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A generalization of the Mingzhe integral inequality

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Abstract: In this article, we present a new Hilbert-type integral inequality involving variable weight functions and an adjustable parameter. It can be described as a generalization of the Mingzhe integral inequality. Some other integral inequalities are also derived. These results provide a flexible framework for obtaining valuable bounds and facilitating further analytical applications.

Keywords: Hilbert integral inequality, Mingzhe integral inequality, Cauchy-Schwarz integral inequality

MSC: 26D15.

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1. Introduction

The classical Hilbert integral inequality is presented below. Let $f, g : [0, +\infty) \to [0, +\infty)$ be two measurable functions such that

$$\int_0^{+\infty} f^2(x)dx < +\infty, \quad \int_0^{+\infty} g^2(y)dy < +\infty.$$

Then we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y} dxdy \leq \pi \left(\int_0^{+\infty} f^2(x)dx \right)^{1/2} \left(\int_0^{+\infty} g^2(y)dy \right)^{1/2}.$$

Moreover, the constant π is the best possible. See [1]. The Hilbert integral inequality has inspired numerous extensions and generalizations. For more information, see the survey [2], the book [3], and the articles [4–18].

For the purposes of this article, we will focus on the work of G. Mingzhe in [10]. The Mingzhe integral inequality, which we shall refer to as such, is described in the theorem below.

Theorem 1. *[10, Theorem 2] Let $f, g : [0, +\infty) \to [0, +\infty)$ be two measurable functions such that*

$$\int_0^{+\infty} f^2(x)dx < +\infty, \quad \int_0^{+\infty} g^2(y)dy < +\infty.$$

For any $x \geq 0$, we set

$$u(x) = 2\arctan\left(\frac{1}{\sqrt{2x+1}}\right).$$

Then we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y+1} dxdy \leq \left(\int_0^{+\infty} f^2(x) \left(\pi - u(x)\right) dx \right)^{1/2} \left(\int_0^{+\infty} g^2(y) \left(\pi - u(y)\right) dy \right)^{1/2}.$$

This theorem is of particular interest because it refines the classical Hilbert integral inequality by introducing a non-constant weight function, i.e., $\pi - u(x)$. Unlike in the classical case, where the constant π is uniform, the present inequality captures a finer structure of the kernel function $1/(x+y+1)$ through the function $u(x)$. Notably, the weight function $\pi - u(x)$, explicitly depends on x and reflects the asymmetry introduced by the shift in the denominator. This can lead to a more precise upper bound in certain situations, highlighting the importance of variable weights in Hilbert-type integral inequalities.

In this article, we generalize this theorem by introducing an adjustable parameter λ, which significantly influences the associated weight function. We analyze the impact of this parameter by allowing it to vary over its admissible range, paying particular attention to the limiting behavior as $\lambda \to +\infty$. Furthermore, in the spirit of the methodology presented in [10], we establish a corollary and derive an additional new theorem.

The remainder of this article is organized as follows: In §2, we present the main results of the article. §3 is devoted to several related results and further developments. Finally, §4 concludes the article with a summary of the findings and a discussion of possible directions for future research.

2. Main result

2.1. Statement and proof

The statement and proof of our main theorem are presented below.

Theorem 2. *Let* $f, g : [0, +\infty) \to [0, +\infty)$ *be two measurable functions such that*

$$\int_0^{+\infty} f^2(x)dx < +\infty, \quad \int_0^{+\infty} g^2(y)dy < +\infty.$$

For any $\lambda > 1$ *and* $x \geq 0$, *we set*

$$v_\lambda(x) = \frac{\sqrt{\lambda x + 1}}{\sqrt{\lambda x + \lambda - 1}}, \quad w_\lambda(x) = 2\arctan\left(\frac{1}{\sqrt{\lambda x + \lambda - 1}}\right).$$

Then, for any $\lambda > 1$, *we have*

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{x+y+1}dxdy \leq \left(\int_0^{+\infty} f^2(x)v_\lambda(x)\left(\pi - w_\lambda(x)\right)dx\right)^{1/2}\left(\int_0^{+\infty} g^2(y)v_\lambda(y)\left(\pi - w_\lambda(y)\right)dy\right)^{1/2}.$$

Proof. Using a suitable decomposition of the integrand via

$$\left(\frac{\lambda x + 1}{\lambda y + 1}\right)^{1/4}\left(\frac{\lambda y + 1}{\lambda x + 1}\right)^{1/4} = 1,$$

which introduces the parameter λ, and the Cauchy-Schwarz integral inequality applied to appropriate functions, we get

$$\begin{aligned}
\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{x+y+1}dxdy &= \int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{x+y+1}\left(\frac{\lambda x + 1}{\lambda y + 1}\right)^{1/4}\left(\frac{\lambda y + 1}{\lambda x + 1}\right)^{1/4} dxdy \\
&= \int_0^{+\infty}\int_0^{+\infty} \frac{f(x)}{(x+y+1)^{1/2}}\left(\frac{\lambda x + 1}{\lambda y + 1}\right)^{1/4} \times \frac{g(y)}{(x+y+1)^{1/2}}\left(\frac{\lambda y + 1}{\lambda x + 1}\right)^{1/4} dxdy \\
&\leq \left(\int_0^{+\infty}\int_0^{+\infty} \frac{f^2(x)}{x+y+1}\sqrt{\frac{\lambda x + 1}{\lambda y + 1}}dxdy\right)^{1/2} \\
&\quad \times \left(\int_0^{+\infty}\int_0^{+\infty} \frac{g^2(y)}{x+y+1}\sqrt{\frac{\lambda y + 1}{\lambda x + 1}}dxdy\right)^{1/2}.
\end{aligned}$$

Using standard primitives and the definitions of the weight functions $v_\lambda(x)$ and $w_\lambda(x)$, we obtain

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f^2(x)}{x+y+1}\sqrt{\frac{\lambda x + 1}{\lambda y + 1}}dxdy = \int_0^{+\infty} f^2(x)\sqrt{\lambda x + 1}\left(\int_0^{+\infty} \frac{1}{(x+y+1)\sqrt{\lambda y + 1}}dy\right)dx$$

$$= \int_0^{+\infty} f^2(x)\sqrt{\lambda x+1}\left[\frac{2}{\sqrt{\lambda x+\lambda-1}}\arctan\left(\frac{\sqrt{\lambda y+1}}{\sqrt{\lambda x+\lambda-1}}\right)\right]_{y=0}^{y\to+\infty} dx$$
$$= \int_0^{+\infty} f^2(x)\frac{\sqrt{\lambda x+1}}{\sqrt{\lambda x+\lambda-1}}\left(\pi - 2\arctan\left(\frac{1}{\sqrt{\lambda x+\lambda-1}}\right)\right) dx$$
$$= \int_0^{+\infty} f^2(x)v_\lambda(x)\left(\pi - w_\lambda(x)\right) dx.$$

Similarly, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{g^2(y)}{x+y+1}\sqrt{\frac{\lambda y+1}{\lambda x+1}}dxdy$$
$$= \int_0^{+\infty} g^2(y)\sqrt{\lambda y+1}\left(\int_0^{+\infty}\frac{1}{(x+y+1)\sqrt{\lambda x+1}}dx\right)dy$$
$$= \int_0^{+\infty} g^2(y)\sqrt{\lambda y+1}\left[\frac{2}{\sqrt{\lambda y+\lambda-1}}\arctan\left(\frac{\sqrt{\lambda x+1}}{\sqrt{\lambda y+\lambda-1}}\right)\right]_{x=0}^{x\to+\infty} dy$$
$$= \int_0^{+\infty} g^2(y)\frac{\sqrt{\lambda y+1}}{\sqrt{\lambda y+\lambda-1}}\left(\pi - 2\arctan\left(\frac{1}{\sqrt{\lambda y+\lambda-1}}\right)\right) dy$$
$$= \int_0^{+\infty} g^2(y)v_\lambda(y)\left(\pi - w_\lambda(y)\right) dy.$$

Combining the above inequalities, we get

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{x+y+1}dxdy \le \left(\int_0^{+\infty} f^2(x)v_\lambda(x)\left(\pi - w_\lambda(x)\right)dx\right)^{1/2}\left(\int_0^{+\infty} g^2(y)v_\lambda(y)\left(\pi - w_\lambda(y)\right)dy\right)^{1/2}.$$

This concludes the proof of the theorem. □

We highlight in particular the introduction of the new weight functions v_λ and w_λ, as well as the presence of the adjustable parameter λ, which provides additional flexibility in the formulation of the inequality. More precisely, the parameter λ allows one to generate a family of inequalities, potentially leading to improved bounds in specific settings. A more detailed discussion of the role and significance of λ is provided below.

2.2. Discussion

Clearly, if we take $\lambda = 2$, then Theorem 2 reduces to [10, Theorem 2], as recalled in Theorem 1.

Let us now investigate the case $\lambda \to +\infty$. We have

$$\lim_{\lambda\to+\infty} v_\lambda(x) = \lim_{\lambda\to+\infty}\frac{\sqrt{\lambda x+1}}{\sqrt{\lambda x+\lambda-1}} = \sqrt{\frac{x}{x+1}},$$

and

$$\lim_{\lambda\to+\infty} w_\lambda(x) = \lim_{\lambda\to+\infty} 2\arctan\left(\frac{1}{\sqrt{\lambda x+\lambda-1}}\right) = 0.$$

In this case, by the Lebesgue dominated convergence theorem, Theorem 2 yields

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{x+y+1}dxdy \le \pi\left(\int_0^{+\infty} f^2(x)\sqrt{\frac{x}{x+1}}dx\right)^{1/2}\left(\int_0^{+\infty} g^2(y)\sqrt{\frac{y}{y+1}}dy\right)^{1/2}.$$

This result is of particular interest as it provides a nontrivial weighted Hilbert-type integral inequality with a shifted kernel function. The appearance of the weight function $\sqrt{x/(x+1)}$ reflects a balance between the decay of the kernel function and the growth of the functions involved. Moreover, the constant π is optimal in many related inequalities of this type, highlighting the interest of the upper bound.

For any $\lambda \in (1, 2]$, we also have $v_\lambda(x) \in [0, 1]$ and $w_\lambda(x) \in [0, \pi]$. Therefore, Theorem 2 implies that

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y+1} dxdy \leq \left(\int_0^{+\infty} f^2(x) v_\lambda(x) \left(\pi - w_\lambda(x)\right) dx \right)^{1/2} \left(\int_0^{+\infty} g^2(y) v_\lambda(y) \left(\pi - w_\lambda(y)\right) dy \right)^{1/2}$$
$$\leq \pi \left(\int_0^{+\infty} f^2(x) dx \right)^{1/2} \left(\int_0^{+\infty} g^2(y) dy \right)^{1/2}.$$

This inequality establishes a clear connection with the classical Hilbert integral inequality. In particular, the parameter λ generates a family of intermediate weighted inequalities that interpolate between refined bounds and the standard Hilbert-type bound.

3. Complementary results

Some results related to Theorem 2 are provided in this section.

3.1. A corollary

A simple corollary to Theorem 2 is given below.

Corollary 1. *Let $f : [0, +\infty) \to [0, +\infty)$ be a measurable function such that*

$$\int_0^{+\infty} f^2(x) dx < +\infty.$$

For any $\lambda > 1$ and $x \geq 0$, we set

$$v_\lambda(x) = \frac{\sqrt{\lambda x + 1}}{\sqrt{\lambda x + \lambda - 1}}, \quad w_\lambda(x) = 2 \arctan\left(\frac{1}{\sqrt{\lambda x + \lambda - 1}} \right).$$

Then, for any $\lambda > 1$, we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)f(y)}{x+y+1} dxdy \leq \int_0^{+\infty} f^2(x) v_\lambda(x) \left(\pi - w_\lambda(x)\right) dx.$$

Proof. The result follows immediately by taking $g = f$ in Theorem 2. This completes the proof. □

3.2. A new theorem

Inspired by Theorem [10, Theorem 4], the theorem below uses Theorem 2 to establish a new integral inequality.

Theorem 3. *Let $h : [0, 1] \to [0, +\infty)$ be a measurable function such that*

$$\int_0^1 h^2(x) dx < +\infty.$$

For any $x \geq 0$, we consider the integral function

$$f(x) = \int_0^1 t^x h(t) dt.$$

For any $\lambda > 1$ and $x \geq 0$, we set

$$v_\lambda(x) = \frac{\sqrt{\lambda x + 1}}{\sqrt{\lambda x + \lambda - 1}}, \quad w_\lambda(x) = 2 \arctan\left(\frac{1}{\sqrt{\lambda x + \lambda - 1}} \right).$$

Then we have

$$\left(\int_0^{+\infty} f^2(x) dx \right)^2 \leq \left(\int_0^{+\infty} f^2(x) v_\lambda(x) \left(\pi - w_\lambda(x)\right) dx \right) \int_0^1 h^2(t) dt.$$

Proof. We write
$$f^2(x) = f(x)\int_0^1 t^x h(t)dt = \int_0^1 f(x)t^x h(t)dt.$$

Using this, the Fubini-Tonelli integral theorem, the Cauchy-Schwarz integral inequality and Corollary 1, we get

$$\begin{aligned}
\left(\int_0^{+\infty} f^2(x)dx\right)^2 &= \left(\int_0^{+\infty}\int_0^1 f(x)t^x h(t)dtdx\right)^2 \\
&= \left(\int_0^1 \left(\int_0^{+\infty} f(x)t^x dx\right) h(t)dt\right)^2 \\
&\leq \left(\int_0^1 \left(\int_0^{+\infty} f(x)t^x dx\right)^2 dt\right)\int_0^1 h^2(t)dt \\
&= \left(\int_0^1 \left(\int_0^{+\infty}\int_0^{+\infty} f(x)f(y)t^x t^y dxdy\right) dt\right)\int_0^1 h^2(t)dt \\
&= \left(\int_0^{+\infty}\int_0^{+\infty} f(x)f(y)\left(\int_0^1 t^{x+y}dt\right) dxdy\right)\int_0^1 h^2(t)dt \\
&= \left(\int_0^{+\infty}\int_0^{+\infty} f(x)f(y)\left[\frac{t^{x+y+1}}{x+y+1}\right]_0^1 dxdy\right)\int_0^1 h^2(t)dt \\
&= \left(\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{x+y+1}dxdy\right)\int_0^1 h^2(t)dt \\
&\leq \left(\int_0^{+\infty} f^2(x)v_\lambda(x)\left(\pi - w_\lambda(x)\right)dx\right)\int_0^1 h^2(t)dt.
\end{aligned}$$

This completes the proof. □

Again, we emphasize the introduction of the new weight functions v_λ and w_λ, as well as the presence of the adjustable parameter λ, which provides additional flexibility in the formulation of the inequality.

4. Conclusion

In this article, we have established a new integral inequality with adjustable weight functions, generalizing and refining the Mingzhe integral inequality. The introduction of the parameter λ provides additional flexibility, allowing for valuable upper bounds and potential applications in analysis. Future work will focus on exploring optimal choices of λ and investigating multidimensional extensions.

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