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A note on the boundedness of the Wolff potential on complete noncompact manifolds in Zygmund spaces

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Abstract: In this note, we prove sharp necessary conditions for the boundedness of the manifold-adapted Wolff potential between Zygmund spaces on complete noncompact Riemannian manifolds with nonnegative Ricci curvature. The nonlinear homogeneity of the operator determines the natural form of the norm inequality, while the Bishop comparison theorem and localized ball tests determine the admissible Sobolev scaling. The boundedness assumption forces Euclidean lower volume growth, identifies the relation between the source and target integrability exponents, and gives the critical logarithmic constraint for the Zygmund indices. In the linear case, the conclusions reduce to the corresponding Riesz-potential scaling with the expected logarithmic refinement.

Keywords: Wolff potential, Zygmund spaces, Riemannian manifolds

MSC: 31C45, 46E30.

1. Introduction

Let M be a complete noncompact Riemannian manifold of dimension n. Denote by $\rho(x,y)$ the Riemannian distance, by $B_x(r)$ the geodesic ball with centre x and radius r, and by

$$V_x(r) = \mathrm{Vol}(B_x(r)),$$

the Riemannian volume of this ball. The interaction between potential estimates and volume growth is a central theme in analysis on manifolds. For Riesz potentials on manifolds with nonnegative Ricci curvature, Li [1] used heat-kernel estimates of Li–Yau [2] to connect boundedness properties of fractional powers of the Laplacian with the Euclidean volume rate. Related geometric inequalities under Ricci lower bounds appear in [3–7].

The purpose of this paper is to identify the corresponding necessary conditions for Wolff potentials acting on Zygmund spaces. The potential is normalized by the intrinsic ball volume, which is the natural choice on spaces where the large-scale volume need not be Euclidean. For $p > 1$ and $0 < \alpha p < n$, define

$$\mathcal{W}_{\alpha,p} f(x) := \int_0^\infty \left(\frac{r^{\alpha p}}{V_x(r)} \int_{B_x(r)} |f(y)|\, dy \right)^{\frac{1}{p-1}} \frac{dr}{r}, \tag{1}$$

for nonnegative measurable functions f; for a general measurable function, $|f|$ is used in place of f. On $\mathbb{R}^n$, where $V_x(r) = \omega_n r^n$, (1) agrees with the classical Wolff potential up to the constant $\omega_n^{-1/(p-1)}$; see, for example, [8]. When $p = 2$, (1) becomes linear and, in Euclidean space,

$$\mathcal{W}_{\alpha,2} f(x) = \frac{1}{\omega_n (n-2\alpha)} \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-2\alpha}}\, dy, \qquad 0 < 2\alpha < n,$$

which is a constant multiple of the Riesz potential of order 2α; see [9].

For $1 \le s < \infty$ and $\gamma \in \mathbb{R}$, the Zygmund space $L^s \log^\gamma L(M)$ is equipped with the Luxemburg-type gauge

$$\|f\|_{L^s \log^\gamma L} := \inf\left\{\lambda > 0 : \int_M \left(\frac{|f(x)|}{\lambda}\right)^s \log^\gamma\left(e + \frac{|f(x)|}{\lambda}\right) dx \le 1\right\}. \tag{2}$$

Equivalent standard monotone gauges give the same space and the same estimates up to constants. Since $\log^0(\cdot) \equiv 1$, one has $L^s \log^0 L(M) = L^s(M)$ with the usual L^s norm.

The estimate studied below is

$$\|\mathcal{W}_{\alpha,p} f\|_{L^q \log^{\alpha_2} L} \le C\, \|f\|^{1/(p-1)}_{L^s \log^{\alpha_1} L}. \tag{3}$$

The exponent $1/(p-1)$ is forced by the homogeneity $\mathcal{W}_{\alpha,p}(tf) = t^{1/(p-1)} \mathcal{W}_{\alpha,p} f$. Thus a linear dependence on $\|f\|$ is compatible with (1) only in the case $p = 2$. The main theorem shows that (3) determines the Euclidean volume rate, the Sobolev exponent, and the admissible logarithmic target scale.

2. Preliminaries

Lemma 1. *Let M be a complete Riemannian manifold with nonnegative Ricci curvature. Then the function $r \mapsto V_x(r)/r^n$ is nonincreasing for each $x \in M$. Consequently, for $0 < r \le R$,*

$$V_x(R) \le \left(\frac{R}{r}\right)^n V_x(r), \qquad V_x(r) \ge \left(\frac{r}{R}\right)^n V_x(R). \tag{4}$$

In particular, $V_x(\lambda r) \le \lambda^n V_x(r)$ for every $\lambda \ge 1$, and

$$\lim_{r \downarrow 0} \frac{V_x(r)}{\omega_n r^n} = 1. \tag{5}$$

Lemma 2. *Let $1 \le s < \infty$, $\gamma \in \mathbb{R}$, and*

$$\ell(m) := 1 + \log(e + m^{-1}), \qquad m > 0.$$

There are constants $c, C > 0$, depending only on s and γ, such that for all measurable sets $E \subset M$ with $0 < |E| < \infty$ and all $A > 0$,

$$c\, A\, |E|^{1/s} \ell(|E|)^{\gamma/s} \le \|A \chi_E\|_{L^s \log^\gamma L} \le C\, A\, |E|^{1/s} \ell(|E|)^{\gamma/s}. \tag{6}$$

Moreover, if $g \ge A$ almost everywhere on E, then

$$\|g\|_{L^s \log^\gamma L} \ge c\, A\, |E|^{1/s} \ell(|E|)^{\gamma/s}. \tag{7}$$

Proof. By homogeneity it is enough to consider $A = 1$. The Luxemburg gauge of χ_E is determined by the unique scale at which $|E|\, t^s \log^\gamma(e + t)$ is comparable to 1. Solving this relation gives $t \asymp |E|^{-1/s} \ell(|E|)^{-\gamma/s}$, which is equivalent to (6). The final assertion follows from the lattice property of Zygmund spaces and (6). □

3. Main Result

Theorem 1. *Let M be a complete noncompact Riemannian manifold with nonnegative Ricci curvature. Let $p > 1$, $0 < \alpha p < n$, $1 \le s < \infty$, $1 < q < \infty$, and $\alpha_1, \alpha_2 \in \mathbb{R}$. Assume that there exists $C > 0$ such that*

$$\|\mathcal{W}_{\alpha,p} f\|_{L^q \log^{\alpha_2} L} \le C\, \|f\|^{1/(p-1)}_{L^s \log^{\alpha_1} L} \qquad \text{for all } f \in L^s \log^{\alpha_1} L(M). \tag{8}$$

Then the following conditions hold:

(i) *there is a constant $c > 0$ such that*

$$V_x(r) \ge c\, r^n \qquad \text{for all } x \in M \text{ and } r > 0; \tag{9}$$

(ii) *the Sobolev scaling relation is*

$$\frac{1}{q} = \frac{1}{s(p-1)} - \frac{\alpha p}{n(p-1)}; \tag{10}$$

(iii) *the Zygmund indices satisfy*

$$\frac{(p-1)\alpha_2}{q} \le \frac{\alpha_1}{s}. \tag{11}$$

In particular, (10) *implies* $s < n/(\alpha p)$, *and the critical logarithmic target exponent is*

$$\alpha_2 = \frac{q}{s(p-1)}\,\alpha_1. \tag{12}$$

Remark 1. If (8) is replaced by an estimate with the first power of $\|f\|_{L^s \log^{\alpha_1} L}$ on the right-hand side, then the homogeneity of (1) forces $p = 2$. Thus Theorem 1 contains the linear Riesz-potential case as the only linearly homogeneous member of the Wolff scale.

Proof of Theorem 1. Fix $x_0 \in M$ and $R > 0$, and set

$$B_R := B_{x_0}(R), \qquad m_R := V_{x_0}(R), \qquad f_R := \chi_{B_R}.$$

The proof tests (8) on the family $\{f_R\}_{R>0}$ and separates three effects: the power of R, the volume ratio $R^n/V_{x_0}(R)$, and the logarithmic factor coming from the Zygmund gauge.

Lower bound for the Wolff potential. Let $z \in B_{x_0}(R/2)$ and $3R/2 \le r \le 2R$. Then $B_R \subset B_z(r)$. Moreover, $B_z(r) \subset B_{x_0}(3R)$, and Lemma 1 gives

$$V_z(r) \le V_{x_0}(3R) \le 3^n V_{x_0}(R) = 3^n m_R.$$

Therefore

$$\begin{aligned} \mathcal{W}_{\alpha,p} f_R(z) &\ge \int_{3R/2}^{2R} \left(\frac{r^{\alpha p}}{V_z(r)} \int_{B_z(r)} \chi_{B_R}(y)\, dy \right)^{1/(p-1)} \frac{dr}{r} \\ &\ge c \int_{3R/2}^{2R} r^{\alpha p/(p-1)} \frac{dr}{r} \ge c\, R^{\alpha p/(p-1)}. \end{aligned} \tag{13}$$

By Lemma 1, $V_{x_0}(R/2) \ge 2^{-n} m_R$, and hence $V_{x_0}(R/2) \asymp m_R$. Applying Lemma 2 to (13) on $B_{x_0}(R/2)$ yields

$$\|\mathcal{W}_{\alpha,p} f_R\|_{L^q \log^{\alpha_2} L} \ge c\, R^{\alpha p/(p-1)} m_R^{1/q} \ell(m_R)^{\alpha_2/q}. \tag{14}$$

Upper bound for the source norm. The characteristic-function estimate in Lemma 2 gives

$$\|f_R\|_{L^s \log^{\alpha_1} L}^{1/(p-1)} \le C\, m_R^{1/[s(p-1)]} \ell(m_R)^{\alpha_1/[s(p-1)]}. \tag{15}$$

Combining (14), (15), and the assumed boundedness (8), and then raising the resulting inequality to the power $p-1$, gives

$$R^{\alpha p} m_R^{\eta} \ell(m_R)^{\theta} \le C, \qquad R > 0, \tag{16}$$

where

$$\eta := \frac{p-1}{q} - \frac{1}{s}, \qquad \theta := \frac{(p-1)\alpha_2}{q} - \frac{\alpha_1}{s}. \tag{17}$$

Determination of the Sobolev exponent. First let $R \downarrow 0$. By (5), $m_R \sim \omega_n R^n$, and $\ell(m_R) \sim n \log(1/R)$. If $\alpha p + n\eta < 0$, then the left-hand side of (16) becomes unbounded as $R \downarrow 0$, which is impossible. Hence

$$\eta \ge -\frac{\alpha p}{n}. \tag{18}$$

We next prove the reverse inequality. Suppose that $\eta > -\alpha p/n$. If $\eta \geq 0$, then for $R \geq 1$ the factor $m_R^\eta \ell(m_R)^\theta$ is bounded below by a positive constant depending only on x_0 and the parameters, while $R^{\alpha p} \to \infty$. This contradicts (16). If $-\alpha p/n < \eta < 0$, then (16) implies, for $R \geq 1$,

$$m_R \geq c_{x_0} R^{\alpha p/(-\eta)}.$$

Because $\alpha p/(-\eta) > n$, this contradicts the Bishop upper bound $m_R \leq CR^n$ for large R. Therefore

$$\eta = -\frac{\alpha p}{n}. \tag{19}$$

Substituting the definition of η from (17) gives (10).

Logarithmic constraint. With (19) imposed, the small-ball asymptotic in (16) reduces to

$$C_0\, \ell(m_R)^\theta \leq C \qquad \text{as } R \downarrow 0.$$

Since $\ell(m_R) \to \infty$ as $R \downarrow 0$, this inequality can hold only if

$$\theta \leq 0. \tag{20}$$

Using (17), (20) is exactly the Zygmund-index condition (11). Equality in (11) corresponds to the critical target exponent (12); strict inequality corresponds to a weaker target logarithmic scale.

Euclidean lower volume growth. It remains to show (9). With (19), the key estimate (16) can be written as

$$\left(\frac{R^n}{V_{x_0}(R)}\right)^{\alpha p/n} \ell(V_{x_0}(R))^\theta \leq C. \tag{21}$$

Set $Y_{x_0}(R) := R^n/V_{x_0}(R)$. By Bishop comparison, $Y_{x_0}(R)$ is nondecreasing in R. Evaluating (21) at $R = 1$ and using $\theta \leq 0$ shows that $Y_{x_0}(1)$ is bounded above by a constant independent of x_0; indeed, the function $Y \mapsto Y^{\alpha p/n}/(1 + \log(e + Y))^{-\theta}$ tends to infinity as $Y \to \infty$. Hence $Y_{x_0}(R) \leq C$ for $0 < R \leq 1$.

For $R \geq 1$, the bound on $Y_{x_0}(1)$ gives $V_{x_0}(R) \geq V_{x_0}(1) \geq c$, so $\ell(V_{x_0}(R))$ is bounded above and below by positive constants. Eq. (21) therefore gives $Y_{x_0}(R) \leq C$ for all $R \geq 1$. Combining the two ranges yields

$$\frac{R^n}{V_{x_0}(R)} \leq C \qquad \text{for all } x_0 \in M \text{ and } R > 0,$$

which is equivalent to (9). The theorem follows. $\square$

4. Discussion

Theorem 1 shows that boundedness of (1) is not a purely functional-analytic property. The estimate detects the large-scale geometry of M: a Zygmund-space bound of the form (8) is possible only when the manifold has Euclidean lower volume growth. Together with the Bishop upper bound, this means that the admissible manifolds have two-sided Euclidean volume growth,

$$cr^n \leq V_x(r) \leq Cr^n, \qquad x \in M, \quad r > 0.$$

Thus the ball-volume normalization in (1) is compatible with boundedness precisely at the Euclidean volume rate.

The exponent relation (10) is dictated by the simultaneous scaling of the Wolff kernel, the volume of test balls, and the nonlinear homogeneity of the operator. Equivalently,

$$q = \frac{ns(p-1)}{n - \alpha ps},$$

so the range $s < n/(\alpha p)$ is necessary. The logarithmic condition (11) has a direct interpretation: the target Zygmund weight may not be stronger than the critical value in (12). At the critical value, the logarithmic powers on both sides of (8) match; below it, the target space is larger and the logarithmic requirement is weaker.

When $p = 2$, the Wolff potential is linearly homogeneous. In the Lebesgue subcase $\alpha_1 = \alpha_2 = 0$, Theorem 1 gives

$$\frac{1}{q} = \frac{1}{s} - \frac{2\alpha}{n},$$

which is the Sobolev relation for the Riesz potential of order 2α. For Zygmund spaces, the critical logarithmic condition becomes $\alpha_2 = (q/s)\alpha_1$, with boundedness into any weaker logarithmic target allowed by the necessary inequality.

5. Conclusion

The boundedness of the manifold-adapted Wolff potential between Zygmund spaces forces three linked conditions: Euclidean lower volume growth of the underlying manifold, the Sobolev exponent relation (10), and the logarithmic constraint (11). The proof uses localized ball tests and Bishop comparison, so the geometric and analytic restrictions arise from the operator itself rather than from an assumed model geometry. The result clarifies the precise role of nonlinear homogeneity and shows how logarithmic Zygmund refinements alter the admissible target scale without changing the underlying Sobolev power law.

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