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Hardy-Hilbert-Mulholland-type integral inequalities

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Abstract: This article introduces what we term Hardy-Hilbert-Mulholland-type integral inequalities, which unify features of Hardy-Hilbert-type and Mulholland-type integral inequalities. These inequalities are parameterized by an adjustable parameter. The obtained constant factors are derived in singular form involving a logarithmic-tangent expression, and their optimality is discussed in detail. Several new secondary inequalities are also established. Complete proofs are provided, including detailed steps and references to intermediate results.

Keywords: Hardy-Hilbert-type integral inequalities, Mulholland-type integral inequalities, optimality, Hölder integral inequality, Fubini-Tonelli integral theorem, Hardy integral inequality

MSC: 26D15, 33E20.

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1. Introduction

Integral inequalities are a fundamental topic in mathematics. They are used in many areas, including differential equations, functional analysis, and approximation theory. A wide array of classical results rely on such inequalities for their theoretical development and practical application. They are particularly important for estimating functions and operators, proving convergence and establishing stability criteria. For comprehensive treatments and examples, see [1–5].

The Hardy-Hilbert integral inequality is one of the most widely studied results in this field. Essentially, it provides sharp upper bounds for a certain class of double integrals. The classical version of the inequality is presented below. Let $p > 1$ and $q = p/(p-1)$ such that $1/p + 1/q = 1$. They will serve as norm parameters. We set $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$ and $\mathbb{R}_+^* = \{x \in \mathbb{R} \mid x > 0\}$. Let $f, g : \mathbb{R}_+^* \to \mathbb{R}_+$ be two functions such that $\int_{\mathbb{R}_+^*} f^p(x)dx < +\infty$ and $\int_{\mathbb{R}_+^*} g^q(y)dy < +\infty$. Then we have

$$\int_{\mathbb{R}_+^*} \int_{\mathbb{R}_+^*} \frac{1}{x+y} f(x)g(y)dxdy \leq \frac{\pi}{\sin(\pi/p)} \left[\int_{\mathbb{R}_+^*} f^p(x)dx\right]^{1/p} \left[\int_{\mathbb{R}_+^*} g^q(y)dy\right]^{1/q}.$$

The constant factor $\pi/\sin(\pi/p)$ is proved to be optimal. Thanks to its elegant structure and broad applicability, this inequality has inspired a wide range of generalizations and extensions. One notable variation is given by

$$\int_{\mathbb{R}_+^*} \int_{\mathbb{R}_+^*} \frac{1}{x+y} f(x)g(y)dxdy \leq \pi \left[\int_{\mathbb{R}_+^*} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\int_{\mathbb{R}_+^*} y^{q/2-1} g^q(y)dy\right]^{1/q}.$$

This form is particularly attractive due to its use of a simpler constant factor, i.e., π, and the appearance of weighted integral norms of f and g. Depending on the analytical context, this variation may be more convenient or more effective than the original Hardy-Hilbert integral inequality. For a comprehensive survey of related results and developments, see [6–8]. Contemporary studies on the topic from 2024-2025 include [9–12].

In addition, the Mulholland integral inequality achieves another objective. It was introduced in [13], and studied in detail in [14]. The version considered in the latter is given below. Let $f, g : (1, +\infty) \to \mathbb{R}_+$ be two functions such that $\int_{(1,+\infty)} f^p(x)dx < +\infty$ and $\int_{(1,+\infty)} g^q(y)dy < +\infty$. Then we have

$$\int_{(1,+\infty)} \int_{(1,+\infty)} \frac{1}{\log(xy)} f(x)g(y)dxdy \leq \frac{\pi}{\sin(\pi/p)} \left[\int_{(1,+\infty)} x^{p-1} f^p(x)dx\right]^{1/p} \left[\int_{(1,+\infty)} y^{q-1} g^q(y)dy\right]^{1/q}.$$

This inequality is characterized by the presence of the logarithmic term $\log(xy)$ in the denominator of the main integrand. This introduces a qualitatively different type of singularity to that observed in the context of the traditional Hardy-Hilbert integral inequality. This structure is particularly relevant to problems involving logarithmic potentials, information theory, and certain classes of integral operators. Furthermore, it is proved that the constant factor $\pi/\sin(\pi/p)$ is optimal. All of these aspects are discussed in detail in [14,15].

For the purposes of this article, we refer to Mulholland-type integral inequalities as those involving a logarithmic term in the denominator of the main integrand. Based on this definition, we examine two new integral inequalities that combine features of Hardy-Hilbert-type and Mulholland-type integral inequalities. These are referred to as Hardy-Hilbert-Mulholland-type integral inequalities. The double integrals considered have the following forms:

$$\int_{\mathbb{R}_+^*} \int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} f(x)g(y)dxdy,$$

and

$$\int_{\mathbb{R}_+^*} \int_{\mathbb{R}_+^*} \frac{(xy)^{\alpha-1/2} - (xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} f(x)g(y)dxdy,$$

where α is an adjustable parameter, with a range of values to be described later. The corresponding kernel functions are given by

$$K_\alpha^{(1)}(x,y) = \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)},$$

and

$$K_\alpha^{(2)}(x,y) = \frac{(xy)^{\alpha-1/2} - (xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)},$$

respectively. For the range of values of α considered, they are defined almost surely on $(\mathbb{R}_+^*)^2$. In both cases, the constant factor obtained in the inequalities is determined by the same expression involving a logarithmic tangent function. It is given precisely by

$$2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right].$$

In line with the approach outlined in [14], we provide a rigorous proof that this constant factor is optimal in both formulations. The remainder of the upper bound is the product of the weighted integral norms of the functions f and g, where the weights are simple power functions of their respective variables. Several auxiliary inequalities are also derived, each of which makes a new contribution to the theory of integral inequalities. These include inequalities involving a single function and inequalities involving the primitives of the main functions. For the sake of completeness, we provide detailed proofs, including step-by-step derivations and references to key intermediary results.

The remainder of the article is as follows: §2 and §3 are devoted to the first and second Hardy-Hilbert-Mulholland-type integral inequalities, respectively, together with related results. §4 provides a conclusion.

2. First Hardy-Hilbert-Mulholland-type integral inequality

2.1. Main result

The theorem below presents the first Hardy-Hilbert-Mulholland-type integral inequality, followed by its detailed proof.

Theorem 1. *Let $p > 1$, $q = p/(p-1)$ and $\alpha \in (1/2, 1)$. Let $f, g : \mathbb{R}_+^* \to \mathbb{R}_+$ be two functions such that*

$$\int_{\mathbb{R}_+^*} x^{p/2-1} f^p(x)dx < +\infty, \quad \int_{\mathbb{R}_+^*} y^{q/2-1} g^q(y)dy < +\infty.$$

Then we have

$$\int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} f(x)g(y)dxdy$$
$$\leq 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\left[\int_{\mathbb{R}_+^*} x^{p/2-1} f^p(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+^*} y^{q/2-1} g^q(y)dy\right]^{1/q}.$$

Proof. Let us begin by observing that, for any $\alpha \in (1/2, 1)$ and $x, y \in \mathbb{R}_+^*$, we have

$$K_\alpha^{(1)}(x,y) = \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} \geq 0,$$

due to the symmetry and monotonicity properties of the involved expressions. Specifically, distinguishing the cases $x/y \geq 1$ and $x/y \in (0,1)$, we verify that the numerator and denominator have the same sign, giving this result. Therefore, since f and g are assumed to be non-negative functions, we have

$$\int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} f(x)g(y)dxdy \geq 0.$$

Moreover, since the integrand is non-negative, it may be raised to any non-negative exponent without affecting the sign of the integral. In light of this structure, we now proceed to decompose the integrand in an appropriate way. Using the equality $1/p + 1/q = 1$, and applying the Hölder integral inequality at the exponent p, we obtain

$$\int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} f(x)g(y)dxdy$$
$$= \int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*} \left\{ x^{1/(2q)} y^{-1/(2p)} \left[\frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)}\right]^{1/p} f(x)\right\}$$
$$\times \left\{ x^{-1/(2q)} y^{1/(2p)} \left[\frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)}\right]^{1/q} g(y)\right\} dxdy$$
$$\leq \mathcal{A}^{1/p}\mathcal{B}^{1/q}, \tag{1}$$

where

$$\mathcal{A} = \int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*} x^{p/(2q)} \frac{1}{\sqrt{y}} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} f^p(x)dxdy,$$

and

$$\mathcal{B} = \int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*} y^{q/(2p)} \frac{1}{\sqrt{x}} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} g^q(y)dxdy.$$

The technical Lemma below is required to define the terms $\mathcal{A}$ and $\mathcal{B}$. It is an integral formula extracted from [16].

Lemma 1. [16, Formula 4.267.10] *Let $\alpha \in (0,1)$. Then we have*

$$\int_{\mathbb{R}_+^*} \frac{x^{\alpha-1} - x^{-\alpha}}{(1+x)\log(x)} dx = 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right].$$

We will use this lemma for $\alpha \in (1/2, 1)$.

Expression for $\mathcal{A}$. Using the Fubini-Tonelli integral theorem to exchange the integral symbols (which is possible because the integrand is non-negative), applying the change of variables $u = y/x$, and using Lemma 1 and the equality $p/(2q) = (p-1)/2$, we obtain

$$\begin{aligned}
\mathcal{A} &= \int_{\mathbb{R}_+^*} x^{p/(2q)} f^p(x) \left[\int_{\mathbb{R}_+^*} \frac{1}{\sqrt{y}} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} dy\right] dx \\
&= \int_{\mathbb{R}_+^*} x^{p/(2q)-1/2} f^p(x) \left[\int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha} - (y/x)^{\alpha-1}}{(x+y)\log(x/y)} dy\right] dx \\
&= \int_{\mathbb{R}_+^*} x^{p/(2q)-1/2} f^p(x) \left[\int_{\mathbb{R}_+^*} \frac{(y/x)^{\alpha-1} - (x/y)^{\alpha}}{(1+y/x)\log(y/x)} \frac{1}{x} dy\right] dx \\
&= \int_{\mathbb{R}_+^*} x^{p/(2q)-1/2} f^p(x) \left[\int_{\mathbb{R}_+^*} \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} du\right] dx \\
&= \int_{\mathbb{R}_+^*} x^{p/(2q)-1/2} f^p(x) \left\{2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\right\} dx \\
&= 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right] \int_{\mathbb{R}_+^*} x^{p/2-1} f^p(x) dx. \qquad (2)
\end{aligned}$$

Expression for $\mathcal{B}$. We proceed as before, but with some crucial differences in terms of parameterization. Using the Fubini-Tonelli integral theorem to exchange the integral symbols (which is possible because the integrand is non-negative), applying the change of variables $v = x/y$, and using Lemma 1 and the equality $q/(2p) = (q-1)/2$, we obtain

$$\begin{aligned}
\mathcal{B} &= \int_{\mathbb{R}_+^*} y^{q/(2p)} g^q(y) \left[\int_{\mathbb{R}_+^*} \frac{1}{\sqrt{x}} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} dx\right] dy \\
&= \int_{\mathbb{R}_+^*} y^{q/(2p)-1/2} g^q(y) \left[\int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1} - (y/x)^{\alpha}}{(x+y)\log(x/y)} dx\right] dy \\
&= \int_{\mathbb{R}_+^*} y^{q/(2p)-1/2} g^q(y) \left[\int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1} - (y/x)^{\alpha}}{(1+x/y)\log(x/y)} \frac{1}{y} dx\right] dy \\
&= \int_{\mathbb{R}_+^*} y^{q/(2p)-1/2} g^q(y) \left[\int_{\mathbb{R}_+^*} \frac{v^{\alpha-1} - v^{-\alpha}}{(1+v)\log(v)} dv\right] dy \\
&= \int_{\mathbb{R}_+^*} y^{q/(2p)-1/2} g^q(y) \left\{2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\right\} dy \\
&= 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right] \int_{\mathbb{R}_+^*} y^{q/2-1} g^q(y) dy. \qquad (3)
\end{aligned}$$

Combining Eqs. (1), (2) and (3), and simplifying via the equality $1/p + 1/q = 1$, we get

$$\begin{aligned}
&\int_{\mathbb{R}_+^*} \int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} f(x)g(y) dx dy \\
&\le \left\{2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right] \int_{\mathbb{R}_+^*} x^{p/2-1} f^p(x) dx\right\}^{1/p} \left\{2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right] \int_{\mathbb{R}_+^*} y^{q/2-1} g^q(y) dy\right\}^{1/q} \\
&= 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right] \left[\int_{\mathbb{R}_+^*} x^{p/2-1} f^p(x) dx\right]^{1/p} \left[\int_{\mathbb{R}_+^*} y^{q/2-1} g^q(y) dy\right]^{1/q}.
\end{aligned}$$

This concludes the proof of Theorem 1. □

To the best of the knowledge of the author, this is the first theorem of its kind to consider a Hardy-Hilbert-Mulholland-type integral inequality. The presence of the parameter α adds a certain degree of flexibility that can be exploited in various mathematical scenarios.

The remainder of this section is devoted to secondary results derived from Theorem 1.

2.2. Secondary results

The proposition below ensures the optimality of the constant factor in Theorem 1.

Proposition 1. *In the context of Theorem 1, the constant factor*

$$2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right],$$

is optimal.

Proof. Our approach is by way of contradiction. We assume that there is a constant

$$v \in \left(0, 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\right)$$

such that, for any $f, g : \mathbb{R}_+^* \to \mathbb{R}_+$ satisfying

$$\int_{\mathbb{R}_+^*} x^{p/2-1} f^p(x)dx < +\infty, \quad \int_{\mathbb{R}_+^*} y^{q/2-1} g^q(y)dy < +\infty,$$

we have

$$\int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} f(x)g(y)dxdy \leq v\left[\int_{\mathbb{R}_+^*} x^{p/2-1} f^p(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+^*} y^{q/2-1} g^q(y)dy\right]^{1/q}. \quad (4)$$

In this way, v is a better constant factor. For any $n \in \mathbb{N}\backslash\{0\}$, we consider $f_n, g_n : \mathbb{R}_+^* \to \mathbb{R}_+$, as follows:

$$f_n(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ x^{-1/2-1/(np)} & \text{if } x \in [1,+\infty), \end{cases} \quad g_n(y) = \begin{cases} 0 & \text{if } y \in [0,1), \\ y^{-1/2-1/(nq)} & \text{if } y \in [1,+\infty). \end{cases}$$

We then calculate

$$\int_{\mathbb{R}_+^*} x^{p/2-1} f_n^p(x)dx = \int_{[1,+\infty)} x^{p/2-1}\left[x^{-1/2-1/(np)}\right]^p dx = \int_{[1,+\infty)} x^{-1/n-1}dx = \left[-nx^{-1/n}\right]_{x=1}^{x\to+\infty} = n,$$

and

$$\int_{\mathbb{R}_+^*} y^{q/2-1} g_n^q(y)dy = \int_{[1,+\infty)} y^{q/2-1}\left[y^{-1/2-1/(nq)}\right]^q dy = \int_{[1,+\infty)} y^{-1/n-1}dy = \left[-ny^{-1/n}\right]_{y=1}^{y\to+\infty} = n.$$

These formulas combined with the equality $1/p + 1/q = 1$ and Eq. (4) ensure that

$$\begin{aligned} v &= v\frac{1}{n}n^{1/p}n^{1/q} = \frac{1}{n}\left\{v\left[\int_{\mathbb{R}_+^*} x^{p/2-1} f_n^p(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+^*} y^{q/2-1} g_n^q(y)dy\right]^{1/q}\right\} \\ &\geq \frac{1}{n}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} f_n(x)g_n(y)dxdy. \end{aligned} \quad (5)$$

Let us now determine this double integral. Using the expressions of f_n and g_n, performing the change of variables $x = uy$, applying the Fubini-Tonelli integral theorem (which is possible because the integrand is non-negative) and using the equality $1/p + 1/q = 1$, we obtain

$$\int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} f_n(x)g_n(y)dxdy$$

$$\begin{aligned}
&= \int_{[1,+\infty)} \int_{[1,+\infty)} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} x^{-1/2-1/(np)} y^{-1/2-1/(nq)} dxdy \\
&= \int_{[1,+\infty)} \left[\int_{[1,+\infty)} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} \frac{1}{\sqrt{xy}} x^{-1/(np)} dx \right] y^{-1/(nq)} dy \\
&= \int_{[1,+\infty)} \left[\int_{(1/y,+\infty)} \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} \frac{1}{y^2} u^{-1/(np)} y^{-1/(np)} (ydu) \right] y^{-1/(nq)} dy \\
&= \int_{[1,+\infty)} \left[\int_{(1/y,+\infty)} \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} u^{-1/(np)} du \right] y^{-(1+1/n)} dy. \qquad (6)
\end{aligned}$$

Using the Chasles integral relation with the threshold value $u = 1$, the Fubini-Tonelli integral theorem (which is possible because the integrand is non-negative), simple integral calculus and the equality $1/p + 1/q = 1$, we get

$$\begin{aligned}
&\int_{[1,+\infty)} \left[\int_{(1/y,+\infty)} \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} u^{-1/(np)} du \right] y^{-(1+1/n)} dy \\
&= \int_{[1,+\infty)} \left[\int_{(1/y,1)} \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} u^{-1/(np)} du \right] y^{-(1+1/n)} dy \\
&\quad + \int_{[1,+\infty)} \left[\int_{[1,+\infty)} \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} u^{-1/(np)} du \right] y^{-(1+1/n)} dy \\
&= \int_{(0,1)} \left[\int_{(1/u,+\infty)} y^{-(1+1/n)} dy \right] \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} u^{-1/(np)} du \\
&\quad + \left[\int_{[1,+\infty)} \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} u^{-1/(np)} du \right] \left[\int_{[1,+\infty)} y^{-(1+1/n)} dy \right] \\
&= \int_{(0,1)} (nu^{1/n}) \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} u^{-1/(np)} du + n \left[\int_{[1,+\infty)} \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} u^{-1/(np)} du \right] \\
&= n \left[\int_{(0,1)} \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} u^{1/(nq)} du + \int_{[1,+\infty)} \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} u^{-1/(np)} du \right]. \qquad (7)
\end{aligned}$$

Combining Eqs. (5), (6) and (7), we have

$$v \geq \int_{(0,1)} \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} u^{1/(nq)} du + \int_{[1,+\infty)} \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} u^{-1/(np)} du.$$

Considering the inferior limit with respect to n denoted $\liminf_{n\to+\infty}$, the Fatou integral lemma, $\liminf_{n\to+\infty} u^{1/(nq)} = 1$ for $u \in (0,1)$, $\liminf_{n\to+\infty} u^{-1/(np)} = 1$ for $u \in [1,+\infty)$, the Chasles integral relation and Lemma 1, we obtain

$$\begin{aligned}
v &\geq \liminf_{n\to+\infty} \int_{(0,1)} \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} u^{1/(nq)} du + \liminf_{n\to+\infty} \int_{[1,+\infty)} \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} u^{-1/(np)} du \\
&\geq \int_{(0,1)} \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} \left[\liminf_{n\to+\infty} u^{1/(nq)} \right] du + \int_{[1,+\infty)} \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} \left[\liminf_{n\to+\infty} u^{-1/(np)} \right] du \\
&= \int_{(0,1)} \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} du + \int_{[1,+\infty)} \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} du = \int_{\mathbb{R}_+^*} \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} du \\
&= 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right].
\end{aligned}$$

A contradiction is with the assumption

$$v \in \left(0, 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\right).$$

Consequently, the constant factor

$$2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right],$$

is optimal. This ends the proof of Proposition 1. □

The result below is an integral inequality derived from Theorem 1 which has the feature of depending on a single function.

Proposition 2. *Let $p > 1$, $q = p/(p-1)$ and $\alpha \in (1/2, 1)$. Let $f : \mathbb{R}_+^* \to \mathbb{R}_+$ be a function such that*

$$\int_{\mathbb{R}_+^*} x^{p/2-1} f^p(x)dx < +\infty.$$

Then we have

$$\int_{\mathbb{R}_+^*} y^{-(q/2-1)(p-1)} \left[\int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} f(x)dx \right]^p dy \le 2^p \log^p \left[\tan\left(\frac{\alpha\pi}{2}\right) \right] \int_{\mathbb{R}_+^*} x^{p/2-1} f^p(x)dx.$$

Proof. For convenience, let us set

$$\mathcal{C} = \int_{\mathbb{R}_+^*} y^{-(q/2-1)(p-1)} \left[\int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} f(x)dx \right]^p dy.$$

Then a simple decomposition gives

$$\begin{aligned} \mathcal{C} &= \int_{\mathbb{R}_+^*} \left[\int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} f(x)dx \right] \times \\ &\quad y^{-(q/2-1)(p-1)} \left[\int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} f(x)dx \right]^{p-1} dy \\ &= \int_{\mathbb{R}_+^*} \int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} f(x) g_\dagger(y) dx dy, \end{aligned} \tag{8}$$

where

$$g_\dagger(y) = \left[y^{-(q/2-1)} \int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} f(x)dx \right]^{p-1}.$$

Applying Theorem 1 to the functions f and $g_\dagger$, we directly obtain

$$\begin{aligned} &\int_{\mathbb{R}_+^*} \int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} f(x) g_\dagger(y) dx dy \\ &\quad \le 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right) \right] \left[\int_{\mathbb{R}_+^*} x^{p/2-1} f^p(x)dx \right]^{1/p} \left[\int_{\mathbb{R}_+^*} y^{q/2-1} g_\dagger^q(y)dy \right]^{1/q}. \end{aligned} \tag{9}$$

Let us now examine the last integral. Using $q(p-1) = p$, we have

$$\begin{aligned} \int_{\mathbb{R}_+^*} y^{q/2-1} g_\dagger^q(y)dy &= \int_{\mathbb{R}_+^*} y^{q/2-1} \left[y^{-(q/2-1)} \int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} f(x)dx \right]^{q(p-1)} dy \\ &= \int_{\mathbb{R}_+^*} y^{q/2-1} \left[y^{-(q/2-1)} \int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} f(x)dx \right]^{p} dy \\ &= \int_{\mathbb{R}_+^*} y^{-(q/2-1)(p-1)} \left[\int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} f(x)dx \right]^{p} dy \\ &= \mathcal{C}. \end{aligned} \tag{10}$$

Combining Eqs. (8), (9) and (10), we get

$$\mathcal{C} \leq 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\left[\int_{\mathbb{R}_+^*} x^{p/2-1} f^p(x)dx\right]^{1/p} \mathcal{C}^{1/q}.$$

This and the equality $1 - 1/q = 1/p$ give

$$\mathcal{C}^{1/p} \leq 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\left[\int_{\mathbb{R}_+^*} x^{p/2-1} f^p(x)dx\right]^{1/p},$$

so, by the definition of $\mathcal{C}$,

$$\int_{\mathbb{R}_+^*} y^{-(q/2-1)(p-1)}\left[\int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} f(x)dx\right]^p dy \leq 2^p \log^p\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\int_{\mathbb{R}_+^*} x^{p/2-1} f^p(x)dx.$$

This ends the proof of Proposition 2. □

Typically, this proposition is used to study the continuity of new types of integral operator depending on a single function.

The result below is an adaptation of Theorem 1, which deals with the primitives of f and g. The proof is based on the classical Hardy integral inequality.

Proposition 3. *Let $p > 1$, $q = p/(p-1)$ and $\alpha \in (1/2, 1)$. Let $f, g : \mathbb{R}_+^* \to \mathbb{R}_+$ be two functions such that*

$$\int_{\mathbb{R}_+^*} f^p(x)dx < +\infty, \quad \int_{\mathbb{R}_+^*} g^q(y)dy < +\infty,$$

and $F, G : \mathbb{R}_+^* \to \mathbb{R}_+$ be their respective primitives, i.e., for any $x, y \in \mathbb{R}_+^*$,

$$F(x) = \int_{(0,x)} f(t)dt, \quad G(y) = \int_{(0,y)} g(t)dt.$$

Then we have

$$\int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} x^{1/p-3/2} y^{1/q-3/2} F(x)G(y)dxdy$$
$$\leq 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\frac{p^2}{p-1}\left[\int_{\mathbb{R}_+^*} f^p(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+^*} g^q(y)dy\right]^{1/q}.$$

Proof. By identification, we can write

$$\int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} x^{1/p-3/2} y^{1/q-3/2} F(x)G(y)dxdy$$
$$= \int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} f_\diamond(x)g_\diamond(y)dxdy, \tag{11}$$

where

$$f_\diamond(x) = x^{1/p-3/2}F(x), \quad g_\diamond(y) = y^{1/q-3/2}G(y).$$

Applying Theorem 1 to $f_\diamond$ and $g_\diamond$, we get

$$\int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2} - (y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} f_\diamond(x)g_\diamond(y)dxdy$$
$$\leq 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\left[\int_{\mathbb{R}_+^*} x^{p/2-1} f_\diamond^p(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+^*} y^{q/2-1} g_\diamond^q(y)dy\right]^{1/q}. \tag{12}$$

Let us now express the two integrals of this upper bound. Using the definition of $f_\diamond$ and the Hardy integral inequality, we obtain

$$\int_{\mathbb{R}_+^*} x^{p/2-1} f_\diamond^p(x)dx = \int_{\mathbb{R}_+^*} x^{p/2-1}\left[x^{1/p-3/2}F(x)\right]^p dx = \int_{\mathbb{R}_+^*} \frac{1}{x^p}F^p(x)dx \le \left(\frac{p}{p-1}\right)^p \int_{\mathbb{R}_+^*} f^p(x)dx. \tag{13}$$

Similarly, using the definition of $g_\diamond$ and $p = q/(q-1)$, we have

$$\int_{\mathbb{R}_+^*} y^{q/2-1} g_\diamond^q(y)dy = \int_{\mathbb{R}_+^*} y^{q/2-1}\left[y^{1/q-3/2}G(y)\right]^q dy = \int_{\mathbb{R}_+^*} \frac{1}{y^q}G^p(y)dy$$
$$\le \left(\frac{q}{q-1}\right)^q \int_{\mathbb{R}_+^*} g^q(y)dy = p^q \int_{\mathbb{R}_+^*} g^q(y)dy. \tag{14}$$

Combining Eqs. (11), (12), (13) and (14), we get

$$\int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*} \frac{(x/y)^{\alpha-1/2}-(y/x)^{\alpha-1/2}}{(x+y)\log(x/y)} x^{1/p-3/2}y^{1/q-3/2}F(x)G(y)dxdy$$
$$\le 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\left[\left(\frac{p}{p-1}\right)^p\int_{\mathbb{R}_+^*} f^p(x)dx\right]^{1/p}\left[p^q\int_{\mathbb{R}_+^*} g^q(y)dy\right]^{1/q}$$
$$= 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\frac{p^2}{p-1}\left[\int_{\mathbb{R}_+^*} f^p(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+^*} g^q(y)dy\right]^{1/q}.$$

This concludes the proof of Proposition 3. □

The upper bound therefore depends on the unweighted integral norms of f and g, and the constant factor is set as precisely as possible, even though optimality is not demonstrated here.

3. Second Hardy-Hilbert-Mulholland-type integral inequality

3.1. Main result

The second Hardy-Hilbert-Mulholland-type integral inequality is presented in the theorem below, followed by its proof.

Theorem 2. *Let $p > 1$, $q = p/(p-1)$ and $\alpha \in (1/2, 1)$. Let $f, g : \mathbb{R}_+^* \to \mathbb{R}_+$ be two functions such that*

$$\int_{\mathbb{R}_+^*} x^{p/2-1} f^p(x)dx < +\infty, \quad \int_{\mathbb{R}_+^*} y^{q/2-1} g^q(y)dy < +\infty.$$

Then we have

$$\int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*} \frac{(xy)^{\alpha-1/2}-(xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} f(x)g(y)dxdy$$
$$\le 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\left[\int_{\mathbb{R}_+^*} x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+^*} y^{q/2-1}g^q(y)dy\right]^{1/q}.$$

Proof. With the same arguments to the proof of Theorem 1, for any $\alpha \in (1/2, 1)$ and $x, y \in \mathbb{R}_+^*$, we have

$$K_\alpha^{(2)}(x,y) = \frac{(xy)^{\alpha-1/2}-(xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} \ge 0,$$

implying that

$$\int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*} \frac{(xy)^{\alpha-1/2}-(xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} f(x)g(y)dxdy \ge 0.$$

Furthermore, thanks to the non-negativity of its main component functions, the integrand can be easily manipulated. By decomposing the integrand using the equality $1/p + 1/q = 1$, and applying the Hölder integral inequality at the exponent p, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*} \frac{(xy)^{\alpha-1/2}-(xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} f(x)g(y)dxdy \\
&= \int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*} \left\{ x^{1/(2q)}y^{-1/(2p)} \left[\frac{(xy)^{\alpha-1/2}-(xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)}\right]^{1/p} f(x)\right\} \\
&\quad \times \left\{ x^{-1/(2q)}y^{1/(2p)} \left[\frac{(xy)^{\alpha-1/2}-(xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)}\right]^{1/q} g(y)\right\} dxdy \le \mathcal{D}^{1/p}\mathcal{E}^{1/q},
\end{aligned} \tag{15}$$

where

$$\mathcal{D} = \int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*} x^{p/(2q)} \frac{1}{\sqrt{y}} \frac{(xy)^{\alpha-1/2}-(xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} f^p(x)dxdy,$$

and

$$\mathcal{E} = \int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*} y^{q/(2p)} \frac{1}{\sqrt{x}} \frac{(xy)^{\alpha-1/2}-(xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} g^q(y)dxdy.$$

Let us now express $\mathcal{D}$ and $\mathcal{E}$.

Expression for $\mathcal{D}$. Using the Fubini-Tonelli integral theorem to exchange the integral symbols (which is possible because the integrand is non-negative), making the change of variables $u = xy$ (with respect to y), and using Lemma 1 and the equality $p/(2q) = (p-1)/2$, we have

$$\begin{aligned}
\mathcal{D} &= \int_{\mathbb{R}_+^*} x^{p/(2q)} f^p(x) \left[\int_{\mathbb{R}_+^*} \frac{1}{\sqrt{y}} \frac{(xy)^{\alpha-1/2}-(xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} dy\right] dx \\
&= \int_{\mathbb{R}_+^*} x^{p/(2q)-1/2} f^p(x) \left[\int_{\mathbb{R}_+^*} \frac{(xy)^{\alpha-1}-(xy)^{-\alpha}}{(1+xy)\log(xy)} xdy\right] dx \\
&= \int_{\mathbb{R}_+^*} x^{p/(2q)-1/2} f^p(x) \left[\int_{\mathbb{R}_+^*} \frac{u^{\alpha-1}-u^{-\alpha}}{(1+u)\log(u)} du\right] dx \\
&= \int_{\mathbb{R}_+^*} x^{p/(2q)-1/2} f^p(x) \left\{2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\right\} dx \\
&= 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right] \int_{\mathbb{R}_+^*} x^{p/2-1} f^p(x)dx.
\end{aligned} \tag{16}$$

Expression for $\mathcal{E}$. We proceed as before, but with some crucial nuances. Using the Fubini-Tonelli integral theorem to exchange the integral symbols (which is possible because the integrand is non-negative), performing the change of variables $v = xy$ (with respect to x), and using Lemma 1, and the equality $q/(2p) = (q-1)/2$, we obtain

$$\begin{aligned}
\mathcal{E} &= \int_{\mathbb{R}_+^*} y^{q/(2p)} g^q(y) \left[\int_{\mathbb{R}_+^*} \frac{1}{\sqrt{x}} \frac{(xy)^{\alpha-1/2}-(xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} dx\right] dy \\
&= \int_{\mathbb{R}_+^*} y^{q/(2p)-1/2} g^q(y) \left[\int_{\mathbb{R}_+^*} \frac{(xy)^{\alpha-1}-(xy)^{-\alpha}}{(1+xy)\log(xy)} ydx\right] dy \\
&= \int_{\mathbb{R}_+^*} y^{q/(2p)-1/2} g^q(y) \left[\int_{\mathbb{R}_+^*} \frac{v^{\alpha-1}-v^{-\alpha}}{(1+v)\log(v)} dv\right] dy \\
&= \int_{\mathbb{R}_+^*} y^{q/(2p)-1/2} g^q(y) \left\{2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\right\} dy \\
&= 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right] \int_{\mathbb{R}_+^*} y^{q/2-1} g^q(y)dy.
\end{aligned} \tag{17}$$

Combining Eqs. (15), (16) and (17), and simplifying via the equality $1/p + 1/q = 1$, we get

$$\begin{aligned}&\int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*}\frac{(xy)^{\alpha-1/2}-(xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)}f(x)g(y)dxdy\\&\leq\left\{2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\int_{\mathbb{R}_+^*}x^{p/2-1}f^p(x)dx\right\}^{1/p}\left\{2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\int_{\mathbb{R}_+^*}y^{q/2-1}g^q(y)dy\right\}^{1/q}\\&=2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\left[\int_{\mathbb{R}_+^*}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+^*}y^{q/2-1}g^q(y)dy\right]^{1/q}.\end{aligned}$$

This concludes the proof of Theorem 2. □

This theorem complements Theorem 1, providing a new Hardy-Hilbert-Mulholland-type integral inequality. In a sense, the sum of the variables is replaced by their product.

The remainder of this section is devoted to secondary results derived from Theorem 2.

3.2. Secondary result

The proposition below ensures the optimality of the constant factor in Theorem 2.

Proposition 4. *In the context of Theorem 2, the constant factor*

$$2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right].$$

is optimal.

Proof. As in the proof of Proposition 1, we proceed by way of contradiction. We assume that there is a constant

$$\omega\in\left(0,2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\right),$$

such that, for any $f,g:\mathbb{R}_+^*\to\mathbb{R}_+$ satisfying

$$\int_{\mathbb{R}_+^*}x^{p/2-1}f^p(x)dx<+\infty,\quad\int_{\mathbb{R}_+^*}y^{q/2-1}g^q(y)dy<+\infty,$$

we have

$$\int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*}\frac{(xy)^{\alpha-1/2}-(xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)}f(x)g(y)dxdy\leq\omega\left[\int_{\mathbb{R}_+^*}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+^*}y^{q/2-1}g^q(y)dy\right]^{1/q}.\quad(18)$$

In this way, ω is a better constant factor. For any $n\in\mathbb{N}\backslash\{0\}$, we consider $f_n,g_n:\mathbb{R}_+^*\to\mathbb{R}_+$ defined by

$$f_n(x)=\begin{cases}x^{-1/2+1/(np)} & \text{if } x\in(0,1),\\0 & \text{if } x\in[1,+\infty),\end{cases}\qquad g_n(y)=\begin{cases}0 & \text{if } y\in[0,1),\\y^{-1/2-1/(nq)} & \text{if } y\in[1,+\infty).\end{cases}$$

We then calculate

$$\int_{\mathbb{R}_+^*}x^{p/2-1}f_n^p(x)dx=\int_{(0,1)}x^{p/2-1}\left[x^{-1/2+1/(np)}\right]^p dx=\int_{(0,1)}x^{1/n-1}dx=\left[nx^{1/n}\right]_{x=0}^{x=1}=n,$$

and

$$\int_{\mathbb{R}_+^*}y^{q/2-1}g_n^q(y)dy=\int_{[1,+\infty)}y^{q/2-1}\left[y^{-1/2-1/(nq)}\right]^q dy=\int_{[1,+\infty)}y^{-1/n-1}dy=\left[-ny^{-1/n}\right]_{y=1}^{y\to+\infty}=n.$$

These formulas combined with the equality $1/p+1/q=1$ and Eq. (18) give

$$\begin{aligned}\omega &= \omega\frac{1}{n}n^{1/p}n^{1/q} = \frac{1}{n}\left\{\omega\left[\int_{\mathbb{R}_+^*} x^{p/2-1}f_n^p(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+^*} y^{q/2-1}g_n^q(y)dy\right]^{1/q}\right\}\\ &\geq \frac{1}{n}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*}\frac{(xy)^{\alpha-1/2}-(xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)}f_n(x)g_n(y)dxdy. \end{aligned} \tag{19}$$

Now, let us work on this double integral. Using the definitions of f_n and g_n, performing the change of variables $x=u/y$, applying the Fubini-Tonelli integral theorem (which is possible because the integrand is non-negative) and using the equality $1/p+1/q=1$, we get

$$\begin{aligned}&\int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*}\frac{(xy)^{\alpha-1/2}-(xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)}f_n(x)g_n(y)dxdy\\ &\quad= \int_{[1,+\infty)}\int_{(0,1)}\frac{(xy)^{\alpha-1/2}-(xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)}x^{-1/2+1/(np)}y^{-1/2-1/(nq)}dxdy\\ &\quad= \int_{[1,+\infty)}\left[\int_{(0,1)}\frac{(xy)^{\alpha-1/2}-(xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)}\frac{1}{\sqrt{xy}}x^{1/(np)}dx\right]y^{-1/(nq)}dy\\ &\quad= \int_{[1,+\infty)}\left[\int_{(0,y)}\frac{u^{\alpha-1}-u^{-\alpha}}{(1+u)\log(u)}u^{1/(np)}y^{-1/(np)}\frac{1}{y}du\right]y^{-1/(nq)}dy\\ &\quad= \int_{[1,+\infty)}\left[\int_{(0,y)}\frac{u^{\alpha-1}-u^{-\alpha}}{(1+u)\log(u)}u^{1/(np)}du\right]y^{-(1+1/n)}dy. \end{aligned} \tag{20}$$

Using the Chasles integral relation with the threshold value $u=1$, the Fubini-Tonelli integral theorem (which is possible because the integrand is non-negative) and the equality $1/p+1/q=1$, we obtain

$$\begin{aligned}&\int_{[1,+\infty)}\left[\int_{(0,y)}\frac{u^{\alpha-1}-u^{-\alpha}}{(1+u)\log(u)}u^{1/(np)}du\right]y^{-(1+1/n)}dy\\ &\quad= \int_{[1,+\infty)}\left[\int_{(0,1)}\frac{u^{\alpha-1}-u^{-\alpha}}{(1+u)\log(u)}u^{1/(np)}du\right]y^{-(1+1/n)}dy\\ &\qquad+\int_{[1,+\infty)}\left[\int_{(1,y)}\frac{u^{\alpha-1}-u^{-\alpha}}{(1+u)\log(u)}u^{1/(np)}du\right]y^{-(1+1/n)}dy\\ &\quad= \left[\int_{(0,1)}\frac{u^{\alpha-1}-u^{-\alpha}}{(1+u)\log(u)}u^{1/(np)}du\right]\left[\int_{[1,+\infty)}y^{-(1+1/n)}dy\right]\\ &\qquad+\int_{[1,+\infty)}\left[\int_{(u,+\infty)}y^{-(1+1/n)}dy\right]\frac{u^{\alpha-1}-u^{-\alpha}}{(1+u)\log(u)}u^{1/(np)}du\\ &\quad= n\left[\int_{(0,1)}\frac{u^{\alpha-1}-u^{-\alpha}}{(1+u)\log(u)}u^{1/(np)}du\right]+\int_{[1,+\infty)}(nu^{-1/n})\frac{u^{\alpha-1}-u^{-\alpha}}{(1+u)\log(u)}u^{1/(np)}du\\ &\quad= n\left[\int_{(0,1)}\frac{u^{\alpha-1}-u^{-\alpha}}{(1+u)\log(u)}u^{1/(np)}du+\int_{[1,+\infty)}\frac{u^{\alpha-1}-u^{-\alpha}}{(1+u)\log(u)}u^{-1/(nq)}du\right]. \end{aligned} \tag{21}$$

Combining Eqs. (19), (20) and (21), we get

$$\omega \geq \int_{(0,1)}\frac{u^{\alpha-1}-u^{-\alpha}}{(1+u)\log(u)}u^{1/(np)}du+\int_{[1,+\infty)}\frac{u^{\alpha-1}-u^{-\alpha}}{(1+u)\log(u)}u^{-1/(nq)}du.$$

Considering the inferior limit with respect to n, the Fatou integral lemma, $\liminf_{n\to+\infty} u^{1/(np)}=1$ for $u\in(0,1)$, $\liminf_{n\to+\infty} u^{-1/(nq)}=1$ for $u\in[1,+\infty)$, the Chasles integral relation and Lemma 1, we obtain

$$\begin{aligned}\omega &\geq \liminf_{n\to+\infty}\int_{(0,1)}\frac{u^{\alpha-1}-u^{-\alpha}}{(1+u)\log(u)}u^{1/(np)}du+\liminf_{n\to+\infty}\int_{[1,+\infty)}\frac{u^{\alpha-1}-u^{-\alpha}}{(1+u)\log(u)}u^{-1/(nq)}du\\ &\geq \int_{(0,1)}\frac{u^{\alpha-1}-u^{-\alpha}}{(1+u)\log(u)}\left[\liminf_{n\to+\infty}u^{1/(np)}\right]du+\int_{[1,+\infty)}\frac{u^{\alpha-1}-u^{-\alpha}}{(1+u)\log(u)}\left[\liminf_{n\to+\infty}u^{-1/(nq)}\right]du\end{aligned}$$

$$= \int_{(0,1)} \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} du + \int_{[1,+\infty)} \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} du$$
$$= \int_{\mathbb{R}_+^*} \frac{u^{\alpha-1} - u^{-\alpha}}{(1+u)\log(u)} du = 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right].$$

A contradiction appears with the assumption

$$\omega \in \left(0, 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\right).$$

Consequently, the constant factor

$$2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right],$$

is optimal. This ends the proof of Proposition 4. □

The result below is a new integral inequality. It is derived from Theorem 2 and depends on a single function.

Proposition 5. *Let $p > 1$, $q = p/(p-1)$ and $\alpha \in (1/2, 1)$. Let $f : \mathbb{R}_+^* \to \mathbb{R}_+$ be a function such that*

$$\int_{\mathbb{R}_+^*} x^{p/2-1} f^p(x) dx < +\infty.$$

Then we have

$$\int_{\mathbb{R}_+^*} y^{-(q/2-1)(p-1)} \left[\int_{\mathbb{R}_+^*} \frac{(xy)^{\alpha-1/2} - (xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} f(x) dx\right]^p dy \le 2^p \log^p\left[\tan\left(\frac{\alpha\pi}{2}\right)\right] \int_{\mathbb{R}_+^*} x^{p/2-1} f^p(x) dx.$$

Proof. To facilitate the development, let us set

$$\mathcal{F} = \int_{\mathbb{R}_+^*} y^{-(q/2-1)(p-1)} \left[\int_{\mathbb{R}_+^*} \frac{(xy)^{\alpha-1/2} - (xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} f(x) dx\right]^p dy.$$

We have

$$\mathcal{F} = \int_{\mathbb{R}_+^*} \left[\int_{\mathbb{R}_+^*} \frac{(xy)^{\alpha-1/2} - (xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} f(x) dx\right] y^{-(q/2-1)(p-1)} \left[\int_{\mathbb{R}_+^*} \frac{(xy)^{\alpha-1/2} - (xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} f(x) dx\right]^{p-1} dy$$
$$= \int_{\mathbb{R}_+^*} \int_{\mathbb{R}_+^*} \frac{(xy)^{\alpha-1/2} - (xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} f(x) g_\ddagger(y) dx dy, \tag{22}$$

where

$$g_\ddagger(y) = \left[y^{-(q/2-1)} \int_{\mathbb{R}_+^*} \frac{(xy)^{\alpha-1/2} - (xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} f(x) dx\right]^{p-1}.$$

Applying Theorem 1 to the functions f and $g_\ddagger$, we get

$$\int_{\mathbb{R}_+^*} \int_{\mathbb{R}_+^*} \frac{(xy)^{\alpha-1/2} - (xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} f(x) g_\ddagger(y) dx dy$$
$$\le 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right] \left[\int_{\mathbb{R}_+^*} x^{p/2-1} f^p(x) dx\right]^{1/p} \left[\int_{\mathbb{R}_+^*} y^{q/2-1} g_\ddagger^q(y) dy\right]^{1/q}. \tag{23}$$

Let us now examine the last integral. Using $q(p-1) = p$, we have

$$\int_{\mathbb{R}_+^*} y^{q/2-1} g_\ddagger^q(y) dy = \int_{\mathbb{R}_+^*} y^{q/2-1} \left[y^{-(q/2-1)} \int_{\mathbb{R}_+^*} \frac{(xy)^{\alpha-1/2} - (xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} f(x) dx\right]^{q(p-1)} dy$$

$$
\begin{aligned}
&= \int_{\mathbb{R}_+^*} y^{q/2-1} \left[y^{-(q/2-1)} \int_{\mathbb{R}_+^*} \frac{(xy)^{\alpha-1/2} - (xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} f(x)dx \right]^p dy \\
&= \int_{\mathbb{R}_+^*} y^{-(q/2-1)(p-1)} \left[\int_{\mathbb{R}_+^*} \frac{(xy)^{\alpha-1/2} - (xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} f(x)dx \right]^p dy \\
&= \mathcal{F}.
\end{aligned} \tag{24}
$$

Combining Eqs. (22), (23) and (24), we obtain

$$
\mathcal{F} \le 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right] \left[\int_{\mathbb{R}_+^*} x^{p/2-1} f^p(x)dx\right]^{1/p} \mathcal{F}^{1/q}.
$$

This and the equality $1 - 1/q = 1/p$ give

$$
\mathcal{F}^{1/p} \le 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right] \left[\int_{\mathbb{R}_+^*} x^{p/2-1} f^p(x)dx\right]^{1/p}.
$$

By the definition of $\mathcal{F}$, we get

$$
\int_{\mathbb{R}_+^*} y^{-(q/2-1)(p-1)} \left[\int_{\mathbb{R}_+^*} \frac{(xy)^{\alpha-1/2} - (xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} f(x)dx \right]^p dy \le 2^p \log^p\left[\tan\left(\frac{\alpha\pi}{2}\right)\right] \int_{\mathbb{R}_+^*} x^{p/2-1} f^p(x)dx.
$$

This concludes the proof of Proposition 5. □

Similar to Proposition 2, this proposition can be used to study the continuity of various types of integral operators depending on a single function. Its original form is primarily the result of the logarithmic factor in the denominator of the integrand.

Following the spirit of Proposition 3, the result below offers a primitive version of Theorem 2.

Proposition 6. *Let $p > 1$, $q = p/(p-1)$ and $\alpha \in (1/2, 1)$. Let $f, g : \mathbb{R}_+^* \to \mathbb{R}_+$ be two functions such that*

$$
\int_{\mathbb{R}_+^*} f^p(x)dx < +\infty, \quad \int_{\mathbb{R}_+^*} g^q(y)dy < +\infty,
$$

and $F, G : \mathbb{R}_+^* \to \mathbb{R}_+$ be their respective primitives, i.e., for any $x, y \in \mathbb{R}_+^*$,

$$
F(x) = \int_{(0,x)} f(t)dt, \quad G(y) = \int_{(0,y)} g(t)dt.
$$

Then we have

$$
\begin{aligned}
&\int_{\mathbb{R}_+^*} \int_{\mathbb{R}_+^*} \frac{(xy)^{\alpha-1/2} - (xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} x^{1/p-3/2} y^{1/q-3/2} F(x)G(y)dxdy \\
&\le 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right] \frac{p^2}{p-1} \left[\int_{\mathbb{R}_+^*} f^p(x)dx\right]^{1/p} \left[\int_{\mathbb{R}_+^*} g^q(y)dy\right]^{1/q}.
\end{aligned}
$$

Proof. Proceeding by identification, we can write

$$
\begin{aligned}
&\int_{\mathbb{R}_+^*} \int_{\mathbb{R}_+^*} \frac{(xy)^{\alpha-1/2} - (xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} x^{1/p-3/2} y^{1/q-3/2} F(x)G(y)dxdy \\
&= \int_{\mathbb{R}_+^*} \int_{\mathbb{R}_+^*} \frac{(xy)^{\alpha-1/2} - (xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)} f_\triangle(x) g_\triangle(y)dxdy,
\end{aligned} \tag{25}
$$

where

$$
f_\triangle(x) = x^{1/p-3/2} F(x), \quad g_\triangle(y) = y^{1/q-3/2} G(y).
$$

Applying Theorem 2 to $f_\triangle$ and $g_\triangle$ yields

$$\int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*}\frac{(xy)^{\alpha-1/2}-(xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)}f_\diamond(x)g_\diamond(y)dxdy \leq 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\left[\int_{\mathbb{R}_+^*}x^{p/2-1}f_\diamond^p(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+^*}y^{q/2-1}g_\diamond^q(y)dy\right]^{1/q}. \quad (26)$$

Let us now express the two integrals of this upper bound. Using the definitions of $f_\triangle$ and $g_\triangle$, and the Hardy integral inequality, we obtain

$$\int_{\mathbb{R}_+^*}x^{p/2-1}f_\triangle^p(x)dx=\int_{\mathbb{R}_+^*}x^{p/2-1}\left[x^{1/p-3/2}F(x)\right]^p dx=\int_{\mathbb{R}_+^*}\frac{1}{x^p}F^p(x)dx\leq\left(\frac{p}{p-1}\right)^p\int_{\mathbb{R}_+^*}f^p(x)dx, \quad (27)$$

and, using $p=q/(q-1)$,

$$\int_{\mathbb{R}_+^*}y^{q/2-1}g_\triangle^q(y)dy=\int_{\mathbb{R}_+^*}y^{q/2-1}\left[y^{1/q-3/2}G(y)\right]^q dy=\int_{\mathbb{R}_+^*}\frac{1}{y^q}G^p(y)dy \leq\left(\frac{q}{q-1}\right)^q\int_{\mathbb{R}_+^*}g^q(y)dy=p^q\int_{\mathbb{R}_+^*}g^q(y)dy. \quad (28)$$

Combining Eqs. (25), (26), (27) and (28), we get

$$\int_{\mathbb{R}_+^*}\int_{\mathbb{R}_+^*}\frac{(xy)^{\alpha-1/2}-(xy)^{-(\alpha-1/2)}}{(1+xy)\log(xy)}x^{1/p-3/2}y^{1/q-3/2}F(x)G(y)dxdy \leq 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\left[\left(\frac{p}{p-1}\right)^p\int_{\mathbb{R}_+^*}f^p(x)dx\right]^{1/p}\left[p^q\int_{\mathbb{R}_+^*}g^q(y)dy\right]^{1/q} = 2\log\left[\tan\left(\frac{\alpha\pi}{2}\right)\right]\frac{p^2}{p-1}\left[\int_{\mathbb{R}_+^*}f^p(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+^*}g^q(y)dy\right]^{1/q}.$$

This completes the proof of Proposition 6. □

As in Proposition 3, the upper bound depends on the unweighted integral norms of f and g, and the constant factor is set as sharp as possible.

4. Conclusion

In conclusion, this article introduces and examines two new Hardy-Hilbert-Mulholland-type integral inequalities, each characterized by the presence of a logarithmic term in the denominator of the integrand and governed by an adjustable parameter. Determining the optimal constant factors adds theoretical depth to the results. Beyond the primary inequalities, several related results enrich the framework of integral inequalities further. Proposed areas for future research include exploring analogous inequalities in discrete settings and higher-dimensional domains, as well as different integrand structures.

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