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Fixed points of differential polynomials generated by solutions of complex linear differential equations

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Abstract: This article concerns the problem on the growth and the oscillation of some differential polynomials generated by solutions of the second order non-homogeneous linear differential equation

$$f'' + P(z)\,e^{a_n z^n} f' + B(z)\,e^{b_n z^n} f = F(z)\,e^{a_n z^n},$$

where a_n, b_n are complex numbers, $P(z)$ $(\not\equiv 0)$ is a polynomial, $B(z)$ $(\not\equiv 0)$ and $F(z)$ $(\not\equiv 0)$ are entire functions with order less than n. Because of the control of differential equation, we can obtain some estimates of their hyper-order and fixed points.

Keywords: differential polynomial, linear differential equations, entire solutions, order of growth, exponent of convergence of zeros, exponent of convergence of distinct zeros

MSC: 30D35, 34M10.

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1. Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the usual notations and basic results of the Nevanlinna's value distribution theory of meromorphic functions [1–3]. In addition, we will use $\lambda(f)$ and $\overline{\lambda}(f)$ to denote respectively the exponents of convergence of the zero-sequence and the sequence of distinct zeros of a meromorphic function f, $\rho(f)$ to denote the order of growth of f. We say that a meromorphic function $a(z)$ is a small function of $f(z)$ if $T(r,a) = o(T(r,f))$ as $r \to +\infty$ outside of a possible exceptional set of finite logarithmic measure. In order to express the rate of growth of meromorphic solutions of infinite order, we recall the following definitions.

Definition 1. ([2–4]) Let f be a meromorphic function. Then the hyper-order $\rho_2(f)$ of f is defined by

$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f. If f is an entire function, then the hyper-order $\rho_2(f)$ of f is defined as follows

$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log\log\log M(r,f)}{\log r},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

Definition 2. ([2–4]) Let f be a meromorphic function. Then the hyper convergence exponents of the zero-sequence and the distinct zeros of f are defined respectively by

$$\lambda_2(f) = \limsup_{r\to+\infty} \frac{\log\log N\left(r,\frac{1}{f}\right)}{\log r},\ \overline{\lambda}_2(f) = \limsup_{r\to+\infty} \frac{\log\log \overline{N}\left(r,\frac{1}{f}\right)}{\log r},$$

where $N\left(r,\frac{1}{f}\right)$ and $\overline{N}\left(r,\frac{1}{f}\right)$ are respectively the integrated counting functions of zeros and distinct zeros of f in $\{z:|z|\leq r\}$.

We now recall some previous results concerning linear differential equations of type

$$f''+e^{-z}f'+B\left(z\right)f=0, \tag{1}$$

where $B\left(z\right)$ is an entire function, it is well-known that each solution f of the Eq. (1) is an entire function, and that if f_1, f_2 are two linearly independent solutions of (1), then by [5], there is at least one of f_1, f_2 of infinite order. Hence, "most" solutions of (1) will have infinite order. But the Eq. (1) with $B(z)=-(1+e^{-z})$ possesses a solution $f\left(z\right)=e^{z}$ of finite order.

In the case when $B\left(z\right)$ is a polynomial, properties of solutions of (1) have been studied, e.g., in [6–9]. Provided that $B\left(z\right)$ is a transcendental entire function and $\rho\left(B\right)=1$, Gundersen pointed out that every nontrivial solution of (1) is of infinite order, see [10]. Chen has considered the case $B(z)=h\left(z\right)e^{bz}$, where $h\left(z\right)$ is a nonzero polynomial and $b\neq-1$, see [11]. More precisely, he proved that every nontrivial solution f of (1) satisfies $\rho_2\left(f\right)=1$. The same paper contains a discussion about more general equations of type

$$f''+A_1\left(z\right)e^{az}f'+A_0\left(z\right)e^{bz}f=0, \tag{2}$$

where the non-vanishing entire functions $A_0\left(z\right)$, $A_1\left(z\right)$ satisfy $\rho\left(A_j\right)<1$, $j=0,1$, and where a,b are complex constants. If $ab\neq0$ and $\arg a\neq\arg b$ or if $a=cb$ for some $c>1$, then all nontrivial solutions f of (2) are of infinite order, see [11]. In [12], Wang and Laine have investigated the growth of solutions of some second order nonhomogenous linear differential equations related to (2) and have obtained the following result.

Theorem 1. [12] *Let $A_j\left(z\right)\ (\not\equiv 0)\ (j=0,1)$ and $H\left(z\right)$ be entire functions with $\max\{\rho\left(A_j\right)(j=0,1),\rho\left(H\right)\}<1$, and let a,b be complex constants that satisfy $ab\neq0$ and $a\neq b$. Then every nontrivial solution f of the Eq.*

$$f''+A_1\left(z\right)e^{az}f'+A_0\left(z\right)e^{bz}f=H, \tag{3}$$

is of infinite order.

Remark 1. If $\rho\left(H\right)=1$, then Eq. (3) can possesses a solution of finite order. For instance the Eq. $f''+z^2e^{-iz}f'+ze^{iz}f=2z^2\cos z$ satisfies $\rho\left(H\right)=\rho\left(2z^2\cos z\right)=1$ and has a finite order solution $f\left(z\right)=z$.

Recently, the author extend the result of Wang and Laine to the case when $\rho(H)=1$ and proved the following result.

Theorem 2. [13] *Let $B\left(z\right)\ (\not\equiv0)$, $F\left(z\right)\ (\not\equiv0)$ be entire functions with*

$$\max\{\rho\left(B\right),\rho\left(F\right)\}<1,$$

and let A,a_1,a_2 be complex numbers such that $Aa_1a_2\neq0$, $a_1\neq a_2$. Then every solution f of the differential equation

$$f''+Ae^{a_1z}f'+B\left(z\right)e^{a_2z}f=F\left(z\right)e^{a_1z},$$

satisfies

$$\overline{\lambda}\left(f\right)=\lambda\left(f\right)=\rho\left(f\right)=+\infty,\ \overline{\lambda}_2\left(f\right)=\lambda_2\left(f\right)=\rho_2(f)\leq1.$$

In this paper, we extend our considerations to non-homogeneous differential equation of type

$$f''+P\left(z\right)e^{a_nz^n}f'+B\left(z\right)e^{b_nz^n}f=F\left(z\right)e^{a_nz^n}. \tag{4}$$

We now proceed to prove three theorems concerning the growth of solutions of (4) and some differential polynomials generated by solutions of this equation. The first main result of this paper states as follows.

Theorem 3. *Let $B(z)\ (\not\equiv 0)$, $F(z)\ (\not\equiv 0)$ be entire functions with*

$$\max\{\rho(B),\rho(F)\}<n,$$

and let a_n, b_n be complex numbers such that $a_nb_n\neq 0$, $a_n\neq b_n$ and $P(z)\ (\not\equiv 0)$ be a polynomial. Then every solution f of the differential equation (4) satisfies

$$\overline{\lambda}(f)=\lambda(f)=\rho(f)=+\infty,\ \overline{\lambda}_2(f)=\lambda_2(f)=\rho_2(f)\leq n.$$

Example 1. Consider the second-order nonhomogeneous differential equation

$$f''+(z+1)\,e^{z^2}f'+(ze^z+3)\,e^{-2z^2}f=\left(z+2e^{-z}\right)e^{z^2}.$$

In this equation, we have

$$P(z)=z+1,\ B(z)=ze^z+3,\ F(z)=z+2e^{-z}.$$

Since

$$\rho(B)=1,\ \rho(F)=1,\ \max\{\rho(B),\rho(F)\}=1<n=2,$$

and the exponential factors satisfy

$$1=a_2\neq b_2=-2,\ a_2b_2=-2\neq 0,$$

all the assumptions of *Theorem* 3 are fulfilled. Therefore, every solution f of the above differential equation satisfies

$$\overline{\lambda}(f)=\lambda(f)=\rho(f)=+\infty,\ \overline{\lambda}_2(f)=\lambda_2(f)=\rho_2(f)\leq 2.$$

Corollary 1. *Under the assumptions of Theorem 3, let $Q(z)=b_nz^n+b_{n-1}z^n+\cdots+b_1z+b_0\ (n\geq 1)$ be a polynomial. Then every solution f of the differential equation*

$$f''+P(z)\,e^{a_nz^n}f'+B(z)\,e^{Q(z)}f=F(z)\,e^{a_nz^n},\tag{5}$$

satisfies $\overline{\lambda}(f)=\lambda(f)=\rho(f)=+\infty,\overline{\lambda}_2(f)=\lambda_2(f)=\rho_2(f)\leq n$.

Proof. We can write (5) as follows

$$f''+P(z)\,e^{a_nz^n}f'+B(z)\,e^{b_nz^n+b_{n-1}z^n+\cdots+b_1z+b_0}f=F(z)\,e^{a_nz^n},$$

that is

$$f''+P(z)\,e^{a_nz^n}f'+C(z)\,e^{b_nz^n}f=F(z)\,e^{a_nz^n},$$

where

$$C(z)=B(z)\,e^{b_{n-1}z^{n-1}+\cdots+b_1z+b_0},$$

and

$$\varrho(C)\leq\max\left\{\rho(B),\rho\left(e^{b_{n-1}z^{n-1}+\cdots+b_1z+b_0}\right)\right\}=\max\{\rho(B),n-1\}<n,$$

so by applying Theorem 3, we get the result. □

Many important results have been obtained on the fixed points of general transcendental meromorphic functions for almost four decades (see [14]). It was in the year 2000 that Z. X. Chen first pointed out the relation between the exponent of convergence of distinct fixed points and the rate of growth of solutions of second order linear differential equations with entire coefficients (see [15]). In [4], Wang and Yi investigated fixed points and hyper order of differential polynomials generated by solutions of second order linear differential equations with meromorphic coefficients. In [16], Laine and Rieppo gave an improvement of the results of [4] by considering fixed points and iterated order. In [17], Liu and Zhang have investigated the fixed points and hyper order of solutions of some higher order linear differential equations with meromorphic coefficients and their derivatives. After that, in [18], Bela ïdi gave an extension of the results of [17].

We know that a differential equation bears a relation to all derivatives of its solutions. Hence, linear differential polynomials generated by its solutions must have special nature because of the control of differential equations, see [4,13,16,19–23].

The second main purpose of this paper is to study the relation between small functions and some differential polynomials generated by solutions of the second order linear differential equation (4). We obtain some estimates of their hyper order and fixed points.

Theorem 4. *Under the assumptions of Theorem 3, let $d_0(z), d_1(z), d_2(z), b(z)$ be entire functions such that at least one of $d_0(z), d_1(z), d_2(z)$ does not vanish identically with $\rho(d_j) < n$ $(j = 0, 1, 2), \rho(b) < \infty$, and let $\varphi(z)$ be an entire function with finite order. If f is a solution of the Eq. (4), then the differential polynomial*

$$g_f = d_2 f'' + d_1 f' + d_0 f + b, \tag{6}$$

satisfies

$$\overline{\lambda}(f) = \lambda(f) = \overline{\lambda}\left(g_f - \varphi\right) = \lambda\left(g_f - \varphi\right) = \rho(f) = +\infty,$$

$$\overline{\lambda}_2(f) = \lambda_2(f) = \overline{\lambda}_2\left(g_f - \varphi\right) = \lambda_2\left(g_f - \varphi\right) = \rho_2(f) \le n.$$

In particular, if f is a solution of Eq. (4), then the differential polynomial $g_f = d_2 f'' + d_1 f' + d_0 f + b$ has infinitely many fixed points and satisfies $\overline{\lambda}\left(g_f - z\right) = \lambda\left(g_f - z\right) = \rho(f) = +\infty$, $\overline{\lambda}_2\left(g_f - z\right) = \lambda_2\left(g_f - z\right) = \rho_2(f) \le n$.

In the next, we investigate the relation between infinite order solutions of a pair non-homogeneous linear differential equations and we obtain the following result.

Theorem 5. *Under the assumptions of Theorem 3, let $F_1 \not\equiv 0$ and $F_2 \not\equiv 0$ be entire functions such that $\max\left\{\rho(F_j) : j = 1, 2\right\} < n$ and $F_1 - KF_2 \not\equiv 0$ for any complex constant K, $\varphi(z)$ is an entire function with finite order. If f_1 is a solution of Eq.*

$$f'' + P(z) e^{a_n z^n} f' + B(z) e^{b_n z^n} f = F_1(z) e^{a_n z^n}, \tag{7}$$

and f_2 is a solution of Eq.

$$f'' + P(z) e^{a_n z^n} f' + B(z) e^{b_n z^n} f = F_2(z) e^{a_n z^n}, \tag{8}$$

then the differential polynomial $g_{f_1 - Kf_2} = d_2\left(f_1'' - Kf_2''\right) + d_1\left(f_1' - Kf_2'\right) + d_0\left(f_1 - Kf_2\right) + b$ satisfies

$$\overline{\lambda}(f_1 - Kf_2) = \lambda(f_1 - Kf_2) = \overline{\lambda}\left(g_{f_1 - Kf_2} - \varphi\right) = \lambda\left(g_{f_1 - Kf_2} - \varphi\right) = \rho(f_1 - Kf_2) = \infty,$$

and

$$\overline{\lambda}_2(f_1 - Kf_2) = \lambda_2(f_1 - Kf_2) = \overline{\lambda}_2\left(g_{f_1 - Kf_2} - \varphi\right) = \lambda_2\left(g_{f_1 - Kf_2} - \varphi\right) = \rho_2(f_1 - Kf_2) \le n,$$

for any complex constant K.

Remark 2. This paper is an improvement of paper [13]. Indeed, when $P(z)$ is a constant, $n = 1$ and $d_2(z) \equiv 0$, we get the results of [13].

2. Some auxiliary lemmas

Lemma 1. [24] *Let $P_1, P_2, ..., P_n$ $(n \ge 1)$ be non-constant polynomials with degree $d_1, d_2, ..., d_n$, respectively, such that $\deg\left(P_i - P_j\right) = \max\left\{d_i, d_j\right\}$ for $i \neq j$. Let $A(z) = \sum_{j=1}^{n} B_j(z) e^{P_j(z)}$, where $B_j(z)$ $(\not\equiv 0)$ are entire functions with $\rho\left(B_j\right) < d_j$. Then $\rho(A) = \max_{1 \le j \le n}\left\{d_j\right\}$.*

Lemma 2. [11] *Suppose that $P(z) = (\alpha + i\beta) z^n + \cdots$ (α, β are real numbers, $|\alpha| + |\beta| \neq 0$) is a polynomial with degree $n \ge 1$, that $A(z)$ $(\not\equiv 0)$ is an entire function with $\rho(A) < n$. Set $g(z) = A(z) e^{P(z)}$, $z = re^{i\theta}$,*

$\delta(P,\theta)=\alpha\cos n\theta-\beta\sin n\theta$. Then for any given $\varepsilon>0$, there is a set $E_1\subset[0,2\pi)$ that has linear measure zero, such that for any $\theta\in[0,2\pi)\setminus(E_1\cup E_2)$, there is $R>0$, such that for $|z|=r>R$, we have

(i) *If $\delta(P,\theta)>0$, then*

$$\exp\{(1-\varepsilon)\delta(P,\theta)r^n\}\le\left|g\left(re^{i\theta}\right)\right|\le\exp\{(1+\varepsilon)\delta(P,\theta)r^n\}.$$

(ii) *If $\delta(P,\theta)<0$, then*

$$\exp\{(1+\varepsilon)\delta(P,\theta)r^n\}\le\left|g\left(re^{i\theta}\right)\right|\le\exp\{(1-\varepsilon)\delta(P,\theta)r^n\},$$

where $E_2=\{\theta\in[0,2\pi):\delta(P,\theta)=0\}$ is a finite set.

Lemma 3. [25] *Let f be a transcendental meromorphic function of finite order ρ. Let $\varepsilon>0$ be a constant, k and j be integers satisfying $k>j\ge0$. Then the following two statements hold:*

(i) *There exists a set $E_3\subset(1,+\infty)$ which has finite logarithmic measure, such that for all z satisfying $|z|\notin E_3\cup[0,1]$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\le|z|^{(k-j)(\rho-1+\varepsilon)}. \tag{9}$$

(ii) *There exists a set $E_4\subset[0,2\pi)$ which has linear measure zero, such that if $\theta\in[0,2\pi)\setminus E_4$, then there is a constant $R=R(\theta)>0$ such that (9) holds for all z satisfying $\arg z=\theta$ and $|z|\ge R$.*

Lemma 4. [26] *Let f be an entire function and suppose that*

$$G(z):=\frac{\log^+\left|f^{(k)}(z)\right|}{|z|^\rho},$$

is unbounded on some ray $\arg z=\theta$ with constant $\rho>0$. Then there exists an infinite sequence of points $z_n=r_ne^{i\theta}$ $(n=1,2,...)$, where $r_n\to+\infty$, such that $G(z_n)\to\infty$ and

$$\left|\frac{f^{(j)}(z_n)}{f^{(k)}(z_n)}\right|\le\frac{1}{(k-j)!}(1+o(1))\,r_n^{k-j},\ j=0,1,...,k-1,$$

as $n\to+\infty$.

Lemma 5. [26] *Let f be an entire function with $\rho(f)=\rho<+\infty$. Suppose that there exists a set $E_5\subset[0,2\pi)$ which has linear measure zero, such that $\log^+\left|f\left(re^{i\theta}\right)\right|\le Mr^\sigma$ for any ray $\arg z=\theta\in[0,2\pi)\setminus E_5$, where M is a positive constant depending on θ, while σ is a positive constant independent of θ. Then $\rho(f)=\rho\le\sigma$.*

Lemma 6. [18,27] *Let $A_j(z)$ $(j=0,1,...,k-1)$, $F(z)\not\equiv0$ be finite order meromorphic functions.*

(i) *If f is a meromorphic solution of the differential equation*

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=F, \tag{10}$$

with $\rho(f)=+\infty$, then f satisfies

$$\overline{\lambda}(f)=\lambda(f)=\rho(f)=+\infty.$$

(ii) *If f is a meromorphic solution of Eq. (10) with $\rho(f)=+\infty$, $\rho_2(f)=\rho$, then f satisfies*

$$\overline{\lambda}(f)=\lambda(f)=\rho(f)=+\infty,\ \overline{\lambda}_2(f)=\lambda_2(f)=\rho_2(f)=\rho.$$

Lemma 7. [21,28] *Let $B_0(z), B_1(z), ..., B_{k-1}(z), H(z)$ be entire functions of finite order. If f is a solution of the equation*

$$f^{(k)} + B_{k-1}(z) f^{(k-1)} + \cdots + B_1(z) f' + B_0(z) f = H(z),$$

then $\rho_2(f) \leq \max\{\rho(B_j) \ (j = 0, 1, ..., k-1), \rho(H)\}$.

Lemma 8. [2] *Let $P(z) = a_n z^n + \cdots + a_0, a_n \neq 0$ be a polynomial with degree $n \geq 1$. Then for every $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that for all sufficiently large $r = |z| > r_\varepsilon$, we have the double inequality*

$$(1 - \varepsilon)|a_n| r^n \leq |P(z)| \leq (1 + \varepsilon)|a_n| r^n.$$

Lemma 9. [19] *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ and $Q(z) = \sum_{i=0}^{n} b_i z^i$ be nonconstant polynomials where $a_i, b_i \ (i = 0, 1, ..., n)$ are complex numbers, $a_n \neq 0, b_n \neq 0$ such that $\arg a_n \neq \arg b_n$ or $a_n = c b_n \ (0 < c < 1)$. We denote index sets by*

$$\Lambda_1 = \{0, P\},$$
$$\Lambda_2 = \{0, P, Q, 2P, P + Q\}.$$

(i) If $H_j \ (j \in \Lambda_1)$ and $H_Q \not\equiv 0$ are all meromorphic functions of orders that are less than n, setting $\Psi_1(z) = \sum_{j \in \Lambda_1} H_j(z) e^j$, then $\Psi_1(z) + H_Q e^Q \not\equiv 0$.

(ii) If $H_j \ (j \in \Lambda_2)$ and $H_{2Q} \not\equiv 0$ are all meromorphic functions of orders that are less than n, setting $\Psi_2(z) = \sum_{j \in \Lambda_2} H_j(z) e^j$, then $\Psi_2(z) + H_{2Q} e^{2Q} \not\equiv 0$.

Lemma 10. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ and $Q(z) = \sum_{i=0}^{n} b_i z^i$ be nonconstant polynomials where $a_i, b_i \ (i = 0, 1, ..., n)$ are complex numbers, $a_n \neq 0, b_n \neq 0$ such that $\arg a_n \neq \arg b_n$ or $a_n = c b_n \ (c > 1)$. We denote index sets by*

$$\Lambda_3 = \{0, P, Q, P + Q, 2P, 2Q\}.$$

Let $H_j \ (j \in \Lambda_3)$ be meromorphic functions of orders that are less than n, setting $\Psi_3(z) = \sum_{j \in \Lambda_3} H_j(z) e^j$. If there exists $j \in \Lambda_3 - \{0\}$ such that $H_j \not\equiv 0$, then $\Psi_3(z) \not\equiv 0$.

Proof. By Lemma 1, we have $\rho(\Psi_3) = n$. Hence, $\Psi_3(z) \not\equiv 0$. □

3. Proof of Theorem 3

We begin by proving that every solution f of Eq. (4) is transcendental.

Let $a = -a_n$ and $b = b_n - a_n$. Then $ab \neq 0$ and $a \neq b$, so Eq. (4) becomes

$$e^{az^n} f'' + P f' + B e^{bz^n} f = F. \tag{11}$$

Our first goal is to show that any solution f of (4) satisfies $\rho(f) \geq n$. Assume, on the contrary, that $\rho(f) < n$. It is clear that $f \not\equiv 0$. Obviously $\rho\left(f^{(j)}\right) < n \ (j = 1, 2)$, $\rho(Bf) < n$. Rewrite (11) as

$$f'' e^{az^n} + B f e^{bz^n} = F - P f'. \tag{12}$$

i) If $f'' \not\equiv 0$, then by (12) and the Lemma 1, we have

$$n = \rho\left\{f'' e^{az^n} + B f e^{bz^n}\right\} = \rho\{F - P f'\} < n.$$

This is a contradiction.

ii) If $f'' \equiv 0$, then by (12) we have

$$n = \rho\left\{B f e^{bz^n}\right\} = \rho\{F - P f'\} < n.$$

This is a contradiction. Thus, $\rho(f) \geq n$, and every solution of f of Eq. (4) must be transcendental.

Now, we prove by contradiction that $\rho(f) = +\infty$. Suppose, on the contrary, that $\rho(f) = \rho < +\infty$. Since $\rho(F) < n$, then for any given ε $(0 < 2\varepsilon < n - \rho(F))$ and sufficiently large r, we have

$$|F(z)| \leq \exp\left\{r^{\rho(F)+\varepsilon}\right\}. \tag{13}$$

By Lemma 2, there exists a set $E \subset [0, 2\pi)$ of linear measure zero, such that whenever $\theta \in [0, 2\pi) \setminus E$, then $\delta(az^n, \theta) \neq 0$, $\delta(bz^n, \theta) \neq 0$ and $\delta(az^n, \theta) \neq \delta(bz^n, \theta)$. By Lemma 3(ii), there exists a set $E_4 \subset [0, 2\pi)$ which has linear measure zero, such that if $\theta \in [0, 2\pi) \setminus E_4$, then there is a constant $R = R(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \geq R$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq |z|^{2\rho},\ 0 \leq i < j \leq 2. \tag{14}$$

For any fixed $\theta \in [0, 2\pi) \setminus (E \cup E_4)$, set

$$\delta_1 = \max\{\delta(az^n, \theta), \delta(bz^n, \theta)\},$$

and

$$\delta_2 = \min\{\delta(az^n, \theta), \delta(bz^n, \theta)\},$$

then $\delta_2 < \delta_1$ and $\delta_1 \neq 0$, $\delta_2 \neq 0$.

We now analyze three cases separately.

Case 1. Suppose that $\delta_1 = \delta(az^n, \theta) > 0$, then $\delta_2 = \delta(bz^n, \theta)$. By Lemma 2, for any given ε with $0 < 2\varepsilon < \min\left\{\frac{\delta_1 - \delta_2}{\delta_1}, 2, n - \rho(F)\right\}$, we obtain

$$\left|e^{az^n}\right| \geq \exp\{(1-\varepsilon)\delta_1 r^n\}, \tag{15}$$

for sufficiently large r. We now prove that $\log^+|f''(z)| / |z|^{\rho(F)+\varepsilon}$ is bounded on the ray $\arg z = \theta$. We assume that $\log^+|f''(z)| / |z|^{\rho(F)+\varepsilon}$ is unbounded on the ray $\arg z = \theta$. Then by Lemma 4, there is a sequence of points $z_m = r_m e^{i\theta}$, such that $r_m \to +\infty$, and that

$$\frac{\log^+|f''(z_m)|}{r_m^{\rho(F)+\varepsilon}} \to +\infty, \tag{16}$$

$$\left|\frac{f^{(j)}(z_m)}{f''(z_m)}\right| \leq \frac{1}{(2-j)!}(1+o(1))\, r_m^{2-j} \leq 2r_m^{2-j},\ \ (j = 0, 1), \tag{17}$$

for m is large enough. From (16) for any sufficiently large number $C > 1$ we have

$$\frac{\log^+|f''(z_m)|}{r_m^{\rho(F)+\varepsilon}} > C, \text{ then } \left|f''(z_m)\right| > \exp\left\{C r_m^{\rho(F)+\varepsilon}\right\} \text{ as } \ m \to +\infty. \tag{18}$$

From (13) and (18), we get

$$\left|\frac{F(z_m)}{f''(z_m)}\right| \leq \frac{\exp\left\{r_m^{\rho(F)+\varepsilon}\right\}}{\exp\left\{C r_m^{\rho(F)+\varepsilon}\right\}} = \frac{1}{\exp\left\{(C-1)\, r_m^{\rho(F)+\varepsilon}\right\}} \to 0, \tag{19}$$

as $m \to +\infty$. From (11), we obtain

$$\left|e^{az^n}\right| \leq |P|\left|\frac{f'}{f''}\right| + \left|Be^{bz^n}\right|\left|\frac{f}{f''}\right| + \left|\frac{F}{f''}\right|. \tag{20}$$

(i) If $\delta_2 > 0$, then by Lemma 2, for ε as above, we obtain

$$\left|B(z)\, e^{bz^n}\right| \leq \exp\{(1+\varepsilon)\delta_2 r^n\}, \tag{21}$$

for sufficiently large r. By using Lemma 8, there exists $\alpha > 0$ such that for sufficiently large $|z| = r$, we have

$$|P(z)| \le \alpha r^k,\ k = \deg P \ge 1. \tag{22}$$

By substituting (15), (17), (19), (21) and (22) into (20), we have

$$\begin{aligned}\exp\{(1-\varepsilon)\delta_1 r_m^n\} \le& \left|e^{az_m^n}\right| \\ \le& |P(z_m)|\left|\frac{f'(z_m)}{f''(z_m)}\right| + \left|B(z_m)e^{bz_m^n}\right|\left|\frac{f(z_m)}{f''(z_m)}\right| + \left|\frac{F(z_m)}{f''(z_m)}\right| \\ \le& \alpha r_m^k(2r_m) + 2r_m^2\exp\{(1+\varepsilon)\delta_2 r_m^n\} + o(1) \\ \le& C_1 r_m^{k+1}\exp\{(1+\varepsilon)\delta_2 r_m^n\},\end{aligned} \tag{23}$$

where $C_1 > 0$ is some constant. By $0 < \varepsilon < \frac{\delta_1-\delta_2}{2\delta_1}$ and (23), we can get

$$\exp\left\{\frac{(\delta_1-\delta_2)^2}{2\delta_1}r_m^n\right\} \le C_1 r_m^{k+1},$$

which is a contradiction.

(ii) If $\delta_2 < 0$, then by Lemma 2, for ε as above, we obtain

$$\left|B(z)e^{bz^n}\right| \le \exp\{(1-\varepsilon)\delta_2 r^n\} < 1, \tag{24}$$

for sufficiently large r. Substituting (15), (17), (19), (22) and (24) into (20), we have

$$\begin{aligned}\exp\{(1-\varepsilon)\delta_1 r_m^n\} \le& \left|e^{az_m^n}\right| \\ \le& |P(z_m)|\left|\frac{f'(z_m)}{f''(z_m)}\right| + \left|B(z_m)e^{bz_m^n}\right|\left|\frac{f(z_m)}{f''(z_m)}\right| + \left|\frac{F(z_m)}{f''(z_m)}\right| \\ \le& 2\alpha r_m^{k+1} + 2r_m^2 + o(1) \le C_2 r_m^{k+1},\end{aligned} \tag{25}$$

where $C_2 > 0$ is some constant, which is a contradiction. Therefore,

$$\log^+|f''(z)| / |z|^{\rho(F)+\varepsilon},$$

is bounded and we have

$$|f''(z)| \le \exp\left\{Mr^{\rho(F)+\varepsilon}\right\}\ (M > 0),$$

on the ray $\arg z = \theta$. Hence, using the same reasoning as in the proof of Lemma 3.1 in [29], by two-fold iterated integration, along the line segment $[0, z]$, we conclude that

$$f(z) = f(0) + f'(0)\frac{z}{1!} + \int_0^z\int_0^t f''(u)\,du\,dt.$$

So, we get for a sufficiently large r

$$\begin{aligned}|f(z)| \le& |f(0)| + |f'(0)|\frac{|z|}{1!} + \left|\int_0^z\int_0^t f''(u)\,du\,dt\right| \\ \le& |f(0)| + |f'(0)|\frac{|z|}{1!} + |f''(z)|\frac{|z|^2}{2!} = \frac{1}{2}(1+o(1))r^2|f''(z)| \\ \le& \frac{1}{2}(1+o(1))r^2\exp\left\{Mr^{\rho(F)+\varepsilon}\right\} \le \exp\left\{r^{\rho(F)+2\varepsilon}\right\},\end{aligned} \tag{26}$$

on the ray $\arg z = \theta$.

Case 2. Suppose that $\delta_1 = \delta\left(bz^n, \theta\right) > 0$, then $\delta_2 = \delta\left(az^n, \theta\right)$. By Lemma 2, for any given ε with $0 < 2\varepsilon < \min\left\{\frac{\delta_1 - \delta_2}{\delta_1}, 2, n - \rho\left(F\right)\right\}$, we obtain

$$\left|B\left(z\right)e^{bz^n}\right| \geq \exp\left\{\left(1-\varepsilon\right)\delta_1 r^n\right\}, \tag{27}$$

for sufficiently large r. We now prove that $\log^+\left|f\left(z\right)\right| / \left|z\right|^{\rho(F)+\varepsilon}$ is bounded on the ray $\arg z = \theta$. We assume that $\log^+\left|f\left(z\right)\right| / \left|z\right|^{\rho(F)+\varepsilon}$ is unbounded on the ray $\arg z = \theta$. Then by Lemma 4, there is a sequence of points $z_m = r_m e^{i\theta}$, such that $r_m \to +\infty$, and that

$$\frac{\log^+\left|f\left(z_m\right)\right|}{r_m^{\rho(F)+\varepsilon}} \to +\infty, \tag{28}$$

for m is large enough. From (13) and (28), we get as in (19)

$$\left|\frac{F\left(z_m\right)}{f\left(z_m\right)}\right| \to 0, \tag{29}$$

for m is large enough. From (11), we obtain

$$\left|Be^{bz^n}\right| \leq \left|e^{az^n}\right|\left|\frac{f''}{f}\right| + \left|P\right|\left|\frac{f'}{f}\right| + \left|\frac{F}{f}\right|. \tag{30}$$

(i) If $\delta_2 > 0$, then by Lemma 2, for ε as above, we obtain

$$\left|e^{az^n}\right| \leq \exp\left\{\left(1+\varepsilon\right)\delta_2 r^n\right\} \tag{31}$$

for sufficiently large r. Substituting (14), (22), (27), (29) and (31) into (30), we have

$$\begin{aligned}\exp\left\{\left(1-\varepsilon\right)\delta_1 r_m^n\right\} \leq& \left|B\left(z_m\right)e^{bz_m^n}\right| \\ \leq& \left|e^{az_m^n}\right|\left|\frac{f''\left(z_m\right)}{f\left(z_m\right)}\right| + \left|P\left(z_m\right)\right|\left|\frac{f'\left(z_m\right)}{f\left(z_m\right)}\right| + \left|\frac{F\left(z_m\right)}{f\left(z_m\right)}\right| \\ \leq& r_m^{2\rho}\exp\left\{\left(1+\varepsilon\right)\delta_2 r_m^n\right\} + \alpha r_m^{k+2\rho} + o\left(1\right) \\ \leq& C_3 r_m^{k+2\rho}\exp\left\{\left(1+\varepsilon\right)\delta_2 r_m^n\right\},\end{aligned} \tag{32}$$

where $C_3 > 0$ is some constant. By $0 < \varepsilon < \frac{\delta_1 - \delta_2}{2\delta_1}$ and (32), we can get

$$\exp\left\{\frac{\left(\delta_1 - \delta_2\right)^2}{2\delta_1} r_m^n\right\} \leq C_3 r_m^{k+2\rho},$$

which is a contradiction.

(ii) If $\delta_2 < 0$, then by Lemma 2, for ε as above, we obtain

$$\left|e^{az^n}\right| \leq \exp\left\{\left(1-\varepsilon\right)\delta_2 r^n\right\} < 1, \tag{33}$$

for sufficiently large r. Substituting (14), (22), (27), (29) and (33) into (30), we have

$$\begin{aligned}\exp\left\{\left(1-\varepsilon\right)\delta_1 r_m^n\right\} \leq& \left|B\left(z_m\right)e^{bz_m^n}\right| \\ \leq& \left|e^{az_m^n}\right|\left|\frac{f''\left(z_m\right)}{f\left(z_m\right)}\right| + \left|P\left(z_m\right)\right|\left|\frac{f'\left(z_m\right)}{f\left(z_m\right)}\right| + \left|\frac{F\left(z_m\right)}{f\left(z_m\right)}\right| \\ \leq& r_m^{2\rho} + \alpha r_m^{k+2\rho} + o\left(1\right) \leq C_4 r_m^{k+2\rho},\end{aligned}$$

where $C_4 > 0$ is some constant, which is a contradiction. Therefore,

$$\log^+ |f(z)| / |z|^{\rho(F)+\varepsilon},$$

is bounded and we have

$$|f(z)| \leq \exp\left\{r^{\rho(F)+2\varepsilon}\right\},$$

on the ray $\arg z = \theta$.

Case 3. Suppose now that $\delta_1 < 0$. From (11) we get

$$-1 = e^{az^n}\frac{f''}{Pf'} + Be^{bz^n}\frac{f}{Pf'} - \frac{F}{Pf'}. \tag{34}$$

By Lemma 2, for any given ε with $0 < 2\varepsilon < n - \rho(F)$, we obtain

$$\left|e^{az^n}\right| \leq \exp\{(1-\varepsilon)\delta(az^n,\theta)r^n\} \leq \exp\{(1-\varepsilon)\delta_1 r^n\}, \tag{35}$$

$$\left|B(z)e^{bz^n}\right| \leq \exp\{(1-\varepsilon)\delta(bz^n,\theta)r^n\} \leq \exp\{(1-\varepsilon)\delta_1 r^n\}, \tag{36}$$

for sufficiently large r. We now prove that $\log^+ |f'(z)| / |z|^{\rho(F)+\varepsilon}$ is bounded on the ray $\arg z = \theta$. We assume that $\log^+ |f'(z)| / |z|^{\rho(F)+\varepsilon}$ is unbounded on the ray $\arg z = \theta$. Then by Lemma 4 there is a sequence of points $z_m = r_m e^{i\theta}$, such that $r_m \to +\infty$, and that

$$\frac{\log^+ |f'(z_m)|}{r_m^{\rho(F)+\varepsilon}} \to +\infty, \tag{37}$$

$$\left|\frac{f(z_m)}{f'(z_m)}\right| \leq (1+o(1))\, r_m \leq 2r_m. \tag{38}$$

From (13) and (37), we have

$$\left|\frac{F(z_m)}{f'(z_m)}\right| \to 0, \tag{39}$$

for m is large enough. By using Lemma 8, there exists $\beta > 0$ such that for sufficiently large $|z| = r$, we have

$$|P(z)| \geq \beta r^k,\ k = \deg P \geq 1. \tag{40}$$

Substituting (14), (35), (36), (38), (39) and (40) into (34), we have

$$\begin{aligned} 1 \leq & \frac{\left|e^{az_m^n}\right|}{|P(z_m)|}\left|\frac{f''(z_m)}{f'(z_m)}\right| + \frac{\left|B(z_m)e^{bz_m^n}\right|}{|P(z_m)|}\left|\frac{f(z_m)}{f'(z_m)}\right| + \frac{1}{|P(z_m)|}\left|\frac{F(z_m)}{f'(z_m)}\right| \\ \leq & \frac{r_m^{2\rho}}{\beta r_m^k}\exp\{(1-\varepsilon)\delta_1 r_m^n\} + 2\frac{r_m}{\beta r_m^k}\exp\{(1-\varepsilon)\delta_1 r_m^n\} + \frac{1}{\beta r_m^k}o(1). \end{aligned} \tag{41}$$

By $\delta_1 < 0$, we have

$$\frac{r_m^{2\rho-k}}{\beta}\exp\{(1-\varepsilon)\delta_1 r_m^n\} + 2\frac{r_m^{1-k}}{\beta}\exp\{(1-\varepsilon)\delta_1 r_m^n\} + \frac{1}{\beta r_m^k}o(1) \to 0,$$

as $r_m \to +\infty$. From (41) we obtain $1 \leq 0$ as $r_m \to +\infty$, which is a contradiction. Therefore, $\log^+ |f'(z)| / |z|^{\rho(F)+\varepsilon}$ is bounded and we have

$$|f'(z)| \leq \exp\left\{Mr^{\rho(F)+\varepsilon}\right\}\ (M > 0),$$

on the ray $\arg z = \theta$. This implies, as in Case 1, that

$$|f(z)| \leq \exp\left\{r^{\rho(F)+2\varepsilon}\right\}. \tag{42}$$

Therefore, for any given $\theta \in [0, 2\pi) \setminus (E \cup E_4)$, we have got (42) on the ray $\arg z = \theta$, provided that r is large enough. Then by Lemma 5, we have $\rho(f) \leq \rho(F) + 2\varepsilon < n$, which is a contradiction. Hence every transcendental solution f of (4) must be of infinite order.

We have

$$\max\left\{\rho\left(Pe^{a_n z^n}\right), \rho\left(Be^{b_n z^n}\right), \rho\left(Fe^{a_n z^n}\right)\right\} = n,$$

so by using Lemma 7, we obtain $\rho_2(f) \leq n$.

Since $F \not\equiv 0$, then by Lemma 6, we get

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty,\ \overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) \leq n.$$

4. Proof of Theorem 4

Suppose that f is a solution of Eq. (4). Then by Theorem 3, we have $\rho(f) = +\infty$ and $\rho_2(f) \leq n$. First, we prove $\rho\left(g_f\right) = \rho(f) = \infty$ and $\rho_2\left(g_f\right) = \rho_2(f) \leq n$. Suppose that $\arg a_n = \arg b_n$ or $a_n = cb_n$, $0 < c < 1$. First, we suppose that $d_2 \not\equiv 0$. Substituting $f'' = Fe^{a_n z^n} - P(z)e^{a_n z^n} f' - B(z)e^{b_n z^n} f$ into g_f, we get

$$g_f - d_2 Fe^{a_n z^n} - b = \left(d_1 - d_2 Pe^{a_n z^n}\right) f' + \left(d_0 - d_2 Be^{b_n z^n}\right) f. \tag{43}$$

Differentiating both sides of Eq. (43) and replacing f'' with $Fe^{a_n z^n} - P(z)e^{a_n z^n} f' - B(z)e^{b_n z^n} f$, we obtain

$$\begin{aligned} g_f' - \left(d_2 Fe^{a_n z^n}\right)' - \left(d_1 - d_2 Pe^{a_n z^n}\right) Fe^{a_n z^n} - b' \\ = \left[d_2 P^2 e^{2a_n z^n} - \left((d_2 P)' + (a_n z^n)' d_2 P + d_1 P\right) e^{a_n z^n} - d_2 Be^{b_n z^n} + d_0 + d_1'\right] f' \\ + \left[d_2 PBe^{(a_n+b_n)z^n} - \left((d_2 B)' + (b_n z^n)' d_2 B + d_1 B\right) e^{b_n z^n} + d_0'\right] f. \end{aligned} \tag{44}$$

Then, by (43) and (44), we have

$$\alpha_1 f' + \alpha_0 f = g_f - d_2 Fe^{a_n z^n} - b, \tag{45}$$

$$\beta_1 f' + \beta_0 f = g_f' - \left(d_2 Fe^{a_n z^n}\right)' - \left(d_1 - d_2 Pe^{a_n z^n}\right) Fe^{a_n z^n} - b'. \tag{46}$$

Set

$$\begin{aligned} h =& \alpha_1\beta_0 - \alpha_0\beta_1 = \left(d_1 - d_2 Pe^{a_n z^n}\right)\left[d_2 PBe^{(a_n+b_n)z^n} - ((d_2 B)' + (b_n z^n)' d_2 B + d_1 B)e^{b_n z^n} + d_0'\right] \\ & - \left(d_0 - d_2 Be^{b_n z^n}\right)\left[d_2 P^2 e^{2a_n z^n} - ((d_2 P)' + (a_n z^n)' d_2 P + d_1 P)e^{a_n z^n} - d_2 Be^{b_n z^n} + d_0 + d_1'\right]. \end{aligned} \tag{47}$$

We prove $h \not\equiv 0$. Now check all the terms of h. Since the term $d_2^2 P^2 Be^{(2a_n+b_n)z^n}$ is eliminated, by (47) we can write $h = \Psi_2(z) - d_2^2 B^2 e^{2b_n z^n}$, where $\Psi_2(z)$ is defined as in Lemma 9 (ii). Thus, by $d_2 \not\equiv 0$ and $B \not\equiv 0$, we see that $h \not\equiv 0$.

Suppose now $a_n = cb_n$, $c > 1$. By (47), we can write

$$h = \Psi_3(z) = H_0 + H_{a_n} e^{a_n z^n} + H_{b_n} e^{b_n z^n} + H_{a_n+b_n} e^{(a_n+b_n)z^n} + H_{2a_n} e^{2a_n z^n} + H_{2b_n} e^{2b_n z^n},$$

where $H_0, H_{a_n}, H_{b_n}, H_{a_n+b_n}, H_{2a_n}, H_{2b_n}$ are entire functions of orders less than n. By $d_2 \not\equiv 0$, $B \not\equiv 0$, we have $H_{2b_n} = -d_2^2 B^2 \not\equiv 0$. Then by Lemma 10, we have $h \not\equiv 0$.

Now suppose $d_2 \equiv 0$, $d_1 \not\equiv 0$, by (47) we can write

$$\begin{aligned} h &= d_1\left(d_1 B e^{b_n z^n} + d_0'\right) - d_0\left(d_1 P e^{a_n z^n} + d_0 + d_1'\right) \\ &= d_1^2 B e^{b_n z^n} - d_0 d_1 P e^{a_n z^n} + d_1 d_0' - d_0^2 - d_0 d_1'. \end{aligned}$$

By Lemma 1 and $d_1^2 B \not\equiv 0$, we have $\rho(h) = n$. Hence, $h \not\equiv 0$.

Finally, if $d_2 \equiv 0$, $d_1 \equiv 0$, $d_0 \not\equiv 0$, then we have $h = -d_0^2 \not\equiv 0$. Hence, $h \not\equiv 0$. By (45), (46) and (47), we obtain

$$f = \frac{\alpha_1\left(g_f' - \left(d_2 F e^{a_n z^n}\right)' - \alpha_1 F e^{a_n z^n} - b'\right) - \beta_1\left(g_f - d_2 F e^{a_n z^n} - b\right)}{h}. \tag{48}$$

If $\rho\left(g_f\right) < \infty$, then by (48) we get $\rho(f) < \infty$ and this is a contradiction. Hence $\rho\left(g_f\right) = \infty$.

Now we prove that $\rho_2\left(g_f\right) = \rho_2(f)$. By (6), we get $\rho_2\left(g_f\right) \le \rho_2(f)$ and by (48) we have $\rho_2(f) \le \rho_2\left(g_f\right)$. This yield $\rho_2\left(g_f\right) = \rho_2(f) \le n$.

Set $w(z) = d_2 f'' + d_1 f' + d_0 f + b - \varphi$. Since $\rho(\varphi) < \infty$, then we have $\rho(w) = \rho\left(g_f\right) = \rho(f) = \infty$ and $\rho_2(w) = \rho_2\left(g_f\right) = \rho_2(f)$. In order to prove $\overline{\lambda}\left(g_f - \varphi\right) = \lambda\left(g_f - \varphi\right) = \infty$ and $\overline{\lambda}_2\left(g_f - \varphi\right) = \lambda_2\left(g_f - \varphi\right) = \rho_2(f)$, we need to prove only $\overline{\lambda}(w) = \lambda(w) = \infty$ and $\overline{\lambda}_2(w) = \lambda_2(w) = \rho_2(f)$. By $g_f = w + \varphi$, we get from (48)

$$f = \frac{\alpha_1\left(w' + \varphi' - \left(d_2 F e^{a_n z^n}\right)' - \alpha_1 F e^{a_n z^n} - b'\right) - \beta_1\left(w + \varphi - d_2 F e^{a_n z^n} - b\right)}{h}. \tag{49}$$

So, we can write

$$f = \frac{\alpha_1 w' - \beta_1 w}{h} + \psi, \tag{50}$$

where

$$\psi(z) = \frac{\alpha_1\left(\varphi' - \left(d_2 F e^{a_n z^n}\right)' - \alpha_1 F e^{a_n z^n} - b'\right) - \beta_1\left(\varphi - d_2 F e^{a_n z^n} - b\right)}{h}.$$

Substituting (50) into Eq. (4), we obtain

$$\frac{\alpha_1}{h} w''' + \phi_2 w'' + \phi_1 w' + \phi_0 w = F e^{a_n z^n} - \left(\psi'' + P(z) e^{a_n z^n} \psi' + B(z) e^{b_n z^n} \psi\right) = G, \tag{51}$$

where ϕ_j $(j = 0,1,2)$ are meromorphic functions with $\rho\left(\phi_j\right) < \infty$ $(j = 0,1,2)$. Since $\rho(\psi) < \infty$, by Theorem 3, it follows that $G \not\equiv 0$. By $\alpha_1 \not\equiv 0$, $h \not\equiv 0$ and Lemma 6, we obtain $\overline{\lambda}(w) = \lambda(w) = \rho(w) = \infty$, $\overline{\lambda}_2(w) = \lambda_2(w) = \rho_2(w) = \rho_2(f)$, i.e., $\overline{\lambda}\left(g_f - \varphi\right) = \lambda\left(g_f - \varphi\right) = \rho\left(g_f\right) = \rho(f) = \infty$ and $\overline{\lambda}_2\left(g_f - \varphi\right) = \lambda_2\left(g_f - \varphi\right) = \rho_2\left(g_f\right) = \rho_2(f) \le n$.

By f is infinite order solution of Eq. (4) and Lemma 6 again, we have

$$\overline{\lambda}(f) = \lambda(f) = \overline{\lambda}\left(g_f - \varphi\right) = \lambda\left(g_f - \varphi\right) = \rho(f) = +\infty,$$

$$\overline{\lambda}_2(f) = \lambda_2(f) = \overline{\lambda}_2\left(g_f - \varphi\right) = \lambda_2\left(g_f - \varphi\right) = \rho_2(f) \le n,$$

which completes the proof. If we put $\varphi(z) = z$, then we get

$$\overline{\lambda}\left(g_f - z\right) = \lambda\left(g_f - z\right) = \rho(f) = +\infty,\ \overline{\lambda}_2\left(g_f - z\right) = \lambda_2\left(g_f - z\right) = \rho_2(f) \le n.$$

5. Proof of Theorem 5

Suppose that f_1 is a solution of Eq. (7) and f_2 is a solution of Eq. (8). Set $w = f_1 - Kf_2$. Then w is a solution of equation

$$w'' + P(z)\,e^{a_n z^n} w' + B(z)\,e^{b_n z^n} w = (F_1 - KF_2)\,e^{a_n z^n}.$$

By $\rho(F_1 - KF_2) < n$, $F_1 - KF_2 \not\equiv 0$ and Theorem 3, we have $\rho(w) = \infty$ and $\rho_2(w) \leq n$. Thus, by using Theorem 4, we obtain

$$\overline{\lambda}(w) = \lambda(w) = \overline{\lambda}(g_w - \varphi) = \lambda(g_w - \varphi) = \rho(w) = +\infty,$$

$$\overline{\lambda}_2(w) = \lambda_2(w) = \overline{\lambda}_2(g_w - \varphi) = \lambda_2(g_w - \varphi) = \rho_2(w) \leq n,$$

that is

$$\overline{\lambda}(f_1 - Kf_2) = \lambda(f_1 - Kf_2) = \overline{\lambda}\left(g_{f_1 - Kf_2} - \varphi\right) = \lambda\left(g_{f_1 - Kf_2} - \varphi\right) = \rho(f_1 - Kf_2) = \infty,$$

and

$$\overline{\lambda}_2(f_1 - Kf_2) = \lambda_2(f_1 - Kf_2) = \overline{\lambda}_2\left(g_{f_1 - Kf_2} - \varphi\right) = \lambda_2\left(g_{f_1 - Kf_2} - \varphi\right) = \rho_2(f_1 - Kf_2) \leq n,$$

for any complex constant K.

6. Conclusion

In this paper, we investigate the growth and oscillation properties of differential polynomials generated by solutions of second-order non-homogeneous linear differential equations. Under suitable conditions on the coefficients, we obtain estimates for their hyper-order and fixed points. We also improve and extend recent results of the author [13].

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