Hausdorff (BCH) Lie group which means that smooth solutions of differential equations are well-defined.

For any complex value z in the boundary of an infinitesimal punctured disk around $z = 0$ in the complex plane, we work on regularized characters $\phi^z \in \mathbb{G}^{\mathfrak{U}}(A_{dr})$ with the general form (44) such that for any stretched graphon W_X , the integral $\phi^z(W_X)$ is meromorphic in z where (i) the coefficients of the Laurent series, which grow at most polynomially, are uniformly bounded, or (ii) the location and order of poles in the Laurent series remain stable under cut-distance convergence of stretched graphons. These characters, as linear and bounded maps, are continuous. These continuous characters, which allow for controlled subtraction of divergences, form a dense subspace in $\mathbb{G}^{\mathfrak{U}}(A_{dr})$ with respect to the pointwise convergence topology.

Let $T_{\phi^z(W_X)}$ be the polar part of the Laurent series expansion $L_{\phi^z(W_X)}$ of $\phi^z(W_X)$ such that the minimal subtraction map R_{ms} removes the polar part $L_{\phi^z(W_X)} - T_{\phi^z(W_X)}$. Following the Atkinson Theorem [26,27], for the algebra A_{dr} of Laurent series with finite pole parts, the Rota–Baxter algebra (A_{dr}, R_{ms}) inherits the (Hopf–)Birkhoff factorization for elements of $\mathbb{G}^{\mathfrak{U}}(A_{dr})$. The unique Hopf–Birkhoff factorization of any $\phi^z \in \mathbb{G}^{\mathfrak{U}}(A_{dr})$, such as (44), is given by the pair (ϕ_-^z, ϕ_+^z) of characters on $\mathcal{H}^{\mathfrak{U},cut}$ such that $\phi^z = (\phi_-^z)^{*core^{-1}} *_{core} \phi_+^z$ with $\phi_-^z \in \mathbb{G}^{\mathfrak{U}}(\mathbb{C}[z^{-1}])$ and $\phi_+^z \in \mathbb{G}^{\mathfrak{U}}(z^{-1}\mathbb{C}[z^{-1}])$.

Thanks to [17,19,22,26,27] and the coproduct (24), we show the compatibility of this factorization with unbounded stretched graphons and infinite direct sums of stretched graphons. For any sequence $\{W_n\}_{n \geq 1}$ of unbounded stretched graphons in $\mathcal{W}_{\approx}^{\mathfrak{U}}(\mathbb{R})$, there exists a subsequence $\{U_{n_i}\}_{i \geq 1}$ which converges to an unbounded stretched graphon V with respect to the cut-distance metric (2). We get $\phi^z(V)$ as the limit of the sequence $\{\phi^z(U_{n_i})\}_{i \geq 1}$ with respect to the topology of pointwise convergence when n tends to infinity. The

renormalized value for V is determined by the limits of the sequences $\{\phi_-^z(U_{n_i})\}_{i \geq 1}$ and $\{\phi_+^z(U_{n_i})\}_{i \geq 1}$ such that

$$\phi_+^z(V) = \left(\phi_-^z *_{core} \phi^z\right)(V) = \lim_{i \rightarrow \infty} \left(\phi_-^z *_{core} \phi^z\right)(U_{n_i}), \tag{47}$$

$$\phi_-^z(U_{n_i}) = -R_{ms} \left(\phi^z(U_{n_i}) + \sum \phi_-^z(Z'_{(n_i)}) \phi^z(Z''_{(n_i)})\right), \tag{48}$$

$$\phi_+^z(U_{n_i}) = \phi^z(U_{n_i}) + \phi_-^z(U_{n_i}) + \sum \phi_-^z(Z'_{(n_i)}) \phi^z(Z''_{(n_i)}), \tag{49}$$

with respect to the Sweedler notation for coproducts. Following a similar process, for any infinite direct sum U of stretched graphons $U_n, n \geq 1$, with the corresponding sequence $\{U^{(m)}\}_{m \geq 1}$ of partial direct sums, we get $\phi^z(U)$ as the limit of the sequence $\{\phi^z(U^{(m)})\}_{m \geq 1}$ such that (i) $\phi^z(U)$ has a meromorphic extension to the whole complex plane and (ii) the renormalized value $\phi_+^z(U)$ exists as the limit of the sequence $\{\phi_+^z(U^{(m)})\}_{m \geq 1}$. \square

Corollary 5. For $p \geq 1$, any unbounded stretched graphon which contributes to the graph limit of a Cauchy sequence of L^p -stretched graphons, with respect to the metric (8), is L^p -renormalizable.

Proof. It is a result of Corollaries 3, 4 such that we need to update the proof of Theorem 2 for the complex Lie group $\mathbb{G}_p^{\mathbb{C}}(A_{dr}) = Hom(\mathcal{H}_p^{\mathbb{C},cut}, A_{dr})$ of regularized characters on $\mathcal{H}_p^{\mathbb{C},cut}$. \square

3.2. Analytic continuation

The analytic extension of the Hopf–Birkhoff factorization via generalized evaluators has been studied in the context of different regularization schemes such as Riesz and covariant regularizations [21]. Here we address a new symplectic geometry model to show the analytic continuation of renormalized values corresponding to any unbounded stretched graphon. This new symplectic model, which is explained for dimensional regularization and minimal subtraction scheme, can be extended to any renormalization scheme in terms of its associated Rota–Baxter algebra.

Definition 3. Thanks to [26,27,31], for the linear functionals \mathcal{R}_{ms} given by (40) and $\tilde{\mathcal{R}} := Id_{L(\mathcal{H}^{\mathbb{C},cut}, A_{dr})} - \mathcal{R}_{ms}$ and any $\lambda \in \mathbb{R}$, the new linear functional $\mathcal{R}_\lambda := \mathcal{R}_{ms} - \lambda \tilde{\mathcal{R}}$ on $L(\mathcal{H}^{\mathbb{C},cut}, A_{dr})$ defines a new associative noncommutative product on $L(\mathcal{H}^{\mathbb{C},cut}, A_{dr})$ given by

$$\phi \circ_\lambda \psi := \mathcal{R}_\lambda(\phi) *_{core} \psi + \phi *_{core} \mathcal{R}_\lambda(\psi) - \mathcal{R}_\lambda(\phi *_{core} \psi). \tag{50}$$

The commutator $[\phi, \psi]_\lambda = \phi \circ_\lambda \psi - \psi \circ_\lambda \phi = [\mathcal{R}_\lambda(\phi), \psi] + [\phi, \mathcal{R}_\lambda(\psi)] - \mathcal{R}_\lambda([\phi, \psi])$ determines a new Poisson structure on the algebra $C_\lambda := (L(\mathcal{H}^{\mathbb{C},cut}, A_{dr}), \circ_\lambda)$.

Lemma 6. Define a new star product on C_λ given by

$$\phi \star_{\lambda, \hbar} \psi = \phi *_{core} \psi + \hbar [\phi, \psi]_\lambda + O(\hbar^2). \tag{51}$$

The Hopf–Birkhoff factorization with respect to $\star_{\lambda, \hbar}$ on $L(\mathcal{H}^{\mathbb{C},cut}, A_{dr})$ determines renormalized values of unbounded stretched graphons in $\mathcal{W}_{\approx}^{\mathbb{C}}(\mathbb{R})$.

Proof. This is a consequence of [32], Theorems 1, 2 and Definition 3. For any unbounded stretched graphon U , as the infinite direct sum of stretched graphons U_n with the corresponding sequence $\{U^{(m)}\}_{m \geq 1}$ of finite partial direct sums, $(\phi \star_{\lambda, \hbar} \psi)(U)$ is the limit of the sequence $\left\{(\phi \star_{\lambda, \hbar} \psi)(U^{(m)})\right\}_{m \geq 1}$ when m tends to infinity. Then, thanks to Theorem 2, for the regularized character $\phi^z : \mathcal{H}^{\mathbb{C},cut} \rightarrow A_{dr}$ given by (44), we get

$$\phi_+^z(U) = \left(\phi_-^z \star_{\lambda, \hbar} \phi^z\right)(U) = \left(\phi_-^z \star_{\lambda, \hbar} \phi^z\right)(\lim_{m \rightarrow \infty} U^{(m)}) = \lim_{m \rightarrow \infty} \left(\phi_-^z \star_{\lambda, \hbar} \phi^z\right)(U^{(m)}). \tag{52}$$

\square

Consider the complex Lie group $char_\lambda \mathcal{H}^{\lambda, cut} \subset C_\lambda$ of characters with the corresponding Lie algebra $\partial char_\lambda \mathcal{H}^{\lambda, cut} \subset C_\lambda$ of infinitesimal characters. There exists the bijection

$$\partial char_\lambda \mathcal{H}^{\lambda, cut} \rightarrow char_\lambda \mathcal{H}^{\lambda, cut}, \quad exp^{\circ\lambda}(f_W) := \sum_{n \geq 0} \frac{f_W^{\circ\lambda n}}{n!}, \tag{53}$$

for any infinitesimal character f_W corresponding to $W \in \mathcal{H}^{\lambda, cut}$.

Remark 2. • For any sequence $\{W_n\}_{n \geq 1}$ of stretched graphons which converges to W with respect to the metric (2), we get $exp^{\circ\lambda}(f_W) = \lim_{n \rightarrow \infty} exp^{\circ\lambda}(f_{W_n})$.

• For any infinite direct sum U of stretched graphons with the corresponding sequence $\{U^{(m)}\}_{m \geq 1}$ of partial direct sums, we get $exp^{\circ\lambda}(f_U) = \lim_{m \rightarrow \infty} exp^{\circ\lambda}(f_{U^{(m)}})$.

Corollary 6. Let U be an infinite direct sum of stretched graphons in $\mathcal{W}_{\approx}^{\lambda}(\mathbb{R})$ with the corresponding sequence $\{U^{(m)}\}_{m \geq 1}$ of finite partial direct sums. Renormalized values $\phi_+^z(U)$ generated by the Hopf–Birkhoff factorization of regularized characters $\phi^z \in \mathbb{G}^{\lambda}(A_{dr})$, $z \in \partial\Delta^*$, have analytic continuation in any infinitesimal punctured disk Δ^* in the complex plane around $z = 0$.

Proof. This is a consequence of Theorems 1, 2, Definition 3, Lemma 6 and Dubois-Violette’s noncommutative differential calculus [33].

Consider Der_λ as the space of derivations $\theta : C_\lambda \rightarrow C_\lambda$ which are linear maps that satisfy Leibniz rule, and Ham^λ as the $Z(C_\lambda)$ -module generated by Hamiltonian derivations $ham(\phi) : \psi \mapsto [\phi, \psi]_\lambda$ for any $\phi \in C_\lambda$ such that $Z(C_\lambda)$ is the center of C_λ . For $n \geq 1$, define $\Omega^n(C_\lambda)$ as the space of all $Z(C_\lambda)$ -multilinear antisymmetric functionals $Ham^\lambda \times \dots \times Ham^\lambda \rightarrow C_\lambda$ such that for $n = 0$, $\Omega^0(C_\lambda) = C_\lambda$.

If $\Omega^\bullet(C_\lambda) = \bigoplus_{n=0}^\infty \Omega^n(C_\lambda)$, then for any $\omega \in \Omega^\bullet(C_\lambda)$ and $\theta_0, \dots, \theta_n \in Der_\lambda$, define

$$(d_\lambda \omega)(\theta_0, \dots, \theta_n) := \sum_{k=0}^n (-1)^k \theta_k \omega(\theta_0, \dots, \hat{\theta}_k, \dots, \theta_n) + \sum_{0 \leq r < s \leq n} (-1)^{r+s} \omega([\theta_r, \theta_s]_\lambda, \theta_0, \dots, \hat{\theta}_r, \dots, \hat{\theta}_s, \dots, \theta_n). \tag{54}$$

The noncommutative deRham complex associated to the differential graded algebra $(\Omega^\bullet(C_\lambda), d_\lambda)$ is given by $DR^\bullet(C_\lambda) := \frac{\Omega^\bullet(C_\lambda)}{[\Omega^\bullet(C_\lambda), \Omega^\bullet(C_\lambda)]_\lambda}$. The symplectic form

$$\omega_\lambda : Ham^\lambda \times Ham^\lambda \rightarrow C_\lambda, \quad (\theta, \theta') = \left(\sum_i u_i \circ_\lambda ham(\phi_i), \sum_j v_j \circ_\lambda ham(\psi_j) \right) \mapsto \sum_{i,j} u_i \circ_\lambda v_j \circ_\lambda [\phi_i, \psi_j]_\lambda, \tag{55}$$

in $\Omega^2(C_\lambda)$ defines the Poisson structure $\{\phi, \psi\}_\lambda = i_{\theta_\phi} i_{\theta_\psi} \omega_\lambda$ on C_λ with the corresponding star products

$$\phi \star_{\{.,.\}_\lambda, z} \psi = \phi *_{core} \psi + |z| \{\phi, \psi\}_\lambda + O(|z|^2), \tag{56}$$

such that z varies in $\partial\Delta^*$ and $\theta_\phi^\lambda, \theta_\psi^\lambda$ are symplectic vector fields associated with $d_\lambda \phi, d_\lambda \psi$.

We have

$$\phi_+^z(U) = \left(\phi_-^z \star_{\{.,.\}_\lambda, z} \phi^z \right)(U) = \left(\phi_-^z \star_{\{.,.\}_\lambda, z} \phi^z \right)(\lim_{m \rightarrow \infty} U^{(m)}) = \lim_{m \rightarrow \infty} \left(\phi_-^z \star_{\{.,.\}_\lambda, z} \phi^z \right)(U^{(m)}). \tag{57}$$

Therefore the continuum limit of $\left\{ \phi_-^z *_{core} \phi^z + |z| \{\phi_-^z, \phi^z\}_\lambda + O(|z|^2) \right\}_{z \rightarrow 0}$ describes the analytic continuation for $\phi_+^z(U)$ in Δ^* . □

Assigning suitable family of graphons is a new way of dealing with boundary problems in physical systems. In this regard, a new solution to the fundamental problem of triviality or "zero charge problem" in quantum theories has been achieved in terms of building a certain family of discrete-time Markov chains of random operators on the spaces of graphons [34]. This investigation leads us to search for possible applications of stochastic processes for the study of the asymptotics of star products in deformation quantization and search

for suitable graphon processes that star products corresponding to unbounded graphons could be definable as limits of star products generated by simpler graphons underlying the renormalization program provided by Theorem 2 and Corollary 6. Here we testify the impact of Corollary 6 to clarify the motivation of working on this unexplored case study.

Deformation quantization is a mathematical method of deforming the pointwise multiplication of a space of functions into a non-commutative star product with a general form

$$f \star g = f \cdot g + \hbar B_1(f, g) + \hbar^2 B_2(f, g) + \dots, \tag{58}$$

such that \hbar is the deformation parameter and B_n are a certain family of bi-differential operators. Generally speaking, thanks to the basic structure of star products in deformation quantization [32,35], let $\{W_n\}_{n \geq 1}$ be a convergent sequence of graphons to the limiting graphon W with respect to the cut-distance topology and suppose each W_n determines a star product \star_n on a suitable algebra of functions given by

$$f \star_n g = \sum_{G \in \mathcal{G}_n} \frac{\hbar^{|G|}}{|Aut(G)|} \omega_n(G) B_{W_n, G}(f, g), \tag{59}$$

such that \mathcal{G}_n is a set of finite weighted graphs associated to the graphon W_n , $\omega_n(G)$ is the weight of graph G determined by the graphon W_n and $B_{W_n, G}(f, g)$ is the bi-differential operator associated to the graph G and graphon W_n . Under some controlled analytic conditions, the limiting star product \star is definable by

$$f \star g = \lim_{n \rightarrow \infty} f \star_n g = \sum_{G \in \mathcal{G}} \frac{\hbar^{|G|}}{|Aut(G)|} \omega(G) B_{W, G}(f, g), \tag{60}$$

such that $\mathcal{G} = \bigsqcup_{n \geq 1} \mathcal{G}_n$ and $\omega(G)$ is the weight of graph G determined by the graphon W . This formulation of star products leads us to interconnect the structure of renormalized star products with the asymptotic behavior of dense graph sequences.

Example 1. Renormalized values corresponding to the BPHZ renormalization of the infinite direct sum $U := W_1 + W_2 + \dots$ of graphons $W_n = n \cdot 1_{[0, \frac{1}{2^n}] \times [0, \frac{1}{2^n}]}$, $n \geq 1$, have analytic continuation with respect to the regulator z in any infinitesimal punctured disk Δ^* in the complex plane around $z = 0$.

Proof. For each n , define $U^{(n)} := U|_{I_n \times I_n} = W_1 + \dots + W_n$ with $I_n = \left[\sum_{k=1}^{n-1} \frac{1}{2^k}, \sum_{k=1}^n \frac{1}{2^k} \right)$ such that the total measure of all intervals is less than 1 while they converge to 1. When n increases, the mass of W_n concentrates in smaller regions which allows to remain small with respect to the cut-distance topology. It can be checked that $\{d_{cut}(U, U^{(n)})\}_n$ converges to zero. Therefore U as the cut-distance convergent limit of the sequence $\{U^{(n)}\}_{n \geq 1}$ of bounded graphons is an unbounded graphon.

Renormalization. We apply a new regulator $z \in \Delta^*$ to define bounded regulated graphons together with related integrals

$$U_z(x, y) := \sum_{n=1}^{\infty} \frac{n}{1 + |z|n} 1_{I_n \times I_n}(x, y), \quad T(z) := \int_{[0,1]^2} U_z(x, y) dx dy, \tag{61}$$

such that $\{T(z)\}_z$ might diverges when the regulator z tends to zero, however, these divergences are isolated as poles in z . The minimal subtraction map R_{ms} removes the divergent part to get

$$T_{ren} = \lim_{z \rightarrow 0} \left(T(z) - \text{divergent part} \right). \tag{62}$$

Consider the regulator z as a deformation parameter, the regulated graphons U_z as deformed versions of U where singularities are untangled, and the minimal subtraction map R_{ms} as a machinery for the removal of the divergent part of the formal series in z which is the same as the subtracting singular terms in a star product expansion. This dictionary leads us to formulate a new family of deformed star products on the algebra \mathcal{H}^U given by

$$W_1 \star_z W_2 := W_1 W_2 + |z| B_1(W_1, W_2) + |z|^2 B_2(W_1, W_2), \tag{63}$$

such that the bi-differential operators B_n , as the analogs of their classical ones, are definable on the basis of homomorphism densities $hom(G, W_i)$ of finite graphs with respect to graphons W_1, W_2 . The renormalized graphon U_{ren} is well-defined by subtraction of divergent terms in the z -expansion of $U_z \star_z U_z$.

The operator $N_U = \sum_{n=1}^{\infty} \frac{1}{n} W_n$ is a Nijenhuis map on the free commutative graded finite type algebra \mathcal{H}^U such that $N_U(W_n) = \frac{1}{n} W_n$, $N_U(W_{i_1} \dots W_{i_k}) = \left(\sum_{j=1}^k \frac{1}{j} \right) W_{i_1} \dots W_{i_k}$ and for any monomial $f(V_{i_1}, \dots, V_{i_k})$ of graphons, $N_U(f) = \sum_{n=1}^{\infty} \frac{1}{n} W_n f$. It defines a new star product

$$W_1 \star_{N_U} W_2 = W_1 W_2 + |z| \{W_1, W_2\}_{N_U} + |z|^2 B_2(W_1, W_2) + \dots, \tag{64}$$

such that the Poisson bracket $\{W_1, W_2\}_{N_U} := \{N_U(W_1), W_2\}_{K,U} + \{W_1, N_U(W_2)\}_{K,U}$ is a deformation of the Poisson bracket

$$\begin{aligned} \{W_1, W_2\}_{K,U}(W) &:= \{F_{W_1}, F_{W_2}\}_{K,U}(W) \\ &= \frac{1}{2} \int_{[0,1]^4} \left(\frac{\delta F_{W_1}}{\delta W}(x, y, W) K U(u, v) \frac{\delta F_{W_2}}{\delta W}(u, v, W) \right. \\ &\quad \left. - \frac{\delta F_{W_2}}{\delta W}(x, y, W) K U(u, v) \frac{\delta F_{W_1}}{\delta W}(u, v, W) \right) dx dy du dv, \end{aligned} \tag{65}$$

with respect to some suitable kernel $K = K(x, y, u, v)$ such that F_{W_i} is the operator on the Hilbert space $L^2(\mathcal{W}_{\approx}^U(\mathbb{R}))$ corresponding to W_i given by

$$f \mapsto F_{W_i}(f) := \int_{\mathcal{W}_{\approx}^U(\mathbb{R})} W_i(x_1, x_2) f(x_2) d\mu_{Borel}(x_2). \tag{66}$$

Here the space of graphons is considered as a metric space with its corresponding Borel σ -algebra or a subset of a L^p -space with its Borel σ -algebra. The limit of the sequence $\{U^{(n)} \star_{N_U} U^{(n)}\}_n$ recovers the renormalization of U . It is enough to add some bounded graphons W_{-1}, W_0 to U to define some new regulated graphons $V_z := |z|^{-1} W_{-1} + W_0 + |z| W_1 + \dots$ such that $R_{ms}(V_z) = W_0 + |z| W_1 + \dots$. We get the renormalized version of the deformed star product (63) given by

$$V_{ren} \tilde{\star}_z V_{ren} := R_{ms}(V_{|z|} V_{|z|}) + |z| R_{ms}(B_1(V_{|z|}, V_{|z|})) + |z|^2 R_{ms}(B_2(V_{|z|}, V_{|z|})) + \dots \tag{67}$$

Applying some regularized character ϕ^z , such as the one given by (44) together with the Hopf–Birkhoff factorization $\phi^z = (\phi_-^z)^{*core^{-1}} *_{core} \phi_+^z$, we get

$$\phi_-^z = |z|^{-1} W_{-1}, \phi_+^z = V_{ren} = R_{ms}(V_z). \tag{68}$$

Analytic continuation. For the sequence $\{W_n\}_{n \geq 1}$ which contributes to the infinite direct sum U , we consider an approximating sequence $\{Z_s\}_{s \geq 1}$ of graphons with nested supports given by

$$Z_s(x, y) = 1_{[0, r_s]^2}(x, y), \quad r_s = 1 - \frac{1}{s+1}, \tag{69}$$

which converges to $1_{[0,1]^2}$. They are nested graphons such that (i) each Z_s can be approximated by some W_k , $k < s$, (ii) any measurable subgraphon of Z_s is a restriction of a smaller square, and (iii) the core coproduct (22) in Theorem 1 on these nested graphons are given by

$$\Delta(Z_s) = \sum_{l=1}^s Z_l \otimes Z_{s-l}. \tag{70}$$

Replace N_U by the new Nijenhuis map $N_{\tilde{U}} = \sum_{n=1}^{\infty} \frac{1}{n} Z_n$ which is an almost everywhere bounded smooth map on $[0, 1]^2$ and it increases when (x, y) tends to $(0, 0)$. In addition, $N_{\tilde{U}}$ obeys the co-Nijenhuis identity

$$\Delta(N_{\tilde{U}}(W)) = \left(N_{\tilde{U}} \otimes id + id \otimes N_{\tilde{U}} - N_{\tilde{U}} \otimes N_{\tilde{U}} \right) (\Delta(W)), \tag{71}$$

with respect to the core coproduct(22). This allows us to lift $N_{\tilde{U}}$ onto the dual space of the free commutative graded finite type topological Hopf algebra $\mathcal{H}^{\mathcal{M}, cut}$. In other words, $N_{\tilde{U}}$ induces a dual map $N_{\tilde{U}}^* : L(\mathcal{H}^{\mathcal{M}, cut}, A_{dr}) \rightarrow L(\mathcal{H}^{\mathcal{M}, cut}, A_{dr})$ given by $N_{\tilde{U}}^*(\phi)(W) = \phi(N_{\tilde{U}}(W))$ which obeys the Nijenhuis identity.

Define the noncommutative associative algebra $C_{N_{\tilde{U}}} := (L(\mathcal{H}^{\mathcal{M}, cut}, A_{dr}), \circ_{N_{\tilde{U}}})$ such that the multiplication $\circ_{N_{\tilde{U}}}$ is a deformation of the convolution product $*_{core}$ with respect to the Nijenhuis map $N_{\tilde{U}}$. Thanks to Corollary 6, the noncommutative symplectic geometry built on the space of Hamiltonian derivations on $C_{N_{\tilde{U}}}$ determines the star product $\star_{\{\dots\}_{N_{\tilde{U}}, z}}$ given by

$$\phi \star_{\{\dots\}_{N_{\tilde{U}}, z}} \psi = \phi *_{core} \psi + |z| \{ \phi, \psi \}_{N_{\tilde{U}}} + O(|z|^2), \tag{72}$$

such that, by applying Theorem 2, the continuum limit of $\left\{ \phi_-^z *_{core} \phi^z + |z| \{ \phi_-^z, \phi^z \}_{N_{\tilde{U}}} + O(|z|^2) \right\}_{z \rightarrow 0}$ describes the analytic continuation for $\phi_+^z(\tilde{U})$ in Δ^* . \square

Example 2. The BPHZ renormalization of the cut-distance convergent limit of the sequence $\{W_n\}_{n \geq 1}$, $W_n(x, y) = n \cdot 1_{[0, 1/n]^2}(x, y)$ of graphons on the Lebesgue measure space $[0, \infty)$ generates a certain family of renormalized start products.

Proof. The sequence $\{W_n\}_{n \geq 1}$, $W_n(x, y) = n \cdot 1_{[0, 1/n]^2}(x, y)$ of graphons represents a concentration into a shrinking infinitesimal square around the origin in the plane. Each W_n has the total integral 1 such that by increasing n the local height inside the infinitesimal square grows but the sequence pointwise tends to zero almost everywhere. For large enough n , each W_n can determine finite graphs with many edges concentrated in an infinitesimal region such that amplitudes generated by these concentrated substructures restore extremely short-distance interactions. In this scenario, if we translate these graphons to Feynman graphons, the corresponding Feynman amplitudes will have ultraviolet divergences. It is possible to assign Laurent series with finite pole parts to these amplitudes in terms of involving a short-distance regulator z .

Rescaling or stretching methods allow us to determine a non-trivial unbounded graphon W defined on $[0, \infty)^2$ as the graph limit of the sequence $\{W_n\}_{n \geq 1}$ when n tends to infinity. Define a regulated weight $\omega_z(W)$ such as $\omega_z(W) = c_{-1}(W)z^{-1} + c_0(W) + O(z)$ for W . Here $c_{-1}(W)$, as the coefficient of the simple pole in the regulator expansion of the regularized amplitudes of W , represents the ultraviolet divergent strength. In addition, $c_0(W)$, as the finite pole part, is the renormalizable finite amplitude which remains after subtracting divergences.

Thanks to Theorem 2, the renormalization of W with respect to the regularized character ϕ is given by the recursive relations (47), (48) and (49) such that

$$\phi_-^z(W) = -R_{ms} \left(\omega_z(W) + \sum \phi_-^z(W') \omega_z(W'') \right), \quad R_{ms}(\omega_z(W)) = c_{-1}(W)z^{-1}, \tag{73}$$

with respect to the Sweedler notation for coproducts.

The Nijenhuis operator $N_1 = 2R_{ms} - Id$ generates a finite weight such that after canceling of z^{-1} terms via the Nijenhuis identities, this finite weight equals the finite coefficient $c_0(W)$ up to $O(z)$. If we replace singular coefficients $\omega_z(W)$ by $c_0(W)$, then we get the renormalized star product

$$f \star_{N_1, ren, W} g = exp \left(\sum_{\tilde{W}} c_0(\tilde{W}) B_{\tilde{W}} \right) (f, g), \tag{74}$$

such that the sum is over the chosen decomposition of the graphon W into its building blocks called graphon-modes.

- For any finite graph G , B_G is a bi-differential operator defined by assigning one derivative to each edge in terms of the combinatorial information of the graph and here $B_{\tilde{W}}$, as the natural continuum analogue, are bi-differential operators associated to the graphon-modes of W .

- If \tilde{W} is the indicator of a block mode, then $B_{\tilde{W}}$ is the block-weighted sum of the corresponding finite-graph bi-differential operators restricted to that region.

- If W concentrates near a point, then $B_{\tilde{W}}$ approximates a local bi-differential operator which generates ultraviolet divergences removable by renormalization.

- The renormalized star product is independent of the choice of the graphon-mode decompositions because different decompositions only affects the intermediate coefficients and operators.

- It is possible to assign different renormalized start products to the renormalization of W in terms of changing the Nijenhuis operator N_1 by N_λ . \square

The description of the renormalized star product built from the graphon W in terms of the convergence of star products built from the approximating sequence $\{W_n\}_{n \geq 1}$ requires additional technical steps such as (i) checking the convergence of the scalar weights after regularization / renormalization, and (ii) checking the convergence of the bi-differential operators B_{W_n} .

Example 3. Consider the graphon $W(x, y) = a.1_{[0,\alpha]^2}(x, y)$, for any fixed real values $0 < \alpha, a \leq 1$, as the graph limit of the sequence $\{W_n\}_{n \geq 1}$ such that W_n is the block-average of W on each $I_{n,i} \times I_{n,j}$ for the partition $\{I_{n,k}\}_{n,k}$ of $[0, 1]$. There exists a renormalized star products built from W which can be described as the limit of the renormalized star products built from the initial approximating sequence.

Proof. Each W_n , as a step graphon, is constant on each block equal to the mean of W over that block. Consider the single graphon-mode $1_{[0,\alpha]^2}$ and the regulated weight $\omega_z^{(n)} = A_n z^{-1} + B_n + O(z)$ such that sequences $\{A_n\}_n, \{B_n\}_n$ converge respectively to A, B . The minimal subtraction map generates the counterterm $-A_n z^{-1}$ where its corresponding renormalized coefficient $c_{0,n}^{ren} = B_n$ tends to B and it defines the limit weight for W . For the Poisson tensor π , consider the Nijenhuis operator $N_1 = 2R_{ms} - Id$ together with the bi-differential operator

$$B_W(f, g)(x) = (a.1_{[0,\alpha]}(x))\pi^{ij}(x)\partial_i f(x)\partial_j g(x), \tag{75}$$

to define

$$f \star_{N_1, ren, n} g = \exp\left(c_{0,n}^{ren} B_{W_n}\right)(f, g) \rightarrow f \star_{N_1, ren, W} g = \exp\left(BB_W\right)(f, g), \tag{76}$$

such that the renormalized star product $\star_{N_1, ren, W}$ can be described as the limit of the renormalized star products built from the initial approximating sequence and Nijenhuis operator N_1 . \square

Example 4. The cut-distance convergent limit of the sequence $\{V_n\}_{n \geq 1}, V_n : [0, 1] \times [0, 1] \rightarrow \mathbb{R}, V_n(x, y) = \min(n, \frac{1}{xy})$, of graphons generates a certain family of Nijenhuis operators which recover renormalization of functions in $L^{[0,1]}$.

Proof. The sequence $\{V_n\}_{n \geq 1}$ is convergent to $V(x, y) = \frac{1}{xy}$ when n tends to infinity. For $n \geq 1$, the bounded operator

$$(N_n f)(x) = \int_{[0,1]} V_n(x, y) f(y) dy, \tag{77}$$

is a Nijenhuis operator on a suitable subalgebra of $L^2([0, 1])$ and it defines the star product

$$f \star_{N_n} g = N_n(f)g + fN_n(g) - N_n(fg). \tag{78}$$

The cut-distance convergence implies the limiting $N_n \rightarrow N$ with respect to the weak operator topology such that the renormalized Nijenhuis operator

$$(N_{ren}f)(x) = Finite - Part \int_{[0,1]} \frac{f(y)}{xy} dy, \tag{79}$$

is extracted by minimal subtraction map. \square

Example 5. The limiting graphon V in Example 4 defines a certain family of renormalized star products on the algebra \mathcal{H}^{st} .

Proof. For any $\epsilon > 0$, define the bounded bilinear integral operator

$$N_{\epsilon,V}(x,y) := \int_{[0,1]^2} \frac{V(u,v)}{(x+\epsilon)(y+\epsilon)(u+\epsilon)(v+\epsilon)} dudv, \tag{80}$$

which acts on $\mathcal{H}^{\text{st,cut}}$. These operators regulate singularities at $x, y, u, v = 0$. When ϵ tends to zero, the kernel $K_\epsilon(x, y, u, v) = \frac{1}{(x+\epsilon)(y+\epsilon)(u+\epsilon)(v+\epsilon)}$ has a Laurent series in ϵ with some divergent terms. Minimal subtraction map removes pole terms to generate the renormalized Nijenhuis operator

$$N_{ren,V}(x,y) = Finite - Part \int_{[0,1]^2} \frac{V(u,v)}{xyuv} dudv. \tag{81}$$

The renormalized star product

$$W_1 \star_{N_{ren,V}} W_2 = N_{ren,V}(W_1)W_2 + W_1N_{ren,V}(W_2) - N_{ren,V}(W_1W_2), \tag{82}$$

encodes the renormalization underlying the limiting $N_{\epsilon,V} \rightarrow N_{ren,V}$. \square

Corollary 7. Consider the decorated version $H_{CK}(\mathbb{N})$ of the Connes–Kreimer Hopf algebra such that vertices of non-planar rooted trees are decorated by natural numbers. For any family $\{\bullet_n\}_{n \geq 1}$ of decorated vertices, consider any recursive equation $E: X = \mathbb{I} + \sum_{n=1}^\infty c^n B_{\bullet_n}^+(X^{n+1})$ with $c \geq 1$ in $H_{CK}(\mathbb{N})$. Its solution is represented by an unbounded stretched graphon W_X in $\mathcal{W}_{\approx}^{\text{st}}(\mathbb{R})$ such that W_X is renormalizable and the map $(W_X, z) \mapsto \phi_+^z(W_X)$ has an analytic continuation in any infinitesimal punctured in the complex plane disk around $z = 0$.

Proof. It is a consequence of Lemmas 2, 3, Theorem 1 and Corollaries 4, 6 where $X = \sum_{n=0}^\infty c^n X_n$ is given by the

recursive relations $X_n = \sum_{j=1}^n B_{\bullet_n}^+ \left(\sum_{k_1+\dots+k_{j+1}=n-j, k_i \geq 0} X_{k_1} \dots X_{k_{j+1}} \right)$ with $X_0 = \mathbb{I}$.

Thanks to [13,22,26,27], the collection $\{X_n\}_{n \geq 1}$ generates a graded Hopf subalgebra H_E of $H_{CK}(\mathbb{N})$ which embeds in $\mathcal{H}^{\text{st,cut}}$ such that $char_\lambda H_E \subset C_{\lambda,E} = (L(H_E, A_{dr}), \circ_\lambda)$. The injective morphism $H_E \rightarrow \mathcal{H}^{\text{st,cut}}$ of Hopf algebras defines a surjective morphism $char_\lambda \mathcal{H}^{\text{st,cut}} \rightarrow char_\lambda H_E$. It allows us to lift the symplectic geometry model introduced in Corollary 6 onto $C_{\lambda,E}$. \square

Thanks to Corollaries 3, 4 and 5, the symplectic geometry model introduced in Corollary 6 can be modified for the L^p -topological Hopf algebra of stretched graphon $\mathcal{H}_p^{\text{st,cut}}$, $p \geq 1$, to show the analytic continuation of renormalized values of any direct sum of L^p -stretched graphons with respect to the method of Hopf–Birkhoff factorization.

This study addresses a new interrelation between the renormalization of unbounded stretched graphons via the method of Hopf–Birkhoff factorization (i.e. Theorems 1, 2 and Corollary 6) and the challenge of the convergence of star products in deformation quantization discussed in [32,35].

Acknowledgments: The author would like to thank the Reviewer because of addressing important comments which were helpful to clarify the results of this work.

Conflicts of Interest: The author declares that he has no known competing financial interests which could influence this paper.

References

- [1] Borgs, C., Chayes, J. T., Lovasz, L., Sos, V. T., & Vesztergombi, K. (2008). Convergent sequences of dense graphs I: Subgraph frequencies, metric properties, and testing. *Advances in Mathematics*, 219(6), 1801–1851.
- [2] Borgs, C., Chayes, J. T., Lovasz, L., Sos, V. T., & Vesztergombi, K. (2012). Convergent sequences of dense graphs II: Multiway cuts and statistical physics. *Annals of Mathematics*, 176(1), 151–219.
- [3] Borgs, C., Chayes, J., Lovasz, L., Sos, V., & Vesztergombi, K. (2011). Limits of randomly grown graph sequences. *European Journal of Combinatorics*, 32(7), 985–999.
- [4] Janson, S. (2013). *Graphons, Cut Norm and Distance, Couplings and Rearrangements* (NYJM Monographs, 4). State University of New York.
- [5] Lovasz, L. (2012). *Large Networks and Graph Limits* (AMS Colloquium Publications, Vol. 60). American Mathematical Society.
- [6] Bollobas, B., & Riordan, O. (2009). Metrics for sparse graphs. In *Surveys in Combinatorics 2009* (London Mathematical Society Lecture Note Series, Vol. 365, pp. 211–287). Cambridge University Press.
- [7] Borgs, C., Chayes, J. T., Cohn, H., & Zhao, Y. (2019). An L^p theory of sparse graph convergence I: Limits, sparse random graph models, and power-law distributions. *Transactions of the American Mathematical Society*, 372(5), 3019–3062.
- [8] Borgs, C., Chayes, J. T., Cohn, H., & Zhao, Y. (2018). An L^p theory of sparse graph convergence II: LD convergence, quotients, and right convergence. *Annals of Probability*, 46(1), 337–396.
- [9] Borgs, C., Chayes, J. T., Cohn, H., & Holden, N. (2017). Sparse exchangeable graphs and their limits via graphon processes. *Journal of Machine Learning Research*, 18, Paper No. 210, 71 pp.
- [10] Gao, S., & Gaines, P. E. (2020). Graphon control of large-scale networks of linear systems. *IEEE Transactions on Automatic Control*, 65(10), 4090–4105.
- [11] Shojaei-Fard, A. (2023). A statistical mechanical model for non-perturbative regimes. *Nuclear Physics B*, 991, 116220, 24 pp.
- [12] Shojaei-Fard, A. (2022). Non-perturbative graph languages, halting problem and complexity. *Forum Mathematicum*, 34(5), 1159–1185.
- [13] Shojaei-Fard, A. (2021). The complexities of nonperturbative computations. *Russian Journal of Mathematical Physics*, 28(3), 358–376.
- [14] Shojaei-Fard, A. (2025). Renormalization bi-Heyting algebra. *Journal of Applied Analysis*. Advance online publication, 18 pp.
- [15] Shojaei-Fard, A. (2024). Subsystems via quantum motions. *Analysis and Mathematical Physics*, 14(3), Article 61, 36 pp.
- [16] Collins, J. C. (2023). *Renormalization: An Introduction to Renormalization, the Renormalization Group, and the Operator-Product Expansion* (Cambridge Monographs on Mathematical Physics). Cambridge University Press.
- [17] Connes, A., & Marcolli, M. (2008). *Noncommutative Geometry, Quantum Fields and Motives* (AMS Colloquium Publications, Vol. 55). American Mathematical Society.
- [18] Kreimer, D. (2003). New mathematical structures in renormalizable quantum field theories. *Annals of Physics*, 303(1), 179–202.
- [19] Zimmermann, W. (1969). Convergence of Bogoliubov’s method of renormalization in momentum space. *Communications in Mathematical Physics*, 15, 208–234.
- [20] Connes, A., & Kreimer, D. (2000). Renormalization in quantum field theory and the Riemann–Hilbert problem I: The Hopf algebra structure of graphs and the main theorem. *Communications in Mathematical Physics*, 210, 249–273.
- [21] Guo, L., Paycha, S., & Zhang, B. (2011). Renormalization by Birkhoff–Hopf factorization and by generalized evaluators: A case study. In *Noncommutative geometry, arithmetic, and related topics* (pp. 183–211). Johns Hopkins University Press.
- [22] Figueroa, H., & Gracia-Bondía, J. M. (2004). The uses of Connes and Kreimer’s algebraic formulation of renormalization theory. *International Journal of Modern Physics A*, 19(16), 2739–2754.
- [23] van Suijlekom, W. D. (2007). Renormalization of gauge fields: A Hopf algebra approach. *Communications in Mathematical Physics*, 276(3), 773–798.
- [24] Connes, A., & Kreimer, D. (2001). Renormalization in quantum field theory and the Riemann–Hilbert problem II: The β -function, diffeomorphisms, and the renormalization group. *Communications in Mathematical Physics*, 216, 215–241.
- [25] Shojaei-Fard, A. (2021). The dynamics of non-perturbative phases via Banach bundles. *Nuclear Physics B*, 969, 115478, 39 pp.
- [26] Ebrahimi-Fard, K., Guo, L., & Kreimer, D. (2004). Integrable renormalization. I. The ladder case. *Journal of Mathematical Physics*, 45(10), 3758–3769.

- [27] Ebrahimi-Fard, K., Guo, L., & Kreimer, D. (2005). Integrable renormalization. II. The general case. *Annales Henri Poincaré*, 6(2), 369–395.
- [28] Kreimer, D., & van Suijlekom, W. D. (2009). Recursive relations in the core Hopf algebra. *Nuclear Physics B*, 820(3), 682–693.
- [29] Bernstein, J. (1972). The analytic continuation of generalized functions with respect to a parameter. *Functional Analysis and Its Applications*, 6(4), 273–285.
- [30] Etingof, P. (1999). Note on dimensional regularization. In *Quantum field theory and strings: A course for mathematicians* (Vol. 1, pp. 595–607). American Mathematical Society.
- [31] Ebrahimi-Fard, K. (n.d.). On the associative Nijenhuis relation. *Electronic Journal of Combinatorics*, 11(1), Research Paper 38, 13 pp.
- [32] Hori, T., & Koikawa, T. (2003). Quantization via star products. *Progress of Theoretical Physics*, 110(1), 127–136.
- [33] Dubois-Violette, M. (2001). Lectures on graded differential algebras and noncommutative geometry. In Y. Maeda, H. Moriyoshi, H. Omori, D. Sternheimer, T. Tate, & S. Watamura (Eds.), *Noncommutative differential geometry and its applications to physics* (Mathematical Physics Studies, Vol. 23, pp. 245–306). Kluwer Academic Publishers.
- [34] Shojaei-Fard, A. (2025). The "zero charge problem" via graphon processes. *Complex Analysis and Operator Theory*, 19(6), Article 168, 53 pp.
- [35] Waldmann, S. (2019). Convergence of star products: From examples to a general framework. *EMS Surveys in Mathematical Sciences*, 6(1/2), 1–31.



© 2025 by the authors; licensee PSRP, Lahore, Pakistan. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC-BY) license (<http://creativecommons.org/licenses/by/4.0/>).