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Combinatorial sums derived from properties of Legendre polynomials

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Abstract: From an identity connecting a combinatorial sum and Legendre polynomials, we derive closed forms for a number of combinatorial sums. Some of them are obtained *via* results about the integrals of functions associated with Legendre polynomials.

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MSC: 05A10, 11B65, 11B83.

1. Introduction

As an introduction and for the convenience of the reader, we start by briefly reviewing some basic properties of the Legendre polynomials. This important class of polynomials is named after A.-M. Legendre, who discovered them around 1780.

First, recall that for a complex number x these polynomials $P_n(x)$, $(n = 0, 1, \ldots)$ are defined by

$$P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}\left((x^2-1)^n\right),$$

which is called Rodrigues' formula for $P_n(x)$. Applying Leibniz's formula to the product $(x+1)^n(x-1)^n$, we get their equivalent form

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}^2 (x+1)^{n-k}(x-1)^k. \tag{1}$$

Thus, $P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{3x^2-1}{2}, \ldots$.

The following important formula (generating series) is well-known:

$$\frac{1}{(z^2-2xz+1)^{1/2}} = \sum_{n=0}^{\infty} P_n(x) z^n. \tag{2}$$

We immediately deduce that for all $n \geq 0$ we have $P_{2n+1}(0) = 0$ and $P_{2n}(0) = (-1)^n 2^{-2n}\binom{2n}{n}$ as well as $P_n(1) = 1$. Note also that $P_n(-x) = (-1)^n P_n(x)$.

By differentiation with respect to z followed by a multiplication by $1 - 2xz + z^2$, we derive the recursion

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \quad (n \geq 1). \tag{3}$$

We also recall the orthogonality property of the Legendre polynomials, that is,

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0 \quad \text{for } n \neq m.$$

In particular with $m = 0$, we obtain that

$$\int_{-1}^{1} P_n(x)\,dx = 0 \quad \text{for } n \geq 1.$$

Although classical, Legendre polynomials are still the subject of mathematical research. Papers covering different aspects of the topic include Klemm and Larsen [1], Wan and Zudilin [2], Diekemaa and Koornwinder [3], Guo [4], Chu and Campbell [5], and Aloui [6], to mention a few. The On-Line Encyclopedia of Integer Sequences (OEIS) [7] contains some number sequences associated with Legendre polynomials: A001801, A008316, A110129 or A330203.

In this article, we first use a coefficient extraction formula to link a class of combinatorial sums to Legendre polynomials. From this connection we deduce a range of combinatorial sums via integrals of certain functions associated with Legendre polynomials.

2. A first result

First we recall some basic facts about the Gamma function and generalized binomial coefficients that will be needed later.

The Gamma function, $\Gamma(z)$, is defined for $Re(z) > 0$ by the integral [8]

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\,dt.$$

The function $\Gamma(z)$ can be extended to the whole complex plane by analytic continuation. The Weierstrass form of the Gamma function shows that it has a simple pole at each of the points $z = \cdots, -3, -2, -1, 0$. In addition, by taking the reciprocal we get [8]

$$\frac{1}{\Gamma(z)} = z \prod_{k=1}^{\infty} \left(\left(1 + \frac{z}{k}\right) \left(1 + \frac{1}{k}\right)^{-z} \right),$$

and see that $1/\Gamma(z)$ is an entire function.

A major property of the Gamma function is that it extends the classical factorial function to the complex plane by $(z-1)! = \Gamma(z)$. This means that for complex numbers r and s, the generalized binomial coefficients can be defined by

$$\binom{r}{s} = \frac{\Gamma(r+1)}{\Gamma(s+1)\Gamma(r-s+1)}.$$

If r and s are nonnegative integers then the binomial coefficients are given by

$$\binom{r}{s} = \begin{cases} \dfrac{r!}{s!(r-s)!}, & r \geq s; \\ 0, & r < s. \end{cases}$$

Let $(c_n)_{n\geq 0}$ be a sequence of complex numbers and let $F(z)$ be its ordinary generating function, i.e., the formal power series

$$F(z) = \sum_{k=0}^{\infty} c_k z^k.$$

Let $[z^n]F(z)$ denote the coefficient in $F(z)$ belonging to z^n. We state the following coefficient extraction identity as a lemma:

Lemma 1. *For all $n \geq 1$ we have*

$$\sum_{k=0}^{n} (-1)^{n-k} \frac{2n}{n+k} \binom{n+k}{2k} c_k = [z^n] \frac{1-z}{1+z} F\left(\frac{z}{(1+z)^2} \right). \tag{4}$$

Proof. Here and in Theorem 1 below, the proof takes place in the frame of formal power series. From

$$\frac{1-z}{1+z}F\left(\frac{z}{(1+z)^2}\right)=\sum_{k=0}^{\infty}\frac{c_k z^k}{(1+z)^{2k+1}}-\sum_{k=0}^{\infty}\frac{c_k z^{k+1}}{(1+z)^{2k+1}},$$

and

$$\frac{1}{(1+z)^{2k+1}}=\sum_{j=0}^{\infty}(-1)^j\binom{2k+j}{2k}z^j,$$

we deduce that

$$\frac{1-z}{1+z}F\left(\frac{z}{(1+z)^2}\right)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}c_k(-1)^j\binom{2k+j}{2k}z^{j+k}-\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}c_k(-1)^j\binom{2k+j}{2k}z^{j+k+1}.$$

Therefore

$$\begin{aligned}[z^n]\frac{1-z}{1+z}F\left(\frac{z}{(1+z)^2}\right)&=\sum_{k=0}^{n}c_k(-1)^{n-k}\left(\binom{2k+n-k}{2k}+\binom{2k+n-k-1}{2k}\right)\\&=\sum_{k=0}^{n}c_k(-1)^{n-k}\frac{(2n)(n+k-1)!}{(2k)!(n-k)!}\\&=\sum_{k=0}^{n}(-1)^{n-k}\frac{2n}{n+k}\binom{n+k}{2k}c_k.\end{aligned}$$

□

Now, our first main result follows.

Theorem 1. *For all $n\geq 1$ we have the polynomial identity*

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{2^{-2k}}{n+k}x^k=\frac{(-1)^n}{2n}\left(P_n\left(-\frac{x+2}{2}\right)-P_{n-1}\left(-\frac{x+2}{2}\right)\right),\tag{5}$$

where $P_n(x)$ is the nth Legendre polynomial.

Proof. We shall use the binomial expansion $(1+z)^{-1/2}=\sum_{k=0}^{\infty}\binom{-1/2}{k}z^k$.
We apply Lemma 1 to

$$F(z)=(1+xz)^{-1/2}=\sum_{k=0}^{\infty}\binom{-1/2}{k}x^k z^k,$$

so that $c_k=\binom{2k}{k}2^{-2k}(-1)^k x^k$.
Then, since

$$\frac{1-z}{1+z}F\left(\frac{z}{(1+z)^2}\right)=(1-z)(z^2+(2+x)z+1)^{-1/2},$$

Lemma 1 and Eq. (2) yield

$$(-1)^n(2n)\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{2^{-2k}}{n+k}x^k=P_n\left(-\frac{x+2}{2}\right)-P_{n-1}\left(-\frac{x+2}{2}\right),$$

the desired identity. □

Remark 1. We note that

$$\binom{n+k}{k}\binom{n}{k}=\binom{n+k}{n}\binom{n}{n-k}=\binom{n+k}{n-k}\binom{2k}{k}=\binom{n+k}{2k}\binom{2k}{k}.$$

Corollary 1. *For all $n \geq 1$*

$$\sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k} \frac{2^{-2k}}{n+k} = \frac{(-1)^n}{2n} (S_n - S_{n-1}), \tag{6}$$

and

$$\sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k} (-1)^k \frac{2^{-2k}}{n+k} = \frac{(-1)^n}{2n} (Q_n - Q_{n-1}), \tag{7}$$

where

$$S_n = \left(-\frac{5}{4}\right)^n \sum_{k=0}^{n} \binom{n}{k}^2 5^{-k},$$

and

$$Q_n = \left(-\frac{3}{4}\right)^n \sum_{k=0}^{n} \binom{n}{k}^2 (-1)^k 3^{-k}.$$

Proof. From Eq. (1), S_n and Q_n are the values for $P_n(-3/2)$ and $P_n(-1/2)$, respectively. □

Corollary 2. *For $n \geq 1$*

$$\sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k} \frac{(-1)^k}{n+k} = 0. \tag{8}$$

Proof. Use Theorem 1 and the fact that $P_n(1) = 1$ for all $n \geq 0$. □

Corollary 3. *For $n \geq 1$*

$$\sum_{k=0}^{2n} \binom{2n+k}{2k} \binom{2k}{k} \frac{(-1)^k 2^{-k}}{2n+k} = \frac{(-1)^n}{4n} 2^{-2n} \binom{2n}{n}, \tag{9}$$

and $n \geq 0$

$$\sum_{k=0}^{2n+1} \binom{2n+1+k}{2k} \binom{2k}{k} \frac{(-1)^k 2^{-k}}{2n+1+k} = \frac{(-1)^n}{2(2n+1)} 2^{-2n} \binom{2n}{n}. \tag{10}$$

Proof. Use the values

$$P_{2n+1}(0) = 0 \qquad \text{and} \qquad P_{2n}(0) = (-1)^n 2^{-2n} \binom{2n}{n}.$$

□

3. Some combinatorial identities derived from integration

Proposition 1. *We have*

$$\sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k} \frac{(-1)^k}{(n+k)(k+1)} = \begin{cases} 1/2, & n = 1 \\ 0, & n \geq 2. \end{cases} \tag{11}$$

Proof. Set $x = -4t$ in Theorem 1 and then rename t as x. Then the main identity becomes

$$\sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k} \frac{(-1)^k}{n+k} x^k = \frac{(-1)^n}{2n} \left(P_n(2x-1) - P_{n-1}(2x-1)\right). \tag{12}$$

The result follows by integrating both sides from 0 to 1 and noting that for $n \geq 1$

$$\int_0^1 P_n(2x-1)dx = \frac{1}{2} \int_{-1}^1 P_n(x)dx = 0.$$

□

Corollary 4. *We have*

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{(n+k)}\frac{k}{k+1}=\begin{cases}-1/2, & n=1,\\ 0, & n\geq 2.\end{cases} \tag{13}$$

Proof. Combine Eq. (11) with Eq. (8). □

Theorem 2. *Let μ be a complex number with $Re(\mu)>0$. Then*

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{(n+k)(2k+2\mu)}=\frac{(-1)^n\Gamma^2(\mu)}{4n}\left(\frac{1}{\Gamma(\mu+n+1)\Gamma(\mu-n)}-\frac{1}{\Gamma(\mu+n)\Gamma(\mu+1-n)}\right), \tag{14}$$

where $\Gamma(z)$ denotes the Gamma function and where $1/\Gamma(z)$ is understood as an entire function.

Proof. We shall use the formula

$$\int_0^1 x^{2\mu-1}P_n(2x^2-1)dx=\frac{\Gamma^2(\mu)}{2}\left(\frac{1}{\Gamma(\mu+n+1)\Gamma(\mu-n)}\right),\quad Re(\mu)>0,$$

directly extracted from Gradshteyn and Ryzhik [9, Equation (7.233)] or easily proved by induction using eq. (3). Again, the right-hand side must be be interpreted using the entire function $1/\Gamma$. Hence, working with Eq. (12), we get

$$\begin{aligned}\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{(n+k)(2k+2\mu)}&=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{(n+k)}\int_0^1 x^{2k+2\mu-1}dx\\ &=\frac{(-1)^n}{2n}\left(\int_0^1 x^{2\mu-1}P_n(2x^2-1)dx-\int_0^1 x^{2\mu-1}P_{n-1}(2x^2-1)dx\right),\end{aligned}$$

and the proof is finished. □

Corollary 5. *For $n\geq 1$ we have*

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{(n+k)(2k+1)}=\frac{2}{(2n-1)(2n+1)}, \tag{15}$$

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{(n+k)(2k+3)}=-\frac{2}{(2n-3)(2n-1)(2n+1)(2n+3)}, \tag{16}$$

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{(n+k)^2}=\frac{(-1)^{n+1}}{n^2\binom{2n}{n}}, \tag{17}$$

and

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{(n+k)(n+1+k)}=\frac{(-1)^{n+1}}{(2n+1)\binom{2n}{n}}. \tag{18}$$

Proof. These identities are special cases of Theorem 2 for $\mu=1/2$, $\mu=3/2$, $\mu=n$, and $\mu=n+1$, respectively. For instance, using the classical properties of the Gamma function (see [8])

$$\Gamma(1/2)=\sqrt{\pi},\quad \Gamma(1+z)=z\Gamma(z),\quad \text{and}\quad \Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)},$$

we calculate

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{(n+k)(2k+1)}=\frac{\pi}{\Gamma(1/2+n)\Gamma(1/2-n)}\frac{(-1)^n}{2n}\left(\frac{1}{2n+1}+\frac{1}{2n-1}\right).$$

But

$$\begin{aligned}\Gamma(1/2+n)\Gamma(1/2-n) &= \frac{\pi}{\sin(\pi/2+\pi n)} \\ &= \frac{\pi}{\sin(\pi/2)\cos(\pi n)+\cos(\pi/2)\sin(\pi n)} \\ &= \frac{\pi}{(-1)^n},\end{aligned}$$

and hence

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{(n+k)(2k+1)} = \frac{1}{2n}\left(\frac{1}{2n+1}+\frac{1}{2n-1}\right).$$

The case $\mu = 3/2$ is similar, noting that $\Gamma(3/2) = \sqrt{\pi}/2$. Setting $\mu = n$ in Theorem 2 produces

$$\begin{aligned}\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{(n+k)^2} &= \frac{(-1)^n\Gamma^2(n)}{2n}\left(\frac{1}{\Gamma(2n+1)\Gamma(0)}-\frac{1}{\Gamma(2n)\Gamma(1)}\right) \\ &= \frac{(-1)^{n+1}}{2n}\frac{(n-1)!^2}{(2n-1)!}.\end{aligned}$$

It is worth remarking that in the last step we used the fact that since $\Gamma(z)$ has simple poles at $z = 0, -1, -2, \ldots$, its reciprocal $1/\Gamma(0)$ is a simple zero and the first term vanishes. Lastly, with $\mu = n+1$, Eq. (14) writes as

$$\begin{aligned}\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{2(n+k)(n+1+k)} &= \frac{(-1)^n(n!)^2}{4n}\left(\frac{1}{(2n+1)!}-\frac{1}{(2n)!}\right) \\ &= \frac{(-1)^{n+1}}{2(2n+1)\binom{2n}{n}},\end{aligned}$$

and provides Eq. (18). □

Corollary 6. *For $n \geq 1$ we have*

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^{k+1}}{(n+k)}\frac{k}{2k+1} = \frac{1}{(2n-1)(2n+1)}, \tag{19}$$

and

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{(n+k)}\frac{k+1}{2k+3} = \frac{1}{(2n-3)(2n-1)(2n+1)(2n+3)}. \tag{20}$$

Proof. Combine the first two identities in Corollary 5 with identity Eq. (8). □

Corollary 7. *For $n \geq 1$ we have*

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{(n+k)^2(n+1+k)} = \frac{(-1)^n(n^2-2n-1)}{n^2(2n+1)\binom{2n}{n}}. \tag{21}$$

Proof. Combine the last two identities in Corollary 5. □

Theorem 3. *Let $m \geq 0$ be an integer such that $m < n-1$. Then*

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{(n+k)(m+k+1)^2} = \frac{(-1)^m(m!)^2(n-m-2)!}{(n+m+1)!}. \tag{22}$$

Proof. Work with Eq. (12) again, multiply both sides by $x^m \ln(1/x)$ and integrate from 0 to 1. This yields

$$\sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{(n+k)}\int_0^1 x^{k+m}\ln(1/x)dx = \frac{(-1)^m}{2n}\left(\int_0^1 x^m \ln(1/x)(P_n(2x-1)-P_{n-1}(2x-1))dx\right).$$

The left-hand side is quickly calculated using the well-known

$$\int_0^1 x^p \ln(1/x)dx = \frac{1}{(p+1)^2},$$

for any nonnegative integer p.

To evaluate the integrals on the right, we make use of Gautschi's formula [10] which states that for $n > m$

$$\int_0^1 x^m \ln(1/x)P_n(2x-1)dx = (-1)^{n-m}(m!)^2\frac{(n-m-1)!}{(n+m+1)!}.$$

From this formula, we deduce that for $n \geq m+2$ the right-hand side is

$$\begin{aligned}
&\frac{(-1)^m}{2n}\left(\int_0^1 x^m \ln(1/x)P_n(2x-1)\,dx - \int_0^1 x^m \ln(1/x)P_{n-1}(2x-1)\,dx\right)\\
&\quad= \frac{(-1)^n}{2n}\left(\frac{(-1)^{n-m}(m!)^2(n-m-1)!}{(n+m+1)!} - \frac{(-1)^{n-m-1}(m!)^2(n-m-2)!}{(n+m)!}\right)\\
&\quad= \frac{(-1)^m}{2n}\cdot\frac{(m!)^2(n-2-m)!}{(n+m)!}\left(\frac{n-m-1}{n+m+1}+1\right)\\
&\quad= \frac{(-1)^m(m!)^2(n-m-2)!}{(n+m+1)!}.
\end{aligned}$$

□

Corollary 8. *We have*

$$\sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{(n+k)(k+1)^2} = \begin{cases} 3/4, & n=1,\\ \frac{1}{(n-1)n(n+1)}, & n\geq 2,\end{cases} \tag{23}$$

and

$$\sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^{k+1}}{(n+k)(k+2)^2} = \begin{cases} -5/36, & n=1,\\ 1/288, & n=2,\\ \frac{1}{(n-2)(n-1)n(n+1)(n+2)}, & n\geq 3.\end{cases} \tag{24}$$

Proof. The first identity is directly verified for $n = 1$ and is immediately deduced by taking $m = 0$ in (3) when $n \geq 2$. Similarly, the second identity is directly verified for $n = 1$ and $n = 2$ and results from the theorem with $m = 1$ when $n \geq 3$. □

Corollary 9. *For $n \geq 2$ we have*

$$\sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}\frac{(-1)^k}{(n+k)(n-1+k)^2} = (-1)^n\frac{2}{(n-1)^2 n\binom{2n}{n}}. \tag{25}$$

Proof. Set $m = n-2$ in Theorem 3 and use

$$\frac{((n-2)!)^2}{(2n-1)!} = \frac{2}{(n-1)^2 n\binom{2n}{n}}.$$

□

We conclude with the following theorem dealing with the square of central binomial coefficients.

Theorem 4. *For $n \geq 1$ we have*

$$\sum_{k=0}^{2n} \binom{2n+k}{2k} \binom{2k}{k}^2 2^{-2k}(2k+1) \frac{(-1)^k}{(2n+k)(k+1)^2} = \frac{2^{-4n}}{n^3} \binom{2(n-1)}{n-1}^2, \tag{26}$$

and

$$\sum_{k=0}^{2n+1} \binom{2n+1+k}{2k} \binom{2k}{k}^2 2^{-2k}(2k+1) \frac{(-1)^k}{(2n+1+k)(k+1)^2} = \frac{2^{-(4n+3)}}{(2n+1)(n+1)^2} \binom{2n}{n}^2. \tag{27}$$

Proof. When $n = 1$, a direct calculation shows that both sides of Eq. (26) equal $\frac{1}{16}$ and that both sides of Eq. (27) equal $\frac{1}{384}$. We suppose that $n \geq 2$ from now on.

Working with Eq. (12) again we see that for any integer $N \geq 2$

$$\sum_{k=0}^{N} \binom{N+k}{2k} \binom{2k}{k} \frac{(-1)^k}{N+k} 2^{-k} (1+x)^k = \frac{(-1)^N}{2N} \left(P_N(x) - P_{N-1}(x) \right),$$

which after multiplying both sides with $\arcsin(x)$ and integrating from -1 to 1 becomes

$$\sum_{k=0}^{N} \binom{N+k}{2k} \binom{2k}{k} \frac{(-1)^k}{N+k} 2^{-k} \int_{-1}^{1} (1+x)^k \arcsin(x)\, dx = \frac{(-1)^N}{2N} \left(I_N - I_{N-1} \right),$$

with

$$I_N = \int_{-1}^{1} P_N(x) \arcsin(x)\, dx.$$

We begin by evaluating the integral on the left-hand side. As $\arcsin(x)$ is an odd function it is readily seen that

$$\int_{-1}^{1} x^k \arcsin(x)\, dx = \begin{cases} 0, & k \text{ even}, \\ 2\int_0^1 x^k \arcsin(x)\, dx, & k \text{ odd}, \end{cases}$$

and hence

$$\int_{-1}^{1} (1+x)^k \arcsin(x)\, dx = 2 \sum_{j=1}^{\lfloor (k+1)/2 \rfloor} \binom{k}{2j-1} \int_0^1 x^{2j-1} \arcsin(x)\, dx.$$

The next step is to use the integral

$$\int_0^1 x^{2j-1} \arcsin(x)\, dx = \frac{\pi}{4j} \left(1 - \frac{1}{2^{2j}} \binom{2j}{j} \right) \quad (j \in \mathbb{N}),$$

which follows from

$$\begin{aligned} \int_0^1 x^{2j-1} \arcsin(x)\, dx &= \int_0^{\pi/2} (\sin u)^{2j-1} u \cos u\, du \\ &= \left[u \cdot \frac{(\sin u)^{2j}}{2j} \right]_0^{\pi/2} - \frac{1}{2j} \int_0^{\pi/2} (\sin u)^{2j}\, du \\ &= \frac{\pi}{2} \cdot \frac{1}{2j} - \frac{1}{2j} \cdot \frac{\pi}{2} \cdot \frac{1}{2^{2j}} \binom{2j}{j}. \end{aligned}$$

This yields

$$\int_{-1}^{1} (1+x)^k \arcsin(x)\, dx = \frac{\pi}{2} \sum_{j=1}^{\lfloor (k+1)/2 \rfloor} \binom{k}{2j-1} \frac{1}{j} \left(1 - 2^{-2j} \binom{2j}{j} \right).$$

The two sums involved can be evaluated in closed form. First, by integration from 0 to 1, the identity

$$\sum_{j=1}^{\lfloor (k+1)/2 \rfloor} \binom{k}{2j-1} x^{2j-1} = \frac{1}{2}((1+x)^k - (1-x)^k)$$

provides

$$\sum_{j=1}^{\lfloor (k+1)/2 \rfloor} \binom{k}{2j-1} \frac{1}{j} = \frac{2^{k+1}-2}{k+1}.$$

Second, the known relation (see Bataille [11])

$$\sum_{j=0}^{\lfloor m/2 \rfloor} \binom{m}{j} \binom{m-j}{j} 2^{m-2j} = \binom{2m}{m} \quad (m \geq 0),$$

leads to

$$\begin{aligned}
\sum_{j=1}^{\lfloor (k+1)/2 \rfloor} \binom{k}{2j-1} \frac{1}{j} 2^{-2j} \binom{2j}{j} &= \frac{2}{k+1} \sum_{j=1}^{\lfloor (k+1)/2 \rfloor} \binom{k+1}{2j} \binom{2j}{j} 2^{-2j} \\
&= \frac{2}{k+1} \sum_{j=1}^{\lfloor (k+1)/2 \rfloor} \binom{k+1}{j} \binom{k+1-j}{j} 2^{-2j} \\
&= \frac{2}{k+1} \left(2^{-(k+1)} \binom{2k+2}{k+1} - 1 \right) \\
&= -\frac{2}{k+1} + \frac{2^{-k+1}}{(k+1)^2} (2k+1) \binom{2k}{k}.
\end{aligned}$$

Finally

$$\int_{-1}^{1} (1+x)^k \arcsin(x)\, dx = \pi \left(\frac{2^k}{k+1} - \frac{2^{-k}}{(k+1)^2} (2k+1) \binom{2k}{k} \right). \tag{28}$$

Using (28) the sum on the left hand side becomes

$$\begin{aligned}
&\sum_{k=0}^{N} \binom{N+k}{2k} \binom{2k}{k} \frac{(-1)^k}{N+k} 2^{-k} \int_{-1}^{1} (1+x)^k \arcsin(x)\, dx \\
&= \pi \sum_{k=0}^{N} \binom{N+k}{2k} \binom{2k}{k} \frac{(-1)^k}{N+k} 2^{-k} \frac{2^k}{k+1} - \pi \sum_{k=0}^{N} \binom{N+k}{2k} \binom{2k}{k} \frac{(-1)^k}{N+k} 2^{-k} \frac{2^{-k}}{(k+1)^2} (2k+1) \binom{2k}{k} \\
&= -\pi \sum_{k=0}^{N} \binom{N+k}{2k} \binom{2k}{k}^2 2^{-2k} (2k+1) \frac{(-1)^k}{(N+k)(k+1)^2},
\end{aligned}$$

where we have applied (11). Now we appeal to Eq. 7.249 (1) in Gradshteyn and Ryzhik [9] which states that

$$I_N = \int_{-1}^{1} P_N(x) \arcsin(x)\, dx = \begin{cases} 0, & N \text{ even} \\ \pi \left(\frac{(N-2)!!}{2^{(N+1)/2} \left(\frac{N+1}{2} \right)!} \right)^2, & N \text{ odd}, \end{cases}$$

and provides

$$\frac{(-1)^N}{2N} (I_N - I_{N-1}) = -\frac{I_{2n-1}}{4n} \qquad \text{or} \qquad -\frac{I_{2n+1}}{2(2n+1)},$$

according as $N = 2n$ or $N = 2n + 1$. Since for $m \geq 0$

$$\left(\frac{((2m+1)-2)!!}{2^{((2m+1)+1)/2}\left(\frac{(2m+1)+1}{2}\right)!} \right)^2 = \left(\binom{2m}{m} \frac{1}{2^{2m+1}(m+1)} \right)^2 = \binom{2m}{m}^2 \frac{1}{2^{4m+2}(m+1)^2},$$

we obtain

$$\sum_{k=0}^{2n} \binom{2n+k}{2k} \binom{2k}{k}^2 2^{-2k}(2k+1) \frac{(-1)^k}{(2n+k)(k+1)^2} = \frac{1}{4n} \binom{2(n-1)}{n-1}^2 \frac{1}{2^{4n-2}n^2},$$

and

$$\sum_{k=0}^{2n+1} \binom{2n+1+k}{2k} \binom{2k}{k}^2 2^{-2k}(2k+1) \frac{(-1)^k}{(2n+1+k)(k+1)^2} = \frac{1}{2(2n+1)} \binom{2n}{n}^2 \frac{1}{2^{4n+2}(n+1)^2}.$$

The identities (26) and (27) follow. □

4. Conclusion

In this article, we firstly used a coefficient extraction formula to link a class of combinatorial sums to Legendre polynomials. This expression is interesting in its own right and can be applied to other combinatorial problems, for instance to sums with Chebyshev polynomials. During the revision of this paper we learned that this formula has been known for more than thirty years. It was derived by Sprugnoli in 1994 using Riordan arrays [12, Equation (2.4)].

We have provided a new application of this very useful result and deduced a range of combinatorial sums via integrals of certain functions associated with Legendre polynomials. Our main identities involving the factors

$$\frac{1}{(n+k)(2k+2\mu)}, \qquad \frac{1}{(n+k)(m+k+1)^2} \qquad \text{and} \qquad \frac{\binom{2k}{k}^2}{(n+k)(k+1)^2},$$

are not mentioned in the literature and seem to be new, although our Eqs. (8) and (11) are rediscoveries of Sprugnoli's results from [12].

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