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Integral inequalities for a class of parameter-dependent weighted integral functionals

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Abstract: This paper studies integral inequalities for a class of parameter-dependent weighted integral functionals involving two non-negative functions. We establish several inequalities describing the behavior of the associated integral functional under various structural assumptions on one of the functions, including monotonicity, convexity, log-convexity, and sub-multiplicativity. These results provide a unified framework that extends and generalizes inequalities obtained previously for certain special functions.

Keywords: parameter-dependent weighted integral functionals, exponential integral function, convexity

MSC: 26D15.

1. Introduction

Integral transforms and parameter-dependent integral functionals play an important role in many areas of mathematical analysis, including functional inequalities, harmonic analysis, and the study of special functions. Such constructions often reveal structural properties of functions by encoding them into classes of integrals depending on a parameter. Studying the behavior of these classes with respect to the parameter can provide insight into monotonicity, convexity, asymptotic behavior, and various functional inequalities. See [1].

For the purposes of this paper, we introduce a class of parameter-dependent weighted integral functionals based on a specific form, which is formally described below. Let $n > 0$, $a, b \geq 0$ with $b > a$, and $f, g : [0, +\infty) \to [0, +\infty)$ be two (non-negative) measurable functions. We define the integral function $\mathcal{T}_n(f, g) : (0, +\infty) \to (0, +\infty)$ by

$$\mathcal{T}_n(f, g)(t) = \int_a^b f(x)^n g(tx) dx, \tag{1}$$

with $t > 0$, provided that the integral converges, which will be assumed throughout the paper whenever it appears.

The function $\mathcal{T}_n(f, g)$ may be viewed as a parameterized integral transform in which the function g is evaluated at a scaled argument while the function f appears with power n as a weight. This structure naturally arises in various analytical contexts such as scaling arguments, convolution-type constructions, and the study of integral inequalities. In particular, the parameter $t > 0$ introduces a dilation in the argument of g, allowing one to investigate how the interaction between the functions f and g changes under scaling. One of most famous example is the Laplace transform of f, defined as in Eq. (1) with $a = 0$, $b = +\infty$, $n = 1$ and $g(x) = e^{-x}$.

The aim of this paper is to establish valuable integral inequalities for functions of the form in Eq. (1) under various assumptions on g, namely when g is non-increasing, or convex, or log-convex, or sub-multiplicative, which are formally described below.

Non-increasing property. We say that g is non-increasing if, for any $x, y \in [0, +\infty)$ with $x \leq y$, we have

$$g(x) \geq g(y).$$

Convex property. We say that g is convex if, for any $x, y \in [0, +\infty)$ and any $\lambda \in [0, 1]$, we have

$$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y).$$

Log-convex property. We say that g is log-convex if $\log(g)$ is convex, that is, for any $x, y \in [0, +\infty)$ and any $\lambda \in [0, 1]$, we have

$$g(\lambda x + (1-\lambda)y) \leq g(x)^{\lambda} g(y)^{1-\lambda}.$$

Sub-multiplicative property. We say that g is sub-multiplicative if, for any $x, y \in [0, +\infty)$, we have

$$g(xy) \leq g(x)g(y).$$

Our inequalities are inspired by, and extend, several previously established results for specific special functions (see [2,4,5]). We bring them together within a unified and more general framework. To the best of our knowledge, such a comprehensive and systematic treatment has not been explored before. The consideration of the sub-multiplicative property in this context is also original.

The rest of the paper is organized as follows: The main results are presented in §2. A conclusion is given in §3.

2. Results

Note that throughout the remainder of the paper, and in the statements of our theorems, it is implicitly assumed that $n > 0$, $a, b \geq 0$ with $b > a$, and that $f, g : [0, +\infty) \to [0, +\infty)$ are nonnegative measurable functions, as required for the definition of $\mathcal{T}_n(f, g)(t)$.

2.1. The monotonicity case

The theorem below presents an inequality involving $\mathcal{T}_n(f, g)$ under a monotonicity assumption on g.

Theorem 1. *Let $n > 0$ and $t_1, t_2 \geq 1$. Let us consider the integral function $\mathcal{T}_n(f, g)$ defined by Eq. (1), assuming that g is non-increasing. Then we have*

$$\mathcal{T}_n(f, g)(t_1 t_2) \leq \mathcal{T}_n(f, g)(\max(t_1, t_2)).$$

Proof. Since g is non-increasing, $t_1, t_2 \geq 1$ and $a \geq 0$, we have $t_1 t_2 x \geq \max(t_1, t_2)x$, which implies that $g(t_1 t_2 x) \leq g(\max(t_1, t_2)x)$, and

$$\begin{aligned}\mathcal{T}_n(f, g)(t_1 t_2) &= \int_a^b f(x)^n g(t_1 t_2 x) dx \leq \int_a^b f(x)^n g(\max(t_1, t_2)x) dx \\ &= \mathcal{T}_n(f, g)(\max(t_1, t_2)).\end{aligned}$$

This completes the proof of the theorem. □

If g is non-decreasing instead of non-increasing, the main inequality is reversed.

In particular, if we take $a > 1$, $f(x) = 1/x$, and $g(x) = e^{-x}$, which is obviously non-increasing, $\mathcal{T}_n(f, g)$ becomes the incomplete exponential integral function E_n, i.e.,

$$E_n(t) = \int_a^b x^n e^{-tx} dx, \tag{2}$$

and we have

$$E_n(t_1 t_2) \leq E_n(\max(t_1, t_2)).$$

This complements the results in [4].

2.2. The convexity case

The theorem below presents an inequality involving $\mathcal{T}_n(f, g)$ under a convexity assumption on g.

Theorem 2. *Let $n > 0$, $t_1, t_2 > 0$ and $\lambda \in [0,1]$. Let us consider the integral function $\mathcal{T}_n(f,g)$ defined by Eq. (1), assuming that g is convex. Then we have*

$$\mathcal{T}_n(f,g)(\lambda t_1 + (1-\lambda)t_2) \leq \lambda \mathcal{T}_n(f,g)(t_1) + (1-\lambda)\mathcal{T}_n(f,g)(t_2).$$

Proof. Since g is convex, we have

$$g((\lambda t_1 + (1-\lambda)t_2)x) = g(\lambda t_1 x + (1-\lambda)t_2 x) \leq \lambda g(t_1 x) + (1-\lambda)g(t_2 x).$$

Therefore, we have

$$\begin{aligned}
\mathcal{T}_n(f,g)(\lambda t_1 + (1-\lambda)t_2) &= \int_a^b f(x)^n g((\lambda t_1 + (1-\lambda)t_2)x)dx \\
&\leq \int_a^b f(x)^n \left(\lambda g(t_1 x) + (1-\lambda)g(t_2 x)\right) dx \\
&= \lambda \int_a^b f(x)^n g(t_1 x)dx + (1-\lambda) \int_a^b f(x)^n g(t_2 x)dx \\
&= \lambda \mathcal{T}_n(f,g)(t_1) + (1-\lambda)\mathcal{T}_n(f,g)(t_2).
\end{aligned}$$

This completes the proof of the theorem. □

In other words, this theorem demonstrates that if g is convex, then $\mathcal{T}_n(f,g)$ is also convex. The well-known convex inequalities can thus be applied on $\mathcal{T}_n(f,g)$. See [3].

In particular, if we take $a > 1$, $f(x) = 1/x$, and $g(x) = e^{-x}$, which is obviously convex, $\mathcal{T}_n(f,g)$ becomes E_n in Eq. (2), and we have

$$E_n(\lambda t_1 + (1-\lambda)t_2) \leq \lambda E_n(t_1) + (1-\lambda)E_n(t_2).$$

This completes the results in [4].

The theorem below presents another inequality involving $\mathcal{T}_n(f,g)$ under a convexity assumption on g.

Theorem 3. *Let $n > 0$, and $t_1, t_2 > 0$. Let us consider the integral function $\mathcal{T}_n(f,g)$ defined by Eq. (1), assuming that g is twice-differentiable,*

$$\int_a^b x f(x)^n \left| g'(t_1 x) - g'\left((t_1 + t_2)x\right)\right| dx, \tag{3}$$

exists with an absolute continuity of the integrand, g is convex and

$$\lim_{t_1 \to \infty} \mathcal{T}_n(f,g)(t_1) = 0. \tag{4}$$

Then we have

$$\mathcal{T}_n(f,g)(t_1 + t_2) \leq \mathcal{T}_n(f,g)(t_1) + \mathcal{T}_n(f,g)(t_2).$$

Proof. For a fixed t_2, let consider the function $\phi : (0, +\infty) \to (0, +\infty)$ defined by

$$\begin{aligned}
\phi(t_1) &= \mathcal{T}_n(f,g)(t_1) + \mathcal{T}_n(f,g)(t_2) - \mathcal{T}_n(f,g)(t_1 + t_2) \\
&= \int_a^b f(x)^n \left(g(t_1 x) + g(t_2 x) - g\left((t_1 + t_2)x\right)\right) dx.
\end{aligned}$$

Using the Leibniz integral rule based on Eq. (3), we have

$$\phi'(t_1) = \int_a^b x f(x)^n \left(g'(t_1 x) - g'\left((t_1 + t_2)x\right)\right) dx.$$

Since g is twice-differentiable convex, g' is non-decreasing. This implies that, for any $t_1, t_2 > 0$ and $x \geq a \geq 0$, $g'(t_1 x) \leq g'((t_1+t_2)x)$. Using this and again $a \geq 0$, we obtain

$$\phi'(t_1) \leq 0.$$

Therefore, ϕ is non-increasing. Using Eq. (4), we get

$$\lim_{t_1 \to \infty} \phi(t_1) = \mathcal{T}_n(f,g)(t_2) \geq 0.$$

Therefore, we have $\phi(t_1) \geq 0$, which gives

$$\mathcal{T}_n(f,g)(t_1+t_2) \leq \mathcal{T}_n(f,g)(t_1) + \mathcal{T}_n(f,g)(t_2).$$

This completes the proof of the theorem. □

In particular, if we take $a > 1$, $f(x) = 1/x$, and $g(x) = e^{-x}$, $\mathcal{T}_n(f,g)$ becomes E_n in Eq. (2), g is convex, Eq. (3) is obviously satisfied and, by the dominated convergence theorem,

$$\lim_{t_1 \to +\infty} \mathcal{T}_n(f,g)(t_1) = \lim_{t_1 \to +\infty} E_n(t_1) = \lim_{t_1 \to +\infty} \int_a^b x^n e^{-t_1 x} dx = \int_a^b x^n \lim_{t_1 \to +\infty} e^{-t_1 x} dx = 0,$$

implying Eq. (4). Therefore, we have

$$E_n(t_1+t_2) \leq E_n(t_1) + E_n(t_2).$$

This complements the results in [4].

2.3. The log-convexity case

The theorem below presents an inequality involving $\mathcal{T}_n(f,g)$ under a log-convexity assumption on g.

Theorem 4. *Let $m, n > 0$, and $t_1, t_2 > 0$. Let $p > 1$ and $q = p/(p-1)$. Let us consider the integral function $\mathcal{T}_n(f,g)$ defined by Eq. (1), assuming that g is log-convex. Then we have*

$$\mathcal{T}_{m+n}(f,g)\left(\frac{t_1}{p} + \frac{t_2}{q}\right) \leq \mathcal{T}_{mp}(f,g)(t_1)^{1/p} \mathcal{T}_{nq}(f,g)(t_2)^{1/q}.$$

Proof. Using the fact that g is log-convex and the Hölder integral inequality applied to the appropriate functions, we get

$$\begin{aligned}
\mathcal{T}_{m+n}(f,g)\left(\frac{t_1}{p} + \frac{t_2}{q}\right) &= \int_a^b f(x)^{m+n} g\left(\left(\frac{t_1}{p} + \frac{t_2}{q}\right)x\right) dx \\
&= \int_a^b f(x)^m f(x)^n g\left(\frac{t_1}{p}x + \frac{t_2}{q}x\right) dx \\
&\leq \int_a^b f(x)^m f(x)^n g(t_1 x)^{1/p} g(t_2 x)^{1/q} dx \\
&\leq \left(\int_a^b f(x)^{mp} g(t_1 x) dx\right)^{1/p} \left(\int_a^b f(x)^{nq} g(t_2 x) dx\right)^{1/q} \\
&= \mathcal{T}_{mp}(f,g)(t_1)^{1/p} \mathcal{T}_{nq}(f,g)(t_2)^{1/q}.
\end{aligned}$$

This completes the proof of the theorem. □

In particular, if we take $p = 2$, then we have

$$\mathcal{T}_{m+n}^2(f,g)\left(\frac{t_1+t_2}{2}\right) \leq \mathcal{T}_{2m}(f,g)(t_1)\mathcal{T}_{2n}(f,g)(t_2).$$

As another remark less general, if we take $a > 1$, $f(x) = 1/x$, and $g(x) = e^{-x}$ which is obviously log-convex, $\mathcal{T}_n(f,g)$ becomes E_n in Eq. (2), and we have

$$E_{m+n}\left(\frac{t_1}{p}+\frac{t_2}{q}\right) \leq E_{mp}(t_1)^{1/p}E_{nq}(t_2)^{1/q}.$$

This corresponds to [4, Theorem 2.2].

A natural generalization of Theorem 4 is given below.

Theorem 5. *Let $r \in \mathbb{N}\backslash\{0\}$, $m_1,\ldots,m_r > 0$, and $t_1,\ldots,t_r > 0$. Let $p_1,\ldots,p_r > 1$ such that $\sum_{i=1}^{r}(1/p_i) = 1$. Let us consider the integral function $\mathcal{T}_n(f,g)$ defined by Eq. (1), assuming that g is log-convex. Then we have*

$$\mathcal{T}_{\sum_{i=1}^{r} m_i}(f,g)\left(\sum_{i=1}^{r}\frac{t_i}{p_i}\right) \leq \prod_{i=1}^{r}\mathcal{T}_{m_ip_i}(f,g)(t_i)^{1/p_i}.$$

Proof. Using the fact that g is log-convex and the generalized Hölder integral inequality applied to the appropriate functions, we get

$$\begin{aligned}
\mathcal{T}_{\sum_{i=1}^{r} m_i}(f,g)\left(\sum_{i=1}^{r}\frac{t_i}{p_i}\right) &= \int_a^b f(x)^{\sum_{i=1}^{r} m_i} g\left(\left(\sum_{i=1}^{r}\frac{t_i}{p_i}\right)x\right)dx \\
&= \int_a^b \prod_{i=1}^{r} f(x)^{m_i} g\left(\sum_{i=1}^{r}\frac{t_ix}{p_i}\right)dx \\
&\leq \int_a^b \prod_{i=1}^{r} f(x)^{m_i} \prod_{i=1}^{r} g(t_ix)^{1/p_i}dx \\
&= \int_a^b \prod_{i=1}^{r}\left(f(x)^{m_i} g(t_ix)^{1/p_i}\right)dx \\
&\leq \prod_{i=1}^{r}\left(\int_a^b f(x)^{m_ip_i} g(t_ix)dx\right)^{1/p_i} \\
&= \prod_{i=1}^{r}\mathcal{T}_{m_ip_i}(f,g)(t_i)^{1/p_i}.
\end{aligned}$$

This completes the proof of the theorem. □

If we take $a > 1$, $f(x) = 1/x$, and $g(x) = e^{-x}$ which is obviously log-convex, $\mathcal{T}_n(f,g)$ becomes E_n in Eq. (2), and we have

$$E_{\sum_{i=1}^{r} m_i}\left(\sum_{i=1}^{r}\frac{t_i}{p_i}\right) \leq \prod_{i=1}^{r} E_{m_ip_i}(t_i)^{1/p_i}.$$

This completes [4, Theorem 2.2].

2.4. The sub-multiplicativity case

The theorem below presents an inequality involving $\mathcal{T}_n(f,g)$ under a sub-multiplicativity assumption on g.

Theorem 6. *Let $n > 0$, and $t_1, t_2 > 0$. Let us consider the integral function $\mathcal{T}_n(f,g)$ defined by Eq. (1), assuming that g is sub-multiplicative. Then we have*

$$\mathcal{T}_n(f,g)(t_1t_2) \leq g(t_1)\mathcal{T}_n(f,g)(t_2),$$

and

$$\mathcal{T}_n(f,g)(t_1t_2) \leq g(t_2)\mathcal{T}_n(f,g)(t_1).$$

Proof. Since g is sub-multiplicative, we have $g(t_1 t_2 x) \leq g(t_1) g(t_2 x)$, which implies that

$$\begin{aligned}\mathcal{T}_n(f,g)(t_1 t_2) &= \int_a^b f(x)^n g(t_1 t_2 x)dx \leq \int_a^b f(x)^n g(t_1) g(t_2 x)dx \\ &= g(t_1) \int_a^b f(x)^n g(t_2 x)dx = g(t_1)\mathcal{T}_n(f,g)(t_2).\end{aligned}$$

Exchanging the roles of t_1 and t_2, we get the second inequality. This completes the proof of the theorem. □

This theorem is a simple result illustrating the interest of using sub-multiplicative functions for some classes of parameter-dependent weighted integral functionals.

3. Conclusion

In this paper, we derive several integral inequalities for a class of parameter-dependent weighted integral functionals, subject to various structural assumptions on the involved functions. These assumptions include monotonicity, convexity, log-convexity, and sub-multiplicativity. The results obtained provide a unified framework that extends and generalizes inequalities previously derived for specific functions, such as the incomplete exponential integral function, as investigated in [4].

Future work could include studying additional properties of these integral functionals, such as monotonicity and convexity with respect to the parameter. It could also involve extending the present results to other classes of functions or related integral transforms that arise in analysis and the theory of special functions.

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