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Existence theorems for the generalized sequential Yeh-Feynman integral

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Abstract: We introduce the concept of a generalized sequential Yeh-Feynman integral for functionals defined on Yeh-Wiener space, formulated via stochastic process Z_h associated with a nonzero function h. Existence theorems and evaluation formulas for generalized sequential Yeh-Feynman integral are established for functionals in the Banach algebra $\hat{\mathcal{S}}(L_2(Q))$ and some related functionals. Furthermore, we show that the class of generalized sequential Yeh-Feynman integrable functionals is strictly larger than $\hat{\mathcal{S}}(L_2(Q))$. Previous results on sequential Yeh-Feynman integral are recovered as corollaries of our results.

Keywords: Yeh-Wiener space, sequential Feynman integral, generalized sequential Yeh-Feynman integral, Banach algebra $\hat{\mathcal{S}}(L_2(Q))$

MSC: 28C20, 46G12.

1. Introduction

The analytic Feynman integral is defined via analytic continuation of the Wiener integral, whereas the sequential Feynman integral is defined as the limit of finite-dimensional Lebesgue integrals. Cameron and Storvick provided a simple definition of the sequential Feynman integral on Wiener space [1]. In [2], they established explicit formulas for the sequential Feynman integral for functionals in the classes containing the Banach algebra $\hat{\mathcal{S}}$ which was introduced in [1]. In [3,4], Yeh extended Wiener space to Yeh-Wiener space, a space of continuous functions of two variables. Various works on the integrals on Yeh-Wiener space have been carried out in [5–9].

On the other hand, the concepts of generalized Wiener integral and generalized analytic Feynman integral were introduced in [10] and further developed in [11]. In [12], the author and a collaborator established the generalized sequential Feynman integral for functionals in $\hat{\mathcal{S}}$ and some related functionals.

This paper introduces the concept of generalized sequential Yeh-Feynman integral (see Eqs. (2) and (3) below), and establish the existence of the integral for functionals in the Banach algebra $\hat{\mathcal{S}}(L_2(Q))$.

We now summarize the results of this paper. Theorem 1 in §3 deals with the existence and evaluation formula for the generalized sequential Yeh-Feynman integral for functionals in the Banach algebra $\hat{\mathcal{S}}(L_2(Q))$ and serves as the key result of the paper. Two additional classes of functionals, important in the application of the Feynman integral to quantum theory are treated in Theorem 2 and Corollary 1. Moreover, we demonstrate via Example 1 that the class of generalized sequential Yeh-Feynman integrable functionals is strictly larger than $\hat{\mathcal{S}}(L_2(Q))$.

2. Preliminaries and some lemmas

Let $C_2(Q)$ denote the Yeh-Wiener space, that is, the space of real valued continuous functions $x(s,t)$ on $Q = [0,S] \times [0,T]$ satisfying the boundary conditions $x(s,0) = x(0,t) = 0$ for all $(s,t) \in Q$. Let a subdivision σ of Q be given:

$$0 = s_0 < s_1 < \cdots < s_l = S, \quad 0 = t_0 < t_1 < \cdots < t_m = T. \tag{1}$$

Let $X = X(s,t)$ be an element in $C_2(Q)$ based on σ and the $l \times m$ matrix of real numbers $\Xi = \{\xi_{j,k}\}$ and defined by

$$\begin{aligned} X(s,t) &= X(s,t;\sigma,\Xi) \\ &= \frac{\xi_{j,k} - \xi_{j-1,k} - \xi_{j,k-1} + \xi_{j-1,k-1}}{(s_j - s_{j-1})(t_k - t_{k-1})}(s - s_{j-1})(t - t_{k-1}) \\ &\quad + \frac{\xi_{j,k-1} - \xi_{j-1,k-1}}{s_j - s_{j-1}}(s - s_{j-1}) + \frac{\xi_{j-1,k} - \xi_{j-1,k-1}}{t_k - t_{k-1}}(t - t_{k-1}) + \xi_{j-1,k-1}, \end{aligned}$$

for $(s,t) \in [s_{j-1}, s_j] \times [t_{k-1}, t_k]$, and $\xi_{0,0} = \xi_{0,k} = \xi_{j,0} = 0$ for $j = 1, 2, \dots, l$ and $k = 1, 2, \cdots, m$. When a sequence of subdivisions $\{\sigma_n\}$ is considered, the corresponding notations σ, l, m, s_j, t_k, and Ξ will be replaced by $\sigma_n, l_n, m_n, s_{n;j}, t_{n;k}$, and Ξ_n, respectively.

For a nonzero function h in $L_2(Q)$, let Z_h be the process on $C_2(Q) \times Q$ defined by

$$Z_h(x;s,t) = \int_0^s \int_0^t h(\tau_1, \tau_2)\, dx(\tau_1, \tau_2),$$

where the integral is understood in the Paley-Wiener-Zygmund sense [13,14]. The process Z_h on Wiener space was introduced by Park and Skoug [15] and used in, for example, [10,12,16,17].

Let $q \neq 0$ be a real number, and let $F(x)$ be a functional defined on a subset of $C_2(Q)$ that contains every path $Z_h(X;\cdot,\cdot)$, where X is the quadratic surface in $C_2(Q)$. Let $\{\sigma_n\}$ be a sequence of subdivisions of Q such that the norm

$$\|\sigma_n\| = \max_{j,k} \sqrt{(s_j - s_{j-1})^2 + (t_k - t_{k-1})^2} \to 0,$$

and let $\{\lambda_n\}$ be a sequence in $\mathbb{C}$ with $\operatorname{Re} \lambda_n > 0$ such that $\lambda_n \to -iq$ as $n \to \infty$. If the integral on the right-hand side of (2) exists, and if the limit exists independently of the particular choices of the sequences $\{\sigma_n\}$ and $\{\lambda_n\}$, then the *generalized sequential Yeh-Feynman integral* with parameter q is said to exist and is denoted by

$$\begin{aligned} \int^{\text{g-syf}_q} F(Z_h(x;\cdot,\cdot))\, dx = \lim_{n\to\infty} \gamma_{\sigma_n,\lambda_n} \int_{\mathbb{R}^{l_n m_n}} \exp\left\{ -\frac{\lambda_n}{2} \int_Q \left[\frac{\partial^2 X}{\partial s\, \partial t}(s,t;\sigma_n,\Xi_n) \right]^2 ds\, dt \right\} \\ \times F(Z_h(X(\cdot,\cdot;\sigma_n,\Xi_n);\cdot,\cdot))\, d\Xi_n, \end{aligned} \tag{2}$$

where

$$\gamma_{\sigma_n,\lambda_n} = \left(\frac{\lambda_n}{2\pi} \right)^{l_n m_n/2} \prod_{j=1}^{l_n} \prod_{k=1}^{m_n} \{(s_{n;j} - s_{n;j-1})(t_{n;k} - t_{n;k-1})\}^{-1/2}.$$

Here, if $l_n m_n$ is odd, we take $\lambda_n^{1/2}$ with positive real part.

Let

$$\begin{aligned} H_{\lambda_n}(\sigma_n, \Xi_n) &\equiv \gamma_{\sigma_n,\lambda_n} \exp\left\{ -\frac{\lambda_n}{2} \int_Q \left[\frac{\partial^2 X}{\partial s\, \partial t}(s,t;\sigma_n,\Xi_n) \right]^2 ds\, dt \right\} \\ &= \left(\frac{\lambda_n}{2\pi} \right)^{l_n m_n/2} \prod_{j=1}^{l_n} \prod_{k=1}^{m_n} \{(s_{n;j} - s_{n;j-1})(t_{n;k} - t_{n;k-1})\}^{-1/2} \\ &\quad \times \exp\left\{ -\frac{\lambda_n}{2} \sum_{j=1}^{l_n} \sum_{k=1}^{m_n} \frac{(\xi_{n;j,k} - \xi_{n;j-1,k} - \xi_{n;j,k-1} + \xi_{n;j-1,k-1})^2}{(s_{n;j} - s_{n;j-1})(t_{n;k} - t_{n;k-1})} \right\}. \end{aligned}$$

Thus in terms of $H_{\lambda_n}(\sigma_n, \Xi_n)$, the generalized sequential Yeh-Feynman integral can be written

$$\int^{\text{g-syf}_q} F(Z_h(x;\cdot,\cdot))\, dx = \lim_{n\to\infty} \int_{\mathbb{R}^{l_n m_n}} H_{\lambda_n}(\sigma_n, \Xi_n) F(Z_h(X(\cdot,\cdot;\sigma_n,\Xi_n);\cdot,\cdot))\, d\Xi_n. \tag{3}$$

It is easy to see that the generalized sequential Yeh-Feynman integral is linear. When $h \equiv 1$ on Q, the generalized sequential Yeh-Feynman integral is reduced to the sequential Yeh-Feynman integral $\int^{\text{syf}_q} F(x)\,dx$ studied in [8,9].

We now introduce the class of functionals considered throughout this paper. Let $D_2(Q)$ denote the class of functions $x \in C_2(Q)$ such that x is absolutely continuous on Q and $\frac{\partial^2 x}{\partial s\,\partial t}(s,t) \in L_2(Q)$. For the definition of absolute continuity on Q, see [8,18].

For $u, v \in L_2(Q)$ and $x \in C_2(Q)$, we let

$$\langle u, v\rangle = \int_Q u(s,t)v(s,t)\,ds\,dt,$$

and

$$\langle u, v\rangle_{j,k} = \int_{t_{k-1}}^{t_k}\int_{s_{j-1}}^{s_j} u(s,t)v(s,t)\,ds\,dt,$$

for $j = 1, \ldots, l$ and $k = 1, \ldots, m$. Thus we have

$$\langle u, v\rangle = \sum_{j=1}^{l}\sum_{k=1}^{m}\langle u, v\rangle_{j,k}.$$

If there exists a sequence of subdivisions $\{\sigma_n\}$, then $\langle u, v\rangle_{j,k}$ will be replaced by $\langle u, v\rangle_{n;j,k}$.

Let $\mathcal{M}(L_2(Q))$ denote the class of complex Borel measures of bounded variation on $L_2(Q)$. A functional F, defined on a subset of $C_2(Q)$ containing $D_2(Q)$, is said to belong to $\hat{\mathcal{S}}(L_2(Q))$ if there exists a measure $f \in \mathcal{M}(L_2(Q))$ such that, for every $x \in D_2(Q)$,

$$F(x) = \int_{L_2(Q)} \exp\left\{ i\left\langle v, \frac{\partial^2 x}{\partial s\,\partial t}\right\rangle\right\} df(v). \tag{4}$$

Note that $\hat{\mathcal{S}}(L_2(Q))$ with the norm $\|F\| = \|f\| = \operatorname{var} f$ is a Banach algebra [5].

Let $v \in L_2(Q)$, and let σ be an arbitrary subdivision as defined in (1). Define the averaged function $v_{h,\sigma}$ for v and h on σ by

$$v_{h,\sigma}(s,t) = \frac{1}{(s_j - s_{j-1})(t_k - t_{k-1})}\langle v, h\rangle_{j,k}, \tag{5}$$

when $s_{j-1} \le s < s_j$ and $t_{k-1} \le t < t_k$ for $j = 1, \ldots, l$ and $k = 1, \ldots, m$, and

$$v_{h,\sigma}(s,t) = 0, \tag{6}$$

when $s = S$ or $t = T$. Then we have

$$\|v_{h,\sigma}\|_2^2 = \sum_{j=1}^{l}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\int_{s_{j-1}}^{s_j}\{v_{h,\sigma}(s,t)\}^2\,dx\,dt = \sum_{j=1}^{l}\sum_{k=1}^{m}\frac{\langle v, h\rangle_{j,k}^2}{(s_j - s_{j-1})(t_k - t_{k-1})}. \tag{7}$$

The following lemmas extend the results established in [6].

Lemma 1. *Let $h \in L_2(Q)$ and $v \in L_2(Q)$. Let $\{\sigma_n\}$ be a sequence of subdivisions of Q such that $\|\sigma_n\| \to 0$ as $n \to \infty$. Then, the sequence of averaged functions converges to vh; that is,*

$$\lim_{n\to\infty} v_{h,\sigma_n}(s,t) = v(s,t)h(s,t), \tag{8}$$

for almost every $(s,t) \in Q$.

Proof. Let $(s,t) \in Q$ be a Lebesgue point of vh, and assume that $s_{j-1} < s \le s_j$ and $t_{k-1} < t \le t_k$ for each n. Since $v,h \in L_2(Q)$, we have $vh \in L_1(Q)$ by the Hölder inequality. Noting that

$$v_{h,\sigma_n}(s,t) = \frac{1}{(s_j - s_{j-1})(t_k - t_{k-1})} \int_{t_{k-1}}^{t_k} \int_{s_{j-1}}^{s_j} v(\tau_1,\tau_2)h(\tau_1,\tau_2)\,d\tau_1\,d\tau_2,$$

we apply the Lebesgue differentiation theorem (e.g., [19, Theorem 7.10]) to the integrable function vh. This directly yields $\lim_{n\to\infty} v_{h,\sigma_n}(s,t) = v(s,t)h(s,t)$, which completes the proof. □

Lemma 2. *Let $h \in L_\infty(Q)$ and $v \in L_2(Q)$. Let $\{\sigma_n\}$ be a sequence of subdivisions of Q satisfying $\|\sigma_n\| \to 0$ as $n \to \infty$. Then we have*

$$\lim_{n\to\infty} \|v_{h,\sigma_n}\|_2^2 = \|vh\|_2^2. \tag{9}$$

Proof. By Lemma 1, $\lim_{n\to\infty} v_{h,\sigma_n}(s,t) = v(s,t)h(s,t)$ for almost every $(s,t) \in Q$, and by Fatou's lemma,

$$\liminf_{n\to\infty} \|v_{h,\sigma_n}\|_2^2 \ge \|vh\|_2^2.$$

On the other hand, by the Schwarz inequality

$$\langle v,h\rangle_{n;j,k}^2 \le (s_{n,j} - s_{n,j-1})(t_{n,k} - t_{n,k-1}) \int_{t_{n,k_n-1}}^{t_{n,k_n}} \int_{s_{n,j_n-1}}^{s_{n,j_n}} \{v(\tau_1,\tau_2)h(\tau_1,\tau_2)\}^2\,d\tau_1\,d\tau_2.$$

Therefore, by (7), we have

$$\|v_{h,\sigma_n}\|_2^2 = \sum_{j=1}^{l_n}\sum_{k=1}^{m_n} \frac{\langle v,h\rangle_{n;j,k}^2}{(s_{n,j} - s_{n,j-1})(t_{n,k} - t_{n,k-1})} \le \|vh\|_2^2.$$

Thus we have

$$\limsup_{n\to\infty} \|v_{h,\sigma_n}\|_2^2 \le \|vh\|_2^2,$$

and this completes the proof. □

3. Existence of the generalized sequential Yeh-Feynman integral

In this section we establish the generalized sequential Yeh-Feynman integrability for functionals in $\hat{\mathcal{S}}(L_2(Q))$ and for some related functionals. Our first theorem shows that every functional in $\hat{\mathcal{S}}(L_2(Q))$ is generalized sequential Yeh-Feynman integrable.

To ensure the existence of various Lebesgue integrals involved, throughout this paper, we assume that h belongs to $L_\infty(Q)$ rather than simply to $L_2(Q)$.

Theorem 1. *If $F \in \hat{\mathcal{S}}(L_2(Q))$ is given by (4), then F is generalized sequential Yeh-Feynman integrable and*

$$\int^{\text{g-syf}_q} F(Z_h(x;\cdot,\cdot))\,dx = \int_{L_2(Q)} \exp\left\{-\frac{i}{2q}\|vh\|_2^2\right\} df(v), \tag{10}$$

for each nonzero real number q.

Proof. Let σ be a subdivision given by (1). Note that $Z_h(X(\cdot,\cdot;\sigma,\Xi);\cdot,\cdot)$ belongs to $D_2(Q)$ and

$$\begin{aligned} F(Z_h(X(\cdot,\cdot;\sigma,\Xi);\cdot,\cdot)) &= \int_{L_2(Q)} \exp\left\{i\left\langle v, \frac{\partial^2}{\partial s\,\partial t} Z_h(X(\cdot,\cdot;\sigma,\Xi);\cdot,\cdot)\right\rangle\right\} df(v) \\ &= \int_{L_2(Q)} \exp\left\{i\sum_{j=1}^{l}\sum_{k=1}^{m} \langle v,h\rangle_{j,k} \frac{\xi_{j,k} - \xi_{j-1,k} - \xi_{j,k-1} + \xi_{j-1,k-1}}{(s_j - s_{j-1})(t_k - t_{k-1})}\right\} df(v). \end{aligned}$$

Let $\lambda \in \mathbb{C}$ satisfy $\operatorname{Re}\lambda > 0$, and let

$$I_{\sigma,\lambda}(F) = \int_{\mathbb{R}^{lm}} H_\lambda(\sigma,\Xi) F(Z_h(X(\cdot,\cdot;\sigma,\Xi);\cdot,\cdot))\, d\Xi.$$

To evaluate the integral on $\mathbb{R}^{lm}$ below, we first consider the $lm \times lm$ matrix T representing the transformation $\mathbb{R}^{lm} \to \mathbb{R}^{lm}$ defined by

$$\eta_{j,k} = \xi_{j,k} - \xi_{j-1,k} - \xi_{j,k-1} + \xi_{j-1,k-1},$$

for $j = 1,2,\ldots,l$ and $k = 1,2,\ldots,m$. This transformation is invertible; in fact we have

$$\xi_{j,k} = \sum_{\alpha=1}^{j} \sum_{\beta=1}^{m} \eta_{\alpha,\beta},$$

for $j = 1,2,\ldots,l$ and $k = 1,2,\ldots,m$. Moreover, T can be expressed via the Kronecker product representation [3] as $T = D_l \otimes D_m$, where D_l denotes the $l \times l$ lower bidiagonal first-difference matrix. Consequently, we obtain

$$\det(T) = \{\det(D_l)\}^m \cdot \{\det(D_m)\}^l = 1.$$

Since

$$\begin{aligned}
&\int_{\mathbb{R}^{lm}} \int_{L_2(Q)} \exp\left\{ -\frac{\operatorname{Re}\lambda}{2} \sum_{j=1}^{l}\sum_{k=1}^{m} \frac{(\xi_{j,k} - \xi_{j-1,k} - \xi_{j,k-1} + \xi_{j-1,k-1})^2}{(s_j - s_{j-1})(t_k - t_{k-1})} \right\} d\Xi\, d|f|(v) \\
&= \int_{L_2(Q)} \int_{\mathbb{R}^{lm}} \exp\left\{ -\frac{\operatorname{Re}\lambda}{2} \sum_{j=1}^{l}\sum_{k=1}^{m} \frac{\eta_{j,k}^2}{(s_j - s_{j-1})(t_k - t_{k-1})} \right\} d\{\eta_{j,k}\}\, d|f|(v) < \infty,
\end{aligned}$$

we apply Fubini's theorem to obtain

$$\begin{aligned}
I_{\sigma,\lambda}(F) &= \gamma_{\sigma,\lambda} \int_{L_2(Q)} \int_{\mathbb{R}^{lm}} \exp\left\{ -\frac{\lambda}{2} \sum_{j=1}^{l}\sum_{k=1}^{m} \frac{(\xi_{j,k} - \xi_{j-1,k} - \xi_{j,k-1} + \xi_{j-1,k-1})^2}{(s_j - s_{j-1})(t_k - t_{k-1})} \right. \\
&\qquad \left. + i \sum_{j=1}^{l}\sum_{k=1}^{m} \langle v,h\rangle_{j,k} \frac{\xi_{j,k} - \xi_{j-1,k} - \xi_{j,k-1} + \xi_{j-1,k-1}}{(s_j - s_{j-1})(t_k - t_{k-1})} \right\} d\Xi\, df(v) \\
&= \gamma_{\sigma,\lambda} \int_{L_2(Q)} \int_{\mathbb{R}^{lm}} \exp\left\{ \sum_{j=1}^{l}\sum_{k=1}^{m} \frac{-\frac{\lambda}{2}\eta_{j,k}^2 + i\langle v,h\rangle_{j,k}\eta_{j,k}}{(s_j - s_{j-1})(t_k - t_{k-1})} \right\} d\{\eta_{j,k}\}\, df(v) \\
&= \int_{L_2(Q)} \exp\left\{ -\frac{1}{2\lambda} \sum_{j=1}^{l}\sum_{k=1}^{m} \frac{\langle v,h\rangle_{j,k}^2}{(s_j - s_{j-1})(t_k - t_{k-1})} \right\} df(v),
\end{aligned}$$

where the last equality results from the integration formula $\int_{\mathbb{R}} e^{-a\eta^2 + ib\eta}\, d\eta = (\frac{\pi}{a})^{1/2} e^{-b^2/4a}$ for $\operatorname{Re} a > 0$. Let $\{\sigma_n\}$ be a sequence of subdivisions of Q with $\|\sigma_n\| \to 0$, and let $\{\lambda_n\}$ be a sequence in $\mathbb{C}$ satisfying $\operatorname{Re}\lambda_n > 0$ and $\lambda_n \to -iq$ as $n \to \infty$. Now let v_{h,σ_n} be the function defined by (5) and (6). Since, for $a \geq 0$ and $\operatorname{Re}\lambda_n > 0$,

$$\left| \exp\left\{ -\frac{a}{2\lambda_n} \right\} \right| = \exp\left\{ -\frac{a}{2} \frac{\operatorname{Re}\lambda_n}{|\lambda_n|^2} \right\} \leq 1,$$

we apply the bounded convergence theorem together with (7) and (9) to obtain

$$\begin{aligned}
I_{\sigma_n,\lambda_n}(F) &= \int_{L_2(Q)} \exp\left\{ -\frac{1}{2\lambda_n} \sum_{j=1}^{l}\sum_{k=1}^{m} \frac{\langle v,h\rangle_{n;j,k}^2}{(s_{n;j} - s_{n;j-1})(t_{n;k} - t_{n;k-1})} \right\} df(v) \\
&\to \int_{L_2(Q)} \exp\left\{ -\frac{i}{2q} \|vh\|_2^2 \right\} df(v),
\end{aligned}$$

as $n \to \infty$. Finally we conclude that

$$\int^{\text{g-syf}_q} F(Z_h(x;\cdot,\cdot))\,dx = \lim_{n\to\infty} I_{\sigma_n,\lambda_n}(F) = \int_{L_2(Q)} \exp\left\{-\frac{i}{2q}\|vh\|_2^2\right\} df(v)$$

and this completes the proof. □

Next we consider two more functionals which are different from but are closely related with the (4). The functional treated in Theorem 2 and Corollary 1 below are functionals on Yeh-Wiener space corresponding to the class of functionals studied in [2,21,22] and [23], respectively, concerning sequential Feynman integrals on Wiener space. In applications of the Feynman integral to quantum mechanics, the function Ψ appearing in Theorem 2 serves as the initial condition for the Schrödinger equation.

Define $\mathcal{T}$ to be the collection of functions Ψ on $\mathbb{R}$ of the form

$$\Psi(r) = \int_{\mathbb{R}} \exp\{ir\xi\}\,d\rho(\xi), \tag{11}$$

where ρ is a complex Borel measure on $\mathbb{R}$ with bounded variation .

For each $\xi \in \mathbb{R}$, let $\phi(\xi)$ denote the function $v \in L_2(Q)$ defined by $v(s,t) = \xi$ for $0 \le s \le S$ and $0 \le t \le T$. Hence, $\phi : \mathbb{R} \to L_2(Q)$ is continuous. Therefore, If E is a Borel measurable subset of $L_2(Q)$, then $\phi^{-1}(E)$ is a Borel measurable subset of $\mathbb{R}$. Let

$$\psi(E) = \rho(\phi^{-1}(E)). \tag{12}$$

Thus ψ is a measure on $L_2(Q)$ and $\psi \in \mathcal{M}(L_2(Q))$. Transforming the right-hand member of Eq. (11), we have for $x \in D_2(Q)$,

$$\Psi(x(S,T)) = \int_{L_2(Q)} \exp\{i\Big\langle v, \frac{\partial^2 x}{\partial s\,\partial t}\Big\rangle\}\,d\psi(v),$$

and $\Psi(x(S,T))$, regarded as a functional of x, belongs to $\hat{\mathcal{S}}(L_2(Q))$.

Theorem 2. *For $x \in D_2(Q)$, define $F(x) = G(x)\Psi(x(S,T))$ where $G \in \hat{\mathcal{S}}(L_2(Q))$ and $\Psi \in \mathcal{T}$ are given by (4) and (11), respectively, with corresponding measure g in $\mathcal{M}(L_2(Q))$. Then F is generalized sequential Yeh-Feynman integrable and*

$$\int^{\text{g-syf}_q} F(Z_h(x;\cdot,\cdot))\,dx = \int_{L_2(Q)}\int_{\mathbb{R}} \exp\left\{-\frac{i}{2q}\|(v+\varphi(\xi))h\|_2^2\right\} d\rho(\xi)\,dg(v), \tag{13}$$

for each nonzero real number q, where $\varphi(\xi)$ is the function in $L_2(Q)$ defined by $\varphi(\xi)(s,t) = \xi$.

Proof. Since $\hat{\mathcal{S}}(L_2(Q))$ is a Banach algebra, and both $G(x)$ and $\Psi(x(S,T))$, viewed as functions of x, belong to $\hat{\mathcal{S}}(L_2(Q))$, it follows that $F \in \hat{\mathcal{S}}(L_2(Q))$. Moreover, by the first part of the proof of Theorem 2.3 in [8], we obtain

$$F(x) = \int_{L_2(Q)} \exp\{i\Big\langle w, \frac{\partial^2 x}{\partial s\,\partial t}\Big\rangle\}\,df_{g,\psi}(w),$$

where $f_{g,\psi}$ is a complex measure on $\mathcal{B}(L_2(Q))$ given by

$$f_{g,\psi}(E) = \int_{L_2(Q)} g(E-u)\,d\psi(u),$$

and ψ is as in (12). Applying Theorem 1, we have

$$\int^{\text{g-syf}_q} F(Z_h(x;\cdot,\cdot))\,dx = \int_{L_2(Q)} \exp\left\{-\frac{i}{2q}\|wh\|_2^2\right\} df_{g,\psi}(w).$$

Applying the unsymmetric Fubini theorem (Theorem 6.1 of [24]) together with the change of variables $v = w - u$, we obtain

$$\int^{\text{g-syf}_q} F(Z_h(x;\cdot,\cdot))\,dx = \int_{L_2(Q)} \int_{L_2(Q)} \exp\left\{-\frac{i}{2q}\|(v+u)h\|_2^2\right\} d\psi(u)\,dg(v).$$

Finally by (12) and the Fubini theorem, we obtain (13). □

Corollary 1. *Let Φ be a bounded measurable functional on $L_2(Q)$, and let*

$$F(x) = \int_{L_2(Q)} \exp\left\{i\left\langle v, \frac{\partial^2 x}{\partial s\,\partial t}\right\rangle\right\} \Phi(v)\,df(v),$$

for $x \in D_2(Q)$. Then F is generalized sequential Yeh-Feynman integrable and

$$\int^{\text{g-syf}_q} F(Z_h(x;\cdot,\cdot))\,dx = \int_{L_2(Q)} \exp\left\{-\frac{i}{2q}\|vh\|_2^2\right\} \Phi(v)\,df(v), \tag{14}$$

for any nonzero real number q.

Proof. Let a measure f_ϕ be defined by $f_\phi(E) = \int_E \Phi(v)\,df(v)$ for $E \in \mathcal{B}(L_2(Q))$. Clearly $f_\phi \in \mathcal{M}(L_2(Q))$ and for $x \in D_2(Q)$,

$$F(x) = \int_{L_2(Q)} \exp\left\{i\left\langle v, \frac{\partial^2 x}{\partial s\,\partial t}\right\rangle\right\} df_\phi(v),$$

so that $F \in \hat{\mathcal{S}}(L_2(Q))$. Applying Theorem 1 and replacing $df_\phi(v)$ by $\Phi(v)\,df(v)$, we complete the proof. □

It is evident that the results established in Theorem 2 and Corollary 1 recover the corresponding results for the sequential Yeh-Feynman integrals. That is, if we take $h \equiv 1$ on Q, (13) and (14) reduced to

$$\int^{\text{g-syf}_q} F(x)\,dx = \int_{L_2(Q)} \int_{\mathbb{R}} \exp\left\{-\frac{i}{2q}\|v+\xi\|_2^2\right\} d\rho(\xi)\,dg(v),$$

and

$$\int^{\text{g-syf}_q} F(x)\,dx = \int_{L_2(Q)} \exp\left\{-\frac{i}{2q}\|v\|_2^2\right\} \Phi(v)\,df(v),$$

which is given in Theorems 2.3 and 2.4 of [8], respectively.

Although $\hat{\mathcal{S}}(L_2(Q))$ is a useful Banach algebra to study generalized sequential Yeh-Feynman integration theory, it does not contain all of the generalized sequential Yeh-Feynman integrable functionals. In Example 1 below we provide an example illustrating that there is a generalized sequential Yeh-Feynman integrable functional which does not belong to $\hat{\mathcal{S}}(L_2(Q))$.

Example 1. Let $F(x) = e^{x(S,T)}$ for $x \in D_2(Q)$. Since F is not bounded on $D_2(Q)$, F does not belong to $\hat{\mathcal{S}}(L_2(Q))$. However, we now show that F is generalized sequential Yeh-Feynman integrable. Let $\{\sigma_n\}$ be a sequence of subdivisions on Q with $\|\sigma_n\| \to 0$ as $n \to \infty$, and let $\{\lambda_n\}$ be a sequence in $\mathbb{C}$ satisfying $\mathrm{Re}\,\lambda_n > 0$ and $\lambda_n \to -iq$ as $n \to \infty$, where q is a nonzero real number. Note that

$$\begin{aligned} F(Z_h(X(\cdot,\cdot;\sigma_n,\Xi_n),\cdot,\cdot)) &= \exp\{Z_h(X(\cdot,\cdot;\sigma_n,\Xi_n);S,T)\} \\ &= \exp\left\{\sum_{j=1}^{l_n}\sum_{k=1}^{m_n} \frac{\xi_{n;j,k} - \xi_{n;j-1,k} - \xi_{n;j,k-1} + \xi_{n;j-1,k-1}}{(s_{n;j} - s_{n,j-1})(t_{n;k} - t_{n;k-1})} \langle h, 1\rangle_{n;j,k}\right\}, \end{aligned}$$

and so we have

$$\begin{aligned}
&\int_{\mathbb{R}^{l_n m_n}} H_{\lambda_n}(\sigma_n, \Xi_n) F(Z_h(X(\cdot,\cdot;\sigma_n,\Xi_n);\cdot,\cdot))\, d\Xi_n \\
&= \gamma_{\sigma_n,\lambda_n} \int_{\mathbb{R}^{l_n m_n}} \exp\left\{ \sum_{j=1}^{l_n} \sum_{k=1}^{m_n} \Big[-\frac{\lambda_n}{2} \frac{(\xi_{n;j,k} - \xi_{n;j-1,k} - \xi_{n;j,k-1} + \xi_{n;j-1,k-1})^2}{(s_{n;j} - s_{n,j-1})(t_{n;k} - t_{n;k-1})} \right. \\
&\quad \left. + \frac{\xi_{n;j,k} - \xi_{n;j-1,k} - \xi_{n;j,k-1} + \xi_{n;j-1,k-1}}{(s_{n;j} - s_{n,j-1})(t_{n;k} - t_{n;k-1})} \langle h, 1 \rangle_{n;j,k} \Big] \right\} d\Xi_n \\
&= \exp\left\{ \frac{1}{2\lambda_n} \sum_{j=1}^{l_n} \sum_{k=1}^{m_n} \frac{\langle h, 1 \rangle_{n;j,k}^2}{(s_{n;j} - s_{n,j-1})(t_{n;k} - t_{n;k-1})} \right\}.
\end{aligned}$$

Taking $v \equiv 1$ in (7) and applying Lemma 2, we see that

$$\sum_{j=1}^{l_n} \sum_{k=1}^{m_n} \frac{\langle h, 1 \rangle_{n;j,k}^2}{(s_{n;j} - s_{n,j-1})(t_{n;k} - t_{n;k-1})} = \|1_{h,\sigma_n}\|_2^2 \to \|h\|_2^2,$$

as $n \to \infty$, and so we conclude that

$$\int^{\text{g-syf}_q} F(Z_h(x;\cdot,\cdot))\, dx = \exp\left\{ \frac{i}{2q} \|h\|_2^2 \right\},$$

and F is generalized sequential Yeh-Feynman integrable.

Remark 1. (1) Example 1 merely demonstrates the existence of a generalized sequential Yeh-Feynman integrable functionals that does not belong to $\hat{\mathcal{S}}(L_2(Q))$. At present, it remains open whether we can define a larger class of generalized sequential Yeh-Feynman integrable functionals than $\hat{\mathcal{S}}(L_2(Q))$ that can be characterized in an explicit functional form.

(2) In our future research, we will seek to identify and construct a larger class of generalized sequential Yeh–Feynman integrable functionals that includes the functional in Example 1.

(3) Furthermore, we plan to extend the results of this manuscript to the framework of the generalized sequential Fourier-Yeh-Feynman transform associated with the process Z_h and a nonzero function h, and to investigate its fundamental properties.

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