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# Green's function and a Lyapunov-type inequality for a generalized boundary value problem

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**Abstract:** This study investigates a generalized fractional Hill differential equation subject to two-point separated homogeneous boundary conditions. First, we formulate the Green's function and detail its fundamental characteristics. Subsequently, we derive a Lyapunov-type inequality for this specific boundary value problem. The article illustrates that these results encompass several established findings in existing literature as special cases. Finally, the work highlights potential avenues for future study through a series of open problems.

**Keywords:** differential equation, generalized derivative, boundary conditions, Green's function, Lyapunov-type inequality

**MSC:** 26A24, 26D10, 34B99.

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## 1. Introduction

Let $p$ be a real and continuous function defined on the interval $[0, T]$. The classical Lyapunov inequality [1] states that if the Hill differential equation

$$y'' + p(t)y = 0, \quad 0 < t < T, \tag{1}$$

subject to Dirichlet boundary conditions

$$y(0) = y(T) = 0, \tag{2}$$

has a non-trivial solution $y$, then

$$\int_0^T |p(t)|\, dt > \frac{4}{T}. \tag{3}$$

Additionally, the constant 4 in this inequality is sharp in the sense that the constant 4 cannot be replaced by a larger number.

Since its inception, the Lyapunov inequality has garnered substantial academic interest, leading to numerous generalizations across various mathematical frameworks. These inequalities are vital to the qualitative study of differential equations, serving as essential tools in oscillation theory, stability analysis, and the investigation of boundary value problems by providing rigorous criteria for the non-existence of non-trivial solutions and establishing bounds for eigenvalues in Sturm–Liouville problems. In practical applications, they are indispensable for evaluating the stability and controllability of dynamical systems within engineering and mathematical physics. Extensive reviews of these inequalities and their applications can be found in [2–10].

The field of fractional calculus [11,12], centered on non-integer order derivatives and integrals, traces its history back to the inception of classical analysis. While rooted in tradition, the discipline has seen an explosion of growth in recent decades, evolving into a highly productive research domain. The expansion of Lyapunov-type inequalities to include fractional operators has significantly broadened their utility, particularly in modeling non-local phenomena and systems with memory-dependent dynamics.

Consequently, a substantial body of recent literature [13–24] has focused on establishing these inequalities for diverse categories of fractional boundary value problems.

While local differential operators first emerged in the 1960s, their formalization was only recently established through the introduction of the conformable derivative [25]. This framework was further expanded in 2018 with the development of the non conformable derivative [26–29], a breakthrough that significantly widened both the theoretical scope and the practical utility of the field. To gain a more comprehensive perspective on these advancements and their many developments, interested readers may refer to [30–33].

Generalized differential operators [34,35] have become a cornerstone in tackling complex problems across mathematical analysis, particularly where traditional calculus falls short. These operators are essential for modeling anomalous diffusion, non-local elliptic and parabolic equations, and integro-differential systems, as they effectively account for memory effects and long-range interactions. Beyond modeling, they are fundamental to the architecture of generalized Sobolev spaces and play a vital role in spectral theory through generalized Sturm–Liouville problems. Their versatility extends to the calculus of variations and non-linear analysis, where they enable the development of generalized Euler–Lagrange equations and fixed-point frameworks.

Building on these motivations, this research explores a version of the Hill differential equation where the traditional second-order derivative is substituted with a generalized derivative of order $\nu \in (1,2)$, governed by two-point separated homogeneous boundary conditions. Our analysis begins with the formulation of the corresponding Green's function and a detailed examination of its fundamental characteristics. These properties then serve as the basis for deriving a new Lyapunov-type inequality for the problem at hand. Furthermore, we demonstrate the generality of our framework by showing that it encompasses several well-known results from previous studies as specific instances. The article wraps up by identifying various unresolved problems to stimulate further investigation in the field.

## 2. Preliminaries

This section presents the preliminary concepts of generalized integrals and derivatives employed in deriving the main results. The notion of a generalized derivative introduced in [29] is further extended in [34,35] to encompass higher-order generalizations.

**Definition 1.** Let $f : [0, +\infty) \to \mathbb{R}$, $\mu \in (0,1)$ and $\Phi(\cdot, \mu)$ be an absolutely continuous real function defined on the interval $[0, +\infty)$. Then, the $N$-derivative of $f$ of order $\mu$ is defined by

$$N_\Phi^\mu f(t) = \lim_{\varepsilon \to 0} \left[ \frac{f(t + \varepsilon \Phi(t, \mu)) - f(t)}{\varepsilon} \right], \quad t > 0.$$

If $\lim_{t \to 0^+} N_\Phi^\mu f(t)$ exists, then

$$N_\Phi^\mu f(0) = \lim_{t \to 0^+} N_\Phi^\mu f(t).$$

**Remark 1.** [36] The generalized derivative defined above contains conformable and non-conformable local derivatives as particular cases.

| $\Phi(t, \mu)$ | Derivative |
|---|---|
| 1 | Classical |
| $e^{t^\mu}$ | Non-conformable [26,28] |
| $e^{(\mu-1)t}$ | Conformable [34] |
| $t^{1-\mu}$ | Conformable [25] |
| $t^\mu$ | Non-conformable [37] |
| $t^{-\mu}$ | Non-conformable [38] |

**Definition 2.** Let $f:[0,+\infty)\to\mathbb{R}$, $n\in\mathbb{N}$, $\mu\in(n-1,n)$ and $\Phi(\cdot,\mu)$ be an absolutely continuous real function defined on the interval $[0,+\infty)$. If $f^{(n-1)}$ exists on $(0,+\infty)$, then the $N$-derivative of $f$ of order $\mu$ is defined by

$$N^{\mu}_{\Phi}f(t)=\lim_{\varepsilon\to 0}\left[\frac{f^{(n-1)}(t+\varepsilon\Phi(t,\mu-n+1))-f^{(n-1)}(t)}{\varepsilon}\right],\quad t>0.$$

Now, we give the definition of a general fractional integral [27,34–36].

**Definition 3.** Let $\mu\in\mathbb{R}$, $K(\cdot,\mu)$ be an absolutely continuous real function defined on the interval $[0,+\infty)$ and $f:[0,+\infty)\to\mathbb{R}$ be a locally integrable function. Then,

1. the left integral of $f$ of order $\mu$ is defined by

$$J^{\mu}_{K,0+}f(t)=\int_0^t\frac{f(s)}{K(t-s,\mu)}ds,\quad t>0; \tag{4}$$

2. the central integral of $f$ of order $\mu$ is defined by

$$J^{\mu}_{K,0}f(t)=\int_0^t\frac{f(s)}{K(s,\mu)}ds,\quad t>0. \tag{5}$$

**Remark 2.** [36] The generalized integrals defined above contain, as particular cases, the integral operators obtained from conformable and non-conformable local derivatives.

The following result is similar to the known result from classical calculus [36].

**Theorem 1.** *Let $f:[0,+\infty)\to\mathbb{R}$, $n\in\mathbb{N}$, $\mu\in(n-1,n)$ and $\Phi(\cdot,\mu)$ be an absolutely continuous real function defined on the interval $[0,+\infty)$. If $f$ is $\mu$-differentiable on $(0,+\infty)$ such that $f^{(n)}$ is locally integrable on $(0,+\infty)$, then*

$$J^{\mu}_{\Phi,0}N^{\mu}_{\Phi}f(t)=f(t)-\left[\sum_{i=0}^{n-1}\frac{f^{(i)}(0)}{i!}t^i\right],\quad t>0.$$

Throughout this article, we assume that $1<\nu<2$ and $F(\cdot,\nu)$ is an absolutely continuous real positive function defined on the interval $[0,T]$.

## 3. Main Results

The outline of this section is as follows: First, we consider the following boundary value problem:

$$\begin{cases}N^{\nu}_{F}x(t)+h(t)=0, & 0<t<T,\\ \alpha x(0)-\beta x'(0)=0,\\ \gamma x(T)+\delta x'(T)=0,\end{cases} \tag{6}$$

where $\alpha,\beta,\gamma,\delta\in\mathbb{R}$ such that $\alpha^2+\beta^2>0$ and $\gamma^2+\delta^2>0$, $T>0$ and $h:[0,T]\to\mathbb{R}$ is a continuous function. We construct the associated Green's function in Theorem 2 and derive a few of its important properties in Theorems 3 and 4. Next, we consider the following boundary value problem

$$\begin{cases}N^{\nu}_{F}x(t)+a(t)x(t)=0, & 0<t<T,\\ \alpha x(0)-\beta x'(0)=0,\\ \gamma x(T)+\delta x'(T)=0,\end{cases} \tag{7}$$

corresponding to (6), and establish a Lyapunov-type inequality in Theorem 7. Here $a$ is a real and continuous function defined on the interval $[0,T]$.

**Remark 3.** If $F(t,\nu)=t^{1-\nu}$, $\alpha=\beta=\gamma=\delta=1$, then (7) reduces to the boundary value problem considered in [15]. If $F(t,\nu)=t^{1-\nu}$, $\alpha=\gamma=1$ and $\beta=\delta=0$, we obtain the boundary value problem studied in [14].

**Theorem 2.** *Assume $\alpha\gamma T+\alpha\delta+\beta\gamma\neq 0$. The unique solution of (6) is given by*

$$x(t)=\int_0^T G(t,s)h(s)ds,\quad 0\leq t\leq T, \tag{8}$$

*where*

$$G(t,s)=\frac{1}{F(s,\nu)}\begin{cases}G_1(t,s), & 0\leq t\leq s\leq T,\\ G_2(t,s), & 0\leq s\leq t\leq T,\end{cases} \tag{9}$$

*is the Green's function associated with (6). Here*

$$G_1(t,s)=\frac{\beta+\alpha t}{\alpha\gamma T+\alpha\delta+\beta\gamma}\left[\gamma(T-s)+\delta\right], \tag{10}$$

*and*

$$G_2(t,s)=G_1(t,s)-(t-s). \tag{11}$$

**Proof.** The general solution of

$$N_F^{\nu}x(t)+h(t)=0,\quad 0<t<T,$$

can be expressed as

$$x(t)=c_1+c_2t-\int_0^t(t-s)\frac{h(s)}{F(s,\nu)}ds,\quad 0\leq t\leq T, \tag{12}$$

where $c_1$ and $c_2$ are arbitrary constants. Then, we have

$$x'(t)=c_2-\int_0^t\frac{h(s)}{F(s,\nu)}ds,\quad 0\leq t\leq T. \tag{13}$$

Using the boundary condition $\alpha x(0)-\beta x'(0)=0$ in (12) - (13), we obtain

$$\alpha c_1-\beta c_2=0. \tag{14}$$

Using the boundary condition $\gamma x(T)+\delta x'(T)=0$ in (12) - (13), we obtain

$$\gamma c_1+(\gamma T+\delta)c_2=\int_0^T\left[\gamma(T-s)+\delta\right]\frac{h(s)}{F(s,\nu)}ds. \tag{15}$$

Solving (14) and (15) for $c_1$ and $c_2$, we get

$$c_1=\frac{\beta}{\alpha\gamma T+\alpha\delta+\beta\gamma}\int_0^T\left[\gamma(T-s)+\delta\right]\frac{h(s)}{F(s,\nu)}ds, \tag{16}$$

and

$$c_2=\frac{\alpha}{\alpha\gamma T+\alpha\delta+\beta\gamma}\int_0^T\left[\gamma(T-s)+\delta\right]\frac{h(s)}{F(s,\nu)}ds. \tag{17}$$

Using (16) and (17) in (12), for $0\leq t\leq T$, we have

$$\begin{aligned}x(t)&=\frac{\beta+\alpha t}{\alpha\gamma T+\alpha\delta+\beta\gamma}\int_0^T\left[\gamma(T-s)+\delta\right]\frac{h(s)}{F(s,\nu)}ds-\int_0^t(t-s)\frac{h(s)}{F(s,\nu)}ds\\&=\int_0^t G_2(t,s)h(s)ds+\int_t^T G_1(t,s)h(s)ds,\end{aligned}$$

the result we were looking for. □

**Remark 4.** If $F(t,\nu)=t^{1-\nu}$, $\alpha=\beta=\gamma=\delta=1$, Theorem 2 becomes Theorem 3.1 of [15]. If $\alpha=\gamma=1$, $\beta=\delta=0$ and $F(t,\nu)=t^{1-\nu}$, we obtain Lemma 3 of [14] from Theorem 2.

**Theorem 3.** *Assume $\alpha, \beta, \gamma, \delta \geq 0$ such that $\alpha\delta + \alpha\gamma T + \beta\gamma > 0$. Then, the Green's function $G(t,s)$ defined in (9) is non-negative for all $(t,s) \in [0,T] \times [0,T]$.*

**Proof.** It follows from $\alpha, \beta, \gamma, \delta \geq 0$ such that $\alpha\delta + \alpha\gamma T + \beta\gamma > 0$ and $0 \leq t \leq s \leq T$ that

$$G_1(t,s) = \frac{\alpha t + \beta}{\alpha\delta + \alpha\gamma T + \beta\gamma} [\gamma(T-s) + \delta] \geq 0.$$

Now, for $0 \leq s \leq t \leq T$, consider

$$\begin{aligned}
G_2(t,s) &= G_1(t,s) - (t-s) \\
&= \frac{\alpha t + \beta}{\alpha\delta + \alpha\gamma T + \beta\gamma} [\gamma(T-s) + \delta] - (t-s) \\
&= \frac{1}{\alpha\delta + \alpha\gamma T + \beta\gamma} [(\alpha t + \beta)[\gamma(T-s) + \delta] - (t-s)(\alpha\delta + \alpha\gamma T + \beta\gamma)] \\
&= \frac{1}{\alpha\delta + \alpha\gamma T + \beta\gamma} \Big[\beta\gamma[(T-s) - (t-s)] + \alpha\gamma[t(T-s) - T(t-s)] \\
&\quad + \alpha\delta[t - (t-s)] + \beta\delta\Big] = \frac{1}{\alpha\delta + \alpha\gamma T + \beta\gamma} [\beta\gamma(T-t) + \alpha\gamma(T-t)s + \alpha\delta s + \beta\delta].
\end{aligned}$$

Since $\alpha, \beta, \gamma, \delta \geq 0$ such that $\alpha\delta + \alpha\gamma T + \beta\gamma > 0$ and $0 \leq s \leq t \leq T$, we obtain $G_2(t,s) \geq 0$. Thus, $G(t,s)$ is non-negative for all $(t,s) \in [0,T] \times [0,T]$. The proof is completed. □

**Remark 5.** Theorem 3 coincides with Lemma 4 of [14] and with Lemma 3.2 of [15], obtained within the framework of the conformable derivative and with particular choices of $\alpha, \beta, \gamma$, and $\delta$.

**Theorem 4.** *Assume $\beta, \delta \geq 0$, $\alpha, \gamma > 0$ such that $\alpha\delta + \alpha\gamma T + \beta\gamma > 0$. Further, assume $\left|\frac{\delta}{\gamma} - \frac{\beta}{\alpha}\right| < T$. For the Green's function $G(t,s)$ defined in (9), we have*

$$\max_{s\in[0,T]} [F(s,\nu)G(t,s)] = F(t,\nu)G(t,t), \quad t \in [0,T],$$

*and*

$$\max_{t\in[0,T]} [F(t,\nu)G(t,t)] = \frac{\alpha\delta + \alpha\gamma T + \beta\gamma}{4\alpha\gamma}.$$

**Proof.** For the first part, we show that for any fixed $t \in [0,T]$, $G(t,s)$ increases in $s$ for $s$ from 0 to $t$, and then decreases in $s$ for $s$ from $t$ to $T$. Let $0 \leq t \leq s \leq T$ and consider

$$\frac{\partial}{\partial s} G_1(t,s) = -\frac{\gamma(\alpha t + \beta)}{\alpha\delta + \alpha\gamma T + \beta\gamma}.$$

Since $\delta \geq 0$, $\alpha, \beta, \gamma > 0$ and $0 \leq t \leq s \leq T$, it follows that $\frac{\partial}{\partial s} G_1(t,s) \leq 0$ implying that $G_1(t,s)$ decreases in $s$ for $s$ from $t$ to $T$. Let $0 \leq s \leq t \leq T$ and consider

$$\frac{\partial}{\partial s} G_2(t,s) = -\frac{\gamma(\alpha t + \beta)}{\alpha\delta + \alpha\gamma T + \beta\gamma} + 1 = \frac{\alpha\gamma(T-t) + \alpha\delta}{\alpha\delta + \alpha\gamma T + \beta\gamma}.$$

Since $\delta \geq 0$, $\alpha, \beta, \gamma > 0$ and $0 \leq s \leq t \leq T$, we obtain $\frac{\partial}{\partial s} G_2(t,s) \geq 0$ implying that $G_2(t,s)$ increases in $s$ for $s$ from 0 to $t$. Thus, we have

$$\max_{s\in[0,T]} [F(s,\nu)G(t,s)] = F(t,\nu)G(t,t), \quad t \in [0,T].$$

To prove the second part, for $t \in [0,T]$, consider

$$F(t,\nu)G(t,t) = \frac{\alpha t + \beta}{\alpha\delta + \alpha\gamma T + \beta\gamma}\left[\gamma(T-t)+\delta\right].$$

We find the maximum value of this expression with respect to $t$ on $[0,T]$. Denote by

$$H(t) = \frac{\alpha t + \beta}{\alpha\delta + \alpha\gamma T + \beta\gamma}\left[\gamma(T-t)+\delta\right].$$

Differentiating $H$ with respect to $t$ and equating it to zero, we obtain

$$t^* = \frac{1}{2}\left[T + \frac{\delta}{\gamma} - \frac{\beta}{\alpha}\right].$$

Since $-T < \frac{\delta}{\gamma} - \frac{\beta}{\alpha} < T$, we have $0 < T + \frac{\delta}{\gamma} - \frac{\beta}{\alpha} < 2T$, and hence

$$0 < \frac{1}{2}\left[T + \frac{\delta}{\gamma} - \frac{\beta}{\alpha}\right] < T,$$

implying that

$$t^* = \frac{1}{2}\left[T + \frac{\delta}{\gamma} - \frac{\beta}{\alpha}\right] \in (0,T).$$

Further

$$H''(t) = -\frac{2\alpha\gamma}{\alpha\delta + \alpha\gamma T + \beta\gamma} < 0,$$

implying that $H(t)$ attains its maximum value at $t = t^*$. Thus,

$$\max_{t\in[0,T]} F(t,\nu)G(t,t) = \max_{t\in[0,T]} H(t) = H(t^*) = \frac{\alpha\delta + \alpha\gamma T + \beta\gamma}{4\alpha\gamma}.$$

The proof is completed. □

**Theorem 5.** *Assume $\gamma > 0$ and $\delta \geq 0$. For the Green's function $G(t,s)$ defined in (9) with $\alpha = 0$ and $\beta = 1$, we have*

$$\max_{s\in[0,T]}[F(s,\nu)G(t,s)] = F(t,\nu)G(t,t), \quad t \in [0,T],$$

*and*

$$\max_{t\in[0,T]}[F(t,\nu)G(t,t)] = T + \frac{\delta}{\gamma}.$$

**Proof.** Let $0 \leq t \leq s \leq T$ and consider

$$\frac{\partial}{\partial s}G_1(t,s) = -1.$$

It follows that $G_1(t,s)$ decreases in $s$ for $s$ from $t$ to $T$. Let $0 \leq s \leq t \leq T$ and consider

$$\frac{\partial}{\partial s}G_2(t,s) = 0.$$

Clearly, $G_2(t,s)$ is a constant function of $s$ for $s$ from 0 to $t$. Thus, we have

$$\max_{s\in[0,T]}[F(s,\nu)G(t,s)] = F(t,\nu)G(t,t), \quad t \in [0,T].$$

To prove the second part, for $t \in [0,T]$, consider

$$F(t,\nu)G(t,t) = \frac{1}{\gamma}\left[\gamma(T-t)+\delta\right].$$

Clearly,

$$\max_{t\in[0,T]} F(t,\nu)G(t,t) = T + \frac{\delta}{\gamma}.$$

The proof is completed. □

**Theorem 6.** *Assume $\alpha > 0$ and $\beta \geq 0$. For the Green's function $G(t,s)$ defined in (9) with $\gamma = 0$ and $\delta = 1$, we have*

$$\max_{s\in[0,T]} [F(s,\nu)G(t,s)] = F(t,\nu)G(t,t), \quad t \in [0,T],$$

*and*

$$\max_{t\in[0,T]} [F(t,\nu)G(t,t)] = T + \frac{\beta}{\alpha}.$$

**Proof.** Let $0 \leq t \leq s \leq T$ and consider

$$\frac{\partial}{\partial s} G_1(t,s) = 0.$$

Clearly, $G_1(t,s)$ is a constant function of $s$ for $s$ from $t$ to $T$. Let $0 \leq s \leq t \leq T$ and consider

$$\frac{\partial}{\partial s} G_2(t,s) = 1.$$

It follows that $G_2(t,s)$ increases in $s$ for $s$ from 0 to $t$. Thus, we have

$$\max_{s\in[0,T]} [F(s,\nu)G(t,s)] = F(t,\nu)G(t,t), \quad t \in [0,T].$$

To prove the second part, for $t \in [0,T]$, consider

$$F(t,\nu)G(t,t) = \frac{\alpha t + \beta}{\alpha}.$$

Clearly,

$$\max_{t\in[0,T]} F(t,\nu)G(t,t) = T + \frac{\beta}{\alpha}.$$

The proof is completed. □

**Remark 6.** If we consider the conformable derivative, then we can derive Lemma 4 of [14] and Lemma 3.2 of [15] from Theorem 4, for appropriate choices of the parameters $\alpha$, $\beta$, $\gamma$, and $\delta$.

We are now able to formulate a Lyapunov-type inequality for (7).

**Theorem 7.** *Assume $\beta, \delta \geq 0$, $\alpha, \gamma > 0$ such that $\alpha\delta + \alpha\gamma T + \beta\gamma > 0$. Further, assume $\left|\frac{\delta}{\gamma} - \frac{\beta}{\alpha}\right| < T$. If (7) has a non-trivial solution $x$, then*

$$\int_0^T \frac{|a(s)|}{F(s,\nu)} ds > \frac{4\alpha\gamma}{\alpha\delta + \alpha\gamma T + \beta\gamma}. \tag{18}$$

**Proof.** Let $\mathcal{B} = C[0,T]$ be the Banach space of real and continuous functions defined on the interval $[0,T]$ endowed with the maximum norm. It follows from Theorem 2 that a solution to (7) satisfies the equation

$$x(t) = \int_0^T G(t,s)a(s)x(s)ds, \quad 0 \leq t \leq T.$$

Hence

$$\begin{aligned}\|x\| &= \max_{t\in[0,T]} |x(t)| \\ &= \max_{t\in[0,T]} \left| \int_0^T G(t,s)a(s)x(s)ds \right| \\ &\le \max_{t\in[0,T]} \left[ \int_0^T G(t,s)|a(s)||x(s)|ds \right] \\ &\le \|x\| \max_{t\in[0,T]} \left[ \int_0^T G(t,s)|a(s)|ds \right] \\ &= \|x\| \max_{t\in[0,T]} \left[ \int_0^T [F(s,\nu)G(t,s)] \frac{|a(s)|}{F(s,\nu)} ds \right] \\ &\le \|x\| \max_{t\in[0,T]} [F(t,\nu)G(t,t)] \left[ \int_0^T \frac{|a(s)|}{F(s,\nu)} ds \right],\end{aligned}$$

or equivalently,

$$1 < \max_{t\in[0,T]} [F(t,\nu)G(t,t)] \left[ \int_0^T \frac{|a(s)|}{F(s,\nu)} ds \right].$$

An application of Theorem 4 yields the result. □

**Theorem 8.** *Assume $\gamma > 0$ and $\delta \ge 0$. If (7) with $\alpha = 0$ and $\beta = 1$ has a non-trivial solution $x$, then*

$$\int_0^T \frac{|a(s)|}{F(s,\nu)} ds > \frac{1}{T + \frac{\delta}{\gamma}}. \tag{19}$$

**Theorem 9.** *Assume $\alpha > 0$ and $\beta \ge 0$. If (7) with $\gamma = 0$ and $\delta = 1$ has a non-trivial solution $x$, then*

$$\int_0^T \frac{|a(s)|}{F(s,\nu)} ds > \frac{1}{T + \frac{\beta}{\alpha}}. \tag{20}$$

**Remark 7.** Theorems 7 - 9 are natural generalizations of Theorem 3 of [14], Theorem 3.3 of [15] and Theorem 2 of [20] and Theorem 3.1 of [10] for appropriate choices of the parameters $\alpha$, $\beta$, $\gamma$, and $\delta$. Obviously, if $F \equiv 1$, from (18), we obtain the classical Lyapunov inequality (3).

## 4. Conclusions

The Lyapunov inequality and its various extensions play a fundamental role in analyzing the qualitative behavior of differential equations, with significant applications in oscillation theory, asymptotic analysis, disconjugacy, and eigenvalue problems. Generalizations of these inequalities to fractional and other generalized differential operators broaden their scope by incorporating nonlocal interactions and memory-dependent phenomena.

In this work, we investigated the Hill differential equation by substituting the classical second-order derivative with a generalized derivative of order $\nu \in (1,2)$, together with two-point separated boundary conditions. By exploiting the properties of the corresponding Green's function, we established a generalized form of the classical Lyapunov inequality. Our approach not only recovers the results obtained in [10,13,17,20, 23,30], but also provides new improvements and extensions of the classical Lyapunov inequality (3).

The nonnegativity property of the Green's function is crucial in determining sufficient conditions that guarantee at least one positive solution for the corresponding boundary value problem. The well-known Guo–Krasnoselskii, Leggett–Williams, and Avery-type fixed point theorems are essential for establishing the existence of multiple positive solutions in a cone of a Banach space.

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