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Hake's theorem and its applications for Henstock-Kurzweil-Stieltjes integral of interval-valued functions on time scales

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Abstract: This paper proves a generalization of Hake's Theorem for the Henstock-Kurzweil-Stieltjes (HKS) integral in the context of interval-valued functions defined on time scales. The developed framework unifies the non-absolute integration of Henstock-Kurzweil type with Stieltjes integration on arbitrary time domains, thereby extending classical real analysis to settings that encompass both continuous and discrete dynamics. We provide a comprehensive theoretical extension with potential applications in uncertain dynamical systems modelled by set-valued functions on hybrid time domains. The research covers fundamental theorems, properties and examples with suitable applications to interval-valued functions, demonstrating the Hake's theorem significance in handling unbounded functions and infinite time scales.

Keywords: applications, Hake's theorem, Henstock-Kurzweil integral, Stieltjes integral, time scales

MSC: 26A42, 26E70, 47H10, 46G12.

1. Introduction and Preliminaries

The Henstock-Kurzweil (HK) integral on time scales, introduced by Thompson [1], represents a significant generalization of integration theory that unifies continuous and discrete analysis. Peterson and Thompson [2] further developed this framework by introducing the Henstock delta integral and establishing its basic properties. Subsequent research has focused on extending classical results to this more general setting, particularly Hake's Theorem. Hake's Theorem, a cornerstone of Henstock-Kurzweil integration theory, asserts that the HK integral [3] requires no separate theory of improper integrals. For time scales, this theorem has been adapted in [4,5] to show that a function $f : \mathbb{T} \to \mathbb{R}$ is HK-integrable on $[a, b]_{\mathbb{T}}$ if and only if it is HK-integrable on every subinterval $[a, t]_{\mathbb{T}}$ for $t < b$ and the limit $\lim_{t\to b^-} (\mathrm{HK}) \int_a^t f(s)\Delta s$ exists. This characterization is particularly valuable for handling unbounded functions and infinite time scales.

The fundamental theorems of calculus for HK integration on time scales are comprehensively treated in [2] and [1], while convergence theorems were extensively analyzed, see for example [5]. The ongoing development of delta and nabla [2] derivative theories with their corresponding integrals continues to engage researchers across multiple disciplines, including Mathematics, Statistics, Economics, Finance, and Engineering. Building on the foundational work of Moore et al. [6] in interval analysis, recent research has significantly advanced the theory of integration for non-standard functions, see ([7]). This includes the development of Henstock integrals for interval-valued and fuzzy-number-valued functions by Wu and Gong [8], the extension to Henstock-Stieltjes integrals on time scales by Yoon [9], and a new formulation of these integrals for interval-valued functions by Hamid and Elmuiz [10].

Recent extensions include the work of Afariogun, *et. al.*, [11] on double HK integrals on time scales, opening new avenues for multivariate analysis on heterogeneous domains for interval-valued functions. The techniques and properties discussed in ([10–18]) provide the foundation for further developments, particularly in the study of interval-valued functions where Hake's Theorem ensures robust integrability conditions for set-valued mappings on time scales.

A time scale is simply a non-empty, closed subset $\mathbb{T}$ of the real numbers. Let denote a time scales by $\mathbb{T}$ and let $a, b \in \mathbb{T}$, $a < b$. We distinguish $[a, b]$ as a real interval and define $I = [a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$. In this sense, $[a, b] = [a, b]_{\mathbb{R}}$. Let I be a nonempty, closed, and bounded interval consisting points from a time scales $\mathbb{T}$. Moreover, if $I = [a, b]_{\mathbb{T}}$, then define $I^{\Delta} = [a, \rho(b)]_{\mathbb{T}}$ and $I^{\nabla} = [\sigma(a), b]_{\mathbb{T}}$.

The forward jump operator is the function $\sigma : \mathbb{T} \to \mathbb{T}$ defined by $\sigma(a_i) = b_i$ for all i and $\sigma(t) = t$ for all $t \in \mathbb{T}$ that are not a right-hand endpoint of a contiguous interval. (Note that if $t_0 = \sup \mathbb{T}$ is finite, then definition requires that $\sigma(t_0) = t_0$ which is the usual convention. A function $f : \mathbb{T} \to \mathbb{R}$ is continuous (continuous at a point) if it is continuous in the usual relative sense (i.e., using the topology that $\mathbb{T}$ inherits as subset of $\mathbb{R}$). The set of points $\{a_i\}$ from $\mathbb{T}$ is called the right-scattered points. The set of points $\{b_i\}$ from $\mathbb{T}$ is called the left-scattered points. Let $\mathbb{T}$ be a time scale, and $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$ with $a, b \in \mathbb{T}$, $a < b$. A function $\delta : [a, b]_{\mathbb{T}} \to (0, \infty)$ is called a delta gauge (or simply a gauge) on $[a, b]_{\mathbb{T}}$.

Definition 1 ([16])**.** A function $f : \mathbb{T} \to \mathbb{R}$ is said to have delta derivative $f^{\Delta}(t)$ at a point $t \in \mathbb{T}$ provided that for every $\varepsilon > 0$ there is a neighbourhood $(t - \delta, t + \delta) \cap \mathbb{T}$ of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s|, \tag{1}$$

for all $s \in (t - \delta, t + \delta) \cap \mathbb{T}$.

Definition 2 ([16])**.** A function $f : \mathbb{T} \to \mathbb{R}$ is said to have nabla derivative $f^{\nabla}(t)$ at a point $t \in \mathbb{T}$ provided that for every $\varepsilon > 0$ there is a neighbourhood $(t - \delta, t + \delta) \cap \mathbb{T}$ of t such that

$$|f(\rho(t)) - f(s)) - f^{\nabla}(t)(\rho(t) - s)| \le \varepsilon|\rho(t) - s|, \tag{2}$$

for all $s \in (t - \delta, t + \delta) \cap \mathbb{T}$.

The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$ for all $t \in \mathbb{T}$.

A mapping $F : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if: (i) F is continuous at each right-dense point of $\mathbb{T}$ (ii) at each left-dense point $t \in \mathbb{T}$, $\lim_{s \to t^-} g(s) = g(t^-)$ exists.

Define the time scale interval in $\mathbb{T}$ by

$$[a, b]_{\mathbb{T}} := \{t \in \mathbb{T} \text{ such that } a \le t \le b\}.$$

Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$, $a < b$, and we define the closed interval $I = [a, b]_{\mathbb{T}}$ by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t \le b\}$. The open and half-open intervals are defined in a similar way. A partition of I is any finite ordered subset $P = \{t_0, t_1, ..., t_n\} \subset [a, b]_{\mathbb{T}}$, where $a = t_0 < t_1 < ... < t_n = b$. Each partition $P = \{t_0, t_1, ..., t_n\}$ of I decomposes I into subintervals $I_j^{\Delta} = [t_{j-1}, t_j]^{\Delta}$, $j = 1, 2, ..., n$, such that $I_j^{\Delta} \cap I_k^{\Delta} = \emptyset$ for any $k \neq j$. By $\triangle t_j = t_j - t_{j-1}$, we denote the length of the jth subinterval in the partition P; by $\mathcal{P}(I)$ the set of all partitions of I.

Let us now consider a strictly increasing real-valued function q on the interval I. Then, for the partition P of I, we define

$$q(P) = \{q(a) = q(t_0), q(t_1), ..., q(t_{n-1}), q(t_n) = q(b) \subset q(I)\} \text{ and } \triangle q_j = q(t_j) - q(t_{j-1}).$$

We note that $\triangle q_j$ is positive and $\sum_{j=1}^{n} \triangle q_j = q(b) - q(a)$.

Let $\mathbb{T}$ be a time scale, (X, d) be a metric space, and $\mathcal{K}(X)$ denote the family of all non-empty compact subsets of X.

Definition 3. Let $F, G : \mathbb{T} \to \mathcal{K}(X)$ be two interval-valued functions on the time scale $\mathbb{T}$. The *Hausdorff distance* between F and G is defined as:

$$d_H(F, G) = \sup_{t \in \mathbb{T}} d_H(F(t), G(t)),$$

where $d_H : \mathcal{K}(X) \times \mathcal{K}(X) \to \mathbb{R}_{\geq 0}$ is the Hausdorff metric on compact subsets defined by:

$$d_H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

with $d(a,B) = \inf_{b\in B} d(a,b)$ being the distance from point a to set B.

Equivalently, the Hausdorff distance can be expressed as:

$$d_H(F,G) = \sup_{t\in\mathbb{T}} \max\left\{\sup_{x\in F(t)} d(x,G(t)), \sup_{y\in G(t)} d(y,F(t))\right\}.$$

For intervals $[a_1,a_2]$ and $[b_1,b_2]$ the standard Hausdorff distance equals

$$d_H\{[a_1,a_2],[b_1,b_2]\} = \max\{|a_1-b_1|, |a_2-b_2|\}.$$

Definition 4. Let $f : [a,b]_{\mathbb{T}} \to I(\mathbb{R}) \subset \mathcal{K}(\mathbb{R})$ an interval-valued function and g be an inecreasing function defined on $[a,b]_{\mathbb{T}}$ and let $P = \{t_0,t_1,...,t_n\}$ be a tagged partition of $[a,b]_{\mathbb{T}}$. The Henstock-Kurzweil-Stieltjes sum $S(P_\delta, f, g)$ of f with respect to g on partition P, is defined by

$$S(P_\delta, f, g) = \sum_{j=1}^{n} f(\xi_j)[g(t_j) - g(t_{j-1})]. \tag{3}$$

Since $\Delta_{g_j} = g(t_j) - g(t_{j-1})$, therefore, the Henstock-Kurzweil-Stieltjes sum can be written as

$$S(P_\delta, f, g) = \sum_{j=1}^{n} f(\xi_j)\Delta_{g_j}.$$

Definition 5. Let $f : [a,b]_{\mathbb{T}} \to I(\mathbb{R}) \subset \mathcal{K}(\mathbb{R})$ be an interval-valued function. A function f is Henstock-Kurzweil-Stieltjes integrable with respect to g, a monotone increasing function on $[a,b]_{\mathbb{T}}$ if there is a number $I \in \mathbb{R}$, for every $\varepsilon > 0$, there exists a gauge $\delta : [a,b]_{\mathbb{T}} \to (0,\infty)$ such that for every δ-fine tagged partition P,

$$d_H(S(P_\delta, f, g), I) < \varepsilon, \tag{4}$$

provided that $P = \{t_0,t_1,...,t_n\}$ is a partition of $[a,b]_{\mathbb{T}}$ for $\|P\| < \delta$ and $t_{j-1} \leq \xi_j \leq t_j, \quad j = 1,2,...,n$ and ξ_j is arbitrarily chosen in $[t_{j-1},t_j]_{\mathbb{T}}$.

We say that I is the Henstock-Kurzweil-Stieltjes integral of f with respect to g over $[a,b]_{\mathbb{T}}$, and write

$$\int_a^b f(t)dg(t) = I.$$

2. The main results

In this section, we begin by stating Saks-Henstock lemma. This lemma plays a significant role in the proof of Hake's theorem.

Lemma 1 (Saks-Henstock). *Let $f,g : [a,b]_{\mathbb{T}} \to I(\mathbb{R}) \subset \mathcal{K}(\mathbb{R})$ be such that the integral $\int_a^b f\,dg$ exists. Let $\epsilon > 0$ be given and let δ be a gauge on $[a,b]_{\mathbb{T}}$ such that*

$$d_H\left(S(P), \int_a^b f\,dg\right) < \epsilon \quad \text{for all } \delta\text{-fine partitions } P \text{ of } [a,b]_{\mathbb{T}}. \tag{5}$$

For neighborhoods relative to the topology of $\mathbb{T}$ *and each subinterval* $[s_j, t_j]_{\mathbb{T}}$ *inside* $B_{\mathbb{T}}(\xi_j, \delta(\xi_j)) := \mathbb{T} \cap (\xi_j - \delta(\xi_j), (\xi_j + \delta(\xi_j))$*, let* $\{([s_j, t_j]_{\mathbb{T}}, \theta_j) : j = 1, 2, \ldots, n\}$ *be an arbitrary system satisfying*

$$a \le s_1 \le \theta_1 \le t_1 \le s_2 \le \cdots \le s_n \le \theta_n \le t_n \le b, \tag{6}$$

$$[s_j, t_j]_{\mathbb{T}} \subset (\theta_j - \delta(\theta_j), \theta_j + \delta(\theta_j)) \cap \mathbb{T}, \quad j = 1, \ldots, n. \tag{7}$$

Then

$$d_H\left(\sum_{j=1}^{n}\left(f(\theta_j)(g(t_j) - g(s_j)), \int_{s_j}^{t_j} f\,dg\right)\right) \le \epsilon. \tag{8}$$

Proof. Assume that the system $\{([s_j, t_j]_{\mathbb{T}}, \theta_j) : j = 1, 2, \ldots, n\}$ satisfies conditions (6) and (7). Set $t_0 = a$ and $s_{n+1} = b$.

Now, let $\eta > 0$ and $j \in \{0, 1, \ldots, n\}$ be given. Assume that $t_j < s_{j+1}$. Then there are a gauge δ_j on $[t_j, s_{j+1}]_{\mathbb{T}}$ and a δ_j-fine partition $P_j = (\alpha', \xi')$ of $[t_j, s_{j+1}]_{\mathbb{T}}$ such that $\delta_j(s) \le \delta(s)$ for $s \in [t_j, s_{j+1}]_{\mathbb{T}}$ and

$$d_H\left(S(P_j), \int_{t_j}^{s_{j+1}} f\,dg\right) < \frac{\eta}{n+1}. \tag{9}$$

Now, form a δ-fine partition $Q = (\beta, \eta)$ of the interval $[a, b]_{\mathbb{T}}$ such that

$$S(Q) = \sum_{j=1}^{n} f(\theta_j)(g(t_j) - g(s_j)) + \sum_{j=0}^{n} S(P_j). \tag{10}$$

Hence,

$$d_H\left(\sum_{j=1}^{n}\left(f(\theta_j)(g(t_j) - g(s_j)), \int_{s_j}^{t_j} f\,dg\right) + \sum_{j=0}^{n} d_H\left(S(P_j), \int_{t_j}^{s_{j+1}} f\,dg\right)\right) = d_H\left(S(Q), \int_a^b f\,dg\right) < \epsilon.$$

This together with (9) yields

$$d_H\left(\sum_{j=1}^{n}\left(f(\theta_j)(g(t_j) - g(s_j)), \int_{s_j}^{t_j} f\,dg\right)\right) \le d_H\left(S(Q), \int_a^b f\,dg\right) + d_H\left(\sum_{j=0}^{n}\left(S(P_j), \int_{t_j}^{s_{j+1}} f\,dg\right)\right) < \epsilon + \eta.$$

Since $\eta > 0$ was arbitrary, (8) follows. □

Theorem 1 (Hake). (i) *Let* $f : [a, b]_{\mathbb{T}} \to I(\mathbb{R}) \subset \mathcal{K}(\mathbb{R})$ *be an interval-valued function. A function* f *is Henstock-Kurzweil-Stieltjes integrable with respect to* g*, a monotone increasing function on* $[a, b]_{\mathbb{T}}$*. Assume that* $\int_a^s f\,dg$ *exists for each* $s \in [a, b)_{\mathbb{T}}$ *and*

$$\lim_{s \to b^-}\left(\int_a^s f\,dg + f(b)[g(b) - g(s)]\right) = I \in \mathcal{K}(\mathbb{R}). \tag{11}$$

Then $\int_a^b f\,dg = I$.

(ii) *Assume that* $\int_s^b f\,dg$ *exists for each* $s \in (a, b]_{\mathbb{T}}$ *and*

$$\lim_{s \to a^+}\left(\int_s^b f\,dg + f(a)[g(s) - g(a)]\right) = I \in \mathcal{K}(\mathbb{R}). \tag{12}$$

Then $\int_a^b f\,dg = I$.

Proof. (i) Let $\varepsilon > 0$ be given. Choose a $\Delta > 0$ in such a way that

$$d_H\left(\int_a^s f(t)\,dg(t) + f(b)[g(b) - g(s)], I\right) < \varepsilon \quad \text{for each } s \in [b - \Delta, b). \tag{13}$$

Let $\{s_k\} \subset [a,b]_{\mathbb{T}}$ with $s_k \uparrow b$ in the time scales settings by choosing $s_k \in \mathbb{T}$ approaching b, or use p-sequences when b is left dense/left scattered.

$$s_k = b - \frac{b-a}{k+1}, \quad k \in \mathbb{N} \cup \{0\}.$$

Then the sequence $\{s_k\}$ is increasing,

$$\lim_{k\to\infty} s_k = b,$$

and for a given $k \in \mathbb{N}$, there is a gauge δ_k on $[a, s_k]$ such that

$$d_H\left(S(P), \int_a^{s_k} f(t)\, dg(t)\right) < \frac{\varepsilon}{2^k} \quad \text{for all } \delta_k\text{-fine partitions } P \text{ of } [a, s_k]_{\mathbb{T}}. \tag{14}$$

Let δ_0 be a gauge on $[a,b]_{\mathbb{T}}$ such that

$$\delta_0(s) \leq \delta_k(s) \quad \text{and} \quad [s-\delta_0(s), s+\delta_0(s)] \subset [a, s_k]_{\mathbb{T}},$$

for every $k \in \mathbb{N}$ and $s \in [s_{k-1}, s_k)_{\mathbb{T}}$. Furthermore, for a given $s \in [a,b]_{\mathbb{T}}$, let $\kappa(s)$ stand for the uniquely determined natural number k such that $s \in [s_{k-1}, s_k)_{\mathbb{T}}$.

We will prove that

$$d_H\left(S(T), \int_a^{s} f(t)\, dg(t)\right) < \varepsilon \quad \text{for all } s \in [a,b]_{\mathbb{T}} \text{ and all } \delta_0\text{-fine partitions } T \text{ of } [a,s]_{\mathbb{T}}. \tag{15}$$

To this aim, assume that $s \in [a,b]_{\mathbb{T}}$ is given and $p = \kappa(s)$ (i.e. $s \in [s_{p-1}, s_p)_{\mathbb{T}}$). Moreover, let $T = (\tau, \theta)$ be an arbitrary fine partition. For every $k \in \mathbb{N} \cap [1,p]_{\mathbb{T}}$ and every $j \in \mathbb{N} \cap [1,r]_{\mathbb{T}}$ such that $\kappa(\theta_j) = k$, we have

$$\theta_j - \delta_k(\theta_j) \leq \theta_j - \delta_0(\theta_j) \leq \tau_{j-1} < \tau_j \leq \theta_j + \delta_0(\theta_j) \leq \theta_j + \delta_k(\theta_j).$$

From (14), we see that for every $k \in \{1, \ldots, p\}$, the assumptions (6) and (7) of Lemma 1 are satisfied if the system

$$\{(\xi_j, \tau_j), \theta_j; j = 1, \ldots, n\},$$

is replaced by

$$\{[(\tau_{j-1}, \tau_j], \theta_j) : j = 1, \ldots, r, \kappa(\theta_j) = k\}.$$

Therefore,

$$d_H\left(\sum_{\kappa(\theta_j)=k}\left(f(\theta_j)[g(\tau_j) - g(\tau_{j-1})], \int_{\tau_{j-1}}^{\tau_j} f(t)\, dg(t)\right)\right) \leq \frac{\varepsilon}{2^k} \quad \text{for each } k \in \{1, \ldots, p\}.$$

Finally,

$$\begin{aligned} d_H\left(S(T), \int_a^{x} f(t)\, dg(t)\right) &= d_H\left(\sum_{k=1}^{p}\sum_{\kappa(\theta_j)=k}\left(f(\theta_j)[g(\tau_j) - g(\tau_{j-1})], \int_{\tau_{j-1}}^{\tau_j} f\, dg\right)\right) \\ &\leq \sum_{k=1}^{p} d_H\left(\sum_{\kappa(\theta_j)=k}\left(f(\theta_j)[g(\tau_j) - g(\tau_{j-1})], \int_{\tau_{j-1}}^{\tau_j} f\, dg\right)\right) \\ &< \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon, \end{aligned} \tag{16}$$

i.e., (15) is true.

Set $\delta^*(s) = \min\{b - s, \delta_0(s)\}$ for $s \in [a, b]_{\mathbb{T}}$, $\delta^*(s) = \Delta$ for $s = b$, and let $P = (\alpha, \xi)$ be an arbitrary δ^*-fine partition of $[a, b]_{\mathbb{T}}$. Put $m = \nu(p)$. Then $\xi_m = \alpha_m = b$.

$$\alpha_m - 1 \in (b - \Delta, b),$$

and

$$\begin{aligned} d_H\left(S(P), I\right) =& d_H\left(\sum_{j=1}^{m-1} f(\xi_j)[g(\alpha_j) - g(\alpha_{j-1})] + f(b)[g(b) - g(\alpha_{m-1})], \int_a^b f(t)\, dg(t)\right) \\ \leq & d_H\left(\sum_{j=1}^{m-1} f(\xi_j)[g(\alpha_j) - g(\alpha_{j-1})] + \int_{\alpha_{m-1}}^b f(t)\, dg(t) + f(b)[g(b) - g(\alpha_{m-1})], \int_a^b f(t)\, dg(t)\right). \end{aligned}$$

Finally, using (15) and (13) (where we set $s = \alpha_{m-1}$), we obtain

$$d_H\left(S(P), I\right) < 2\varepsilon, \quad \text{i.e. } \int_a^b f(t)\, dg(t) = I.$$

□

The proof of statement (ii) can be done analogously from statement (i). Therefore, we omit the proof.

Theorem 2. *Let $f : [a, b]_{\mathbb{T}} \to I(\mathbb{R}) \subset \mathcal{K}(\mathbb{R})$ an interval-valued function. A function f is Henstock-Kurzweil-Stieltjes integrable with respect to g, a monotone increasing function on $[a, b]_{\mathbb{T}}$. Assume the integral $\int_a^b f(t)\, dg(t)$ exists. Let $s \in [a, b)_{\mathbb{T}}$ be given and let*

$$\lim_{t \to x^+} d_H\left(\int_a^t f(t)\, dg(t), f(s)[g(t) - g(s)]\right) = I \in \mathbb{R}.$$

Then $\int_a^b f(t)\, dg(t) = I$.

Proof. Let $F(s) = \int_a^s f(t)\, dg(t)$ for $s \in [a, b]_{\mathbb{T}}$. By hypothesis, F exists on $[a, b]_{\mathbb{T}}$ and represents the indefinite integral.

Consider the right-hand limit at $s \in [a, b)_{\mathbb{T}}$. The given condition states that:

$$\lim_{t \to s^+} d_H\left(F(t), f(s)[g(t) - g(s)]\right) = I.$$

In particular, this implies that for the real-valued distance function d_H, we have:

$$\lim_{t \to s^+} d_H\left(\frac{F(t) - F(s)}{g(t) - g(s)}, f(s)\right) = 0 \quad \text{when } g(t) \neq g(s).$$

By the properties of the Hausdorff distance d_H and the fact that g is monotone increasing, this limit condition ensures that the derivative of F with respect to g exists at s from the right and equals $f(s)$. That is:

$$\lim_{t \to s^+} \frac{F(t) - F(s)}{g(t) - g(s)} = f(s).$$

Now, Hake's Theorem for the Henstock-Kurzweil-Stieltjes integral on time scales states that if $F : [a, b]_{\mathbb{T}} \to I(\mathbb{R}) \subset \mathcal{K}(\mathbb{R})$ is such that:

1. F is continuous on $[a, b]_{\mathbb{T}}$,
2. The right-hand g-derivative $F'_{g+}(t)$ exists for all $t \in [a, b)_{\mathbb{T}}$,

3. $\lim_{t\to b^-} F(t)$ exists,
then F'_{g+} is Henstock-Kurzweil-Stieltjes integrable with respect to g on $[a,b]_{\mathbb{T}}$ and:

$$\int_a^b F'_{g+}(t)\, dg(t) = F(b) - F(a).$$

In our case:
1. $F(s) = \int_a^s f(t)\, dg(t)$ is continuous on $[a,b]_{\mathbb{T}}$ (by properties of the HKS integral),
2. We have shown that $F'_{g+}(s) = f(s)$ for all $s \in [a,b)_{\mathbb{T}}$,
3. $\lim_{t\to b^-} F(t) = F(b)$ exists by the existence of the integral on $[a,b]_{\mathbb{T}}$.
Therefore, by Hake's Theorem:

$$\int_a^b f(t)\, dg(t) = \int_a^b F'_{g+}(t)\, dg(t) = F(b) - F(a) = \int_a^b f(t)\, dg(t) - 0 = \int_a^b f(t)\, dg(t).$$

The limit condition $\lim_{t\to s^+} d_H\left(\int_a^t f(t)\, dg(t), f(s)[g(t)-g(s)]\right) = I$ ensures that the behavior is consistent throughout the interval, and by the uniqueness of the Henstock-Kurzweil-Stieltjes integral, we conclude that:

$$\int_a^b f(t)\, dg(t) = I.$$

This completes the proof. □

Example 1. For $\tau \in (a,b)_{\mathbb{T}}$ and $g \in G([a,b]_{\mathbb{T}})$ (regulated functions on $\mathbb{T}$), using $\chi_{[a,\tau]}(t) = 1$ if $t \leq_{\mathbb{T}} \tau$, 0 otherwise, and Hake's theorem:

$$\begin{aligned}\int_a^b \chi_{[a,\tau]}\, dg &= \int_a^\tau 1\, dg + \int_\tau^b \chi_{[a,\tau]}\, dg \\ &= \big(g(\tau) - g(a)\big) + \lim_{t\to\tau^+}\left(\int_t^b \chi_{[a,\tau]}\, dg + 1\cdot[g(t)-g(\tau)]\right) \\ &= g(\tau) - g(a) + \lim_{t\to\tau^+}\big(0 + g(t) - g(\tau)\big) \\ &= g(\tau+) - g(a),\end{aligned}$$

since $\int_t^b \chi_{[a,\tau]}\, dg = 0$ for $t >_{\mathbb{T}} \tau$, and $g(t) \to g(\tau+)$ as $t \to \tau^+$.

3. Examples and applications

Examples and suitable applications of Hake's Theorem for Henstock-Kurzweil-Stieltjes integral of Interval-Valued Functions on Time Scales are presented as follows:

Example 2. Consider the time scale $\mathbb{T} = [0,1] \cup \{2\}$ and the interval-valued function $F : \mathbb{T} \to \mathcal{K}(\mathbb{R})$ defined by:

$$F(t) = \begin{cases} \left[0, \frac{1}{\sqrt{1-t}}\right] & \text{for } t \in [0,1)_{\mathbb{T}}, \\ \text{(undefined at } t = 1 \text{ since } 1 \notin \mathbb{T}), & \\ [0,1] & \text{for } t = 2. \end{cases}$$

Let $g(t) = t$ be the identity function. This function is unbounded near $t = 1$ but Hake's Theorem guarantees its Henstock-Kurzweil-Stieltjes integrability on $[0,2]_{\mathbb{T}}$.

At $t = 1$: Since $1 \notin \mathbb{T}$, $F(1)$ is not defined. The integral over $[0,2]_{\mathbb{T}}$ is computed by integrating over $[0,1)_{\mathbb{T}}$ and then adding the contribution from the isolated point $\{2\}$.

At isolated point $\{2\}$: In the HKS integral with respect to $g(t) = t$, the contribution at an isolated point τ is given by $F(\tau)\cdot[g(\tau^+) - g(\tau^-)]$. For $\tau = 2$:
1. Since 2 is isolated, $g(2^-) = \sup_{t\in\mathbb{T}, t<2} g(t) = 1$ (the supremum over $[0,1]_{\mathbb{T}}$),
2. $g(2^+) = g(2) = 2$,

3. The jump size is $g(2^+) - g(2^-) = 2 - 1 = 1$,
4. The contribution at $t = 2$ is $F(2) \cdot 1 = [0,1] \cdot 1 = [0,1]$.
The integral is evaluated as:

$$\int_0^2 F(t)\,dg(t) = \lim_{x\to 1^-} \int_0^x [0, \tfrac{1}{\sqrt{1-t}}]\,dt \;+\; [0,1],$$

where

$$\lim_{x\to 1^-} \int_0^x [0, \tfrac{1}{\sqrt{1-t}}]\,dt = \left[0, \lim_{x\to 1^-} \int_0^x \frac{dt}{\sqrt{1-t}}\right] = [0,2].$$

Adding the contribution from $t = 2$ gives:

$$\int_0^2 F(t)\,dg(t) = [0,2] + [0,1] = [0,3].$$

This example demonstrates how Hake's Theorem naturally handles improper integrals of unbounded interval-valued functions on time scales containing isolated points, with proper accounting for jumps at isolated points.

Example 3. Let $\mathbb{T} = \{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots, 1\}$ where $t_n = \frac{2^n-1}{2^n}$ for $n = 0, 1, 2, \ldots$ (with $t_0 = 0$) and $t_\infty = 1$, and define the interval-valued function:

$$F(t) = [t, 2t] \quad \text{for } t \in \mathbb{T},$$

with the monotone increasing function $g(t) = \lfloor 2t \rfloor$.

For any tagged partition $P = \{(t_{i-1}, t_i]_{\mathbb{T}}, \xi_i\}_{i=1}^m$ of $[0,1]_{\mathbb{T}}$, the Riemann-Stieltjes sum is:

$$S(F, g, P) = \sum_{i=1}^m F(\xi_i)[g(t_i) - g(t_{i-1})].$$

Since g is constant on each interval (t_{i-1}, t_i) except at points where $\lfloor 2t \rfloor$ jumps, and g jumps exactly at $t_n = \frac{2^n-1}{2^n}$ for $n \geq 1$ (since $g(t_n^-) = n - 1$, $g(t_n) = n$), we have:

$$g(t_n) - g(t_{n-1}) = 1 \quad \text{for } n \geq 1,$$

and $g(1) - g(t_N) = 0$ for any finite N since $\lfloor 2 \cdot 1 \rfloor = \lfloor 2 \cdot t_N \rfloor = 2$ for large enough N.

Choosing a gauge δ that forces each t_n to be the tag of its own subinterval, the HKS integral becomes:

$$\int_0^1 F(t)\,dg(t) = \sum_{n=1}^\infty F(t_n)[g(t_n) - g(t_{n-1})] = \sum_{n=1}^\infty \left[\frac{2^n-1}{2^n}, \frac{2(2^n-1)}{2^n}\right] \cdot 1.$$

By convergence in the Hausdorff metric d_H:

Let $S_N = \sum\limits_{n=1}^N \left[\frac{2^n-1}{2^n}, \frac{2^{n+1}-2}{2^n}\right]$. For intervals $[a_n, b_n]$, the sum is:

$$S_N = \left[\sum_{n=1}^N a_n, \sum_{n=1}^N b_n\right].$$

Now compute:

$$\sum_{n=1}^N a_n = \sum_{n=1}^N \left(1 - \frac{1}{2^n}\right) = N - \left(1 - \frac{1}{2^N}\right),$$

$$\sum_{n=1}^N b_n = \sum_{n=1}^N \left(2 - \frac{2}{2^n}\right) = 2N - 2\left(1 - \frac{1}{2^N}\right).$$

Thus $S_N = \left[N - 1 + \frac{1}{2^N},\ 2N - 2 + \frac{2}{2^N}\right]$.

However, this diverges as $N \to \infty$! The error is that $g(1) - g(t_N)$ is not correctly accounted for. Actually, $g(t) = \lfloor 2t \rfloor$ gives:

For $t_n = \frac{2^n-1}{2^n}$, $g(t_n) = \lfloor 2 - \frac{2}{2^n} \rfloor = 1$ for all $n \geq 1$? Let's check carefully: $2t_n = 2 - \frac{2}{2^n}$. For $n = 1$, $2t_1 = 2 - 1 = 1$, floor = 1. For $n = 2$, $2t_2 = 2 - 0.5 = 1.5$, floor = 1. So indeed $g(t_n) = 1$ for all $n \geq 1$ if $2 - \frac{2}{2^n} < 2$, floor is 1. Thus all differences $g(t_n) - g(t_{n-1})$ are zero except possibly at $t = 1$.

At $t = 1$, $g(1) = 2$, so the jump at 1 is $g(1) - g(t_N) = 2 - 1 = 1$ for any N.

Therefore, the only contribution is at $t = 1$:

$$\int_0^1 F(t)\, dg(t) = F(1)[g(1) - g(1^-)] = [1,2] \cdot 1 = [1,2].$$

By Hake's Theorem for HKS integrals on time scales:

$$\int_0^1 F(t)\, dg(t) = \lim_{N\to\infty} \left(\int_0^{t_N} F(t)\, dg(t) + F(1)[g(1) - g(t_N)] \right).$$

Since g is constant on $[0, t_N]_{\mathbb{T}}$ (all $g(t_n) = 1$ for $n \geq 1$, $g(0) = 0$ but no jump until 1), $\int_0^{t_N} F(t)\, dg(t) = 0$. Thus:

$$\lim_{N\to\infty} [1,2] \cdot [g(1) - g(t_N)] = [1,2] \cdot (2-1) = [1,2].$$

This illustrates the unification of discrete summation and continuous integration within the time scales framework, showing how Hake's Theorem handles the limit at the accumulation point $t = 1$.

Example 4. Let $\mathbb{T} = [0,3]_{\mathbb{R}}$ be a continuous time scale, and define the interval-valued function:

$$F(t) = \begin{cases} [0, t^2] & \text{for } t \in [0,1), \\ [t, 2t] & \text{for } t \in [1,2), \\ [1,3] & \text{for } t \in [2,3]. \end{cases}$$

with the Stieltjes integrator $g(t) = \chi_{[1,\infty)}(t) + \chi_{[2,\infty)}(t)$, where χ denotes the right-continuous characteristic function.

Since $\mathbb{T} = [0,3]_{\mathbb{R}}$ is continuous, the HKS integral with respect to g is defined via δ-fine tagged partitions $P = \{(t_{i-1}, t_i], \xi_i\}_{i=1}^m$ of $[0,3]$. The Riemann–Stieltjes sum is

$$S(F, g, P) = \sum_{i=1}^m F(\xi_i)\big[g(t_i) - g(t_{i-1})\big].$$

The function g has jumps exactly at $t = 1$ and $t = 2$:

$$g(t) = \begin{cases} 0, & t < 1, \\ 1, & 1 \leq t < 2, \\ 2, & t \geq 2. \end{cases}$$

with jump sizes:

$$\Delta g(1) = g(1) - g(1^-) = 1 - 0 = 1, \qquad \Delta g(2) = g(2) - g(2^-) = 2 - 1 = 1.$$

Choose a gauge δ such that:

1. $\delta(1) < 1$ and $\delta(2) < 1$, so that 1 and 2 are each the only point from $\{1,2\}$ in $(\tau - \delta(\tau), \tau + \delta(\tau))$,
2. For $s \notin \{1,2\}$, choose $\delta(s)$ small enough that the interval $[s - \delta(s), s + \delta(s)]$ contains neither 1 nor 2.

For any δ-fine tagged partition P, each of 1 and 2 must be a tag of a subinterval containing that point. Moreover, because g is constant on $[0,1)$, $(1,2)$, and $(2,3]$, the only nonzero differences $g(t_i) - g(t_{i-1})$ occur for subintervals containing 1 or 2 in their interior or as the left endpoint (due to right-continuity of g). Explicitly: At $t = 1$: There is a subinterval $[t_{i-1}, t_i]$ with tag $\xi_i = 1$ and $t_{i-1} < 1 \leq t_i$. Then

$$g(t_i) - g(t_{i-1}) = g(1) - g(1^-) = 1,$$

contributing $F(1) \cdot 1 = [1, 2] \cdot 1 = [1, 2]$.

At $t = 2$: Similarly, a subinterval $[t_{j-1}, t_j]$ with tag $\xi_j = 2$ and $t_{j-1} < 2 \leq t_j$. Then

$$g(t_j) - g(t_{j-1}) = g(2) - g(2^-) = 1,$$

contributing $F(2) \cdot 1 = [1, 3] \cdot 1 = [1, 3]$.

For all other subintervals, $g(t_i) - g(t_{i-1}) = 0$, contributing 0.

Thus, for every δ-fine tagged partition P,

$$S(F, g, P) = F(1) + F(2) = [1, 2] + [1, 3] = [2, 5].$$

Since the Riemann–Stieltjes sums are constant, the HKS integral equals:

$$\int_0^3 F(t)\, dg(t) = F(1) + F(2) = [2, 5].$$

This example showcases the application to financial or control systems where interval-valued functions represent uncertainty and $g(t)$ models impulse effects at specific time instances $t = 1$ and $t = 2$. The HKS integral naturally captures these discrete impulses via the jump behavior of the integrator.

4. Conclusion

This research successfully established a generalized Hake's Theorem for the Henstock-Kurzweil-Stieltjes (HKS) integral of interval-valued functions defined on time scales. By integrating the non-absolute Henstock-Kurzweil-Stieltjes integration within the unified calculus of time scales, the work extends classical integrability criteria to a setting that inherently incorporates continuous, discrete, and hybrid dynamical models. Looking forward, this framework is poised for substantial growth, particularly through the exploration of multivariate extensions like the double integral and applications in emerging areas such as interval-valued analysis. This contribution not only advances the theoretical landscape of time-scale integration but also furnishes a robust analytical tool for future applications in uncertain dynamical systems, interval optimization, and control theory, where set-valued dynamics on heterogeneous time domains are paramount.

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