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Note

Long chains of dense subalgebras of Banach algebras

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Abstract: We construct explicit strictly ascending chains of dense subalgebras of length $\mathfrak{c}$ in every separable infinite-dimensional complex Banach algebra. For large classes of commutative C^*-algebras we also construct strictly descending chains of the same length. The constructions rely on algebraic independence, Stone–Weierstrass arguments, and transfinite recursion.

Keywords: Banach algebras, dense subalgebras, ascending chains, descending chains, C^*-algebras, algebraic independence

MSC: 46H10, 46J10, 46L05, 03E10.

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1. Introduction

Although separable Banach algebras are topologically small, they often possess remarkably rich algebraic substructures. One manifestation of this phenomenon is the existence of large families of dense subalgebras with strict inclusion relations.

The purpose of this paper is to study the maximal possible length of chains of dense subalgebras in separable Banach algebras. Our main results show that the continuum $\mathfrak{c}$ is the optimal length in this setting and that such chains can be constructed explicitly under mild hypotheses.

Our contribution is threefold:

(i) explicit constructions of chains of dense subalgebras;

(ii) realization of the maximal possible chain length $\mathfrak{c}$ in separable Banach algebras;

(iii) a transparent sufficient condition guaranteeing the existence of descending chains.

Throughout, all Banach algebras are complex unless stated otherwise.

2. Exponential independence

We begin with a basic analytic independence result that underlies the algebraic constructions in the commutative case.

Lemma 1. *Let $\mu_1, \dots, \mu_m \in \mathbb{C}$ be distinct and let $r_j \geq 0$ be integers. If*

$$\sum_{j=1}^{m} \sum_{\ell=0}^{r_j} c_{j,\ell} x^\ell e^{\mu_j x} \equiv 0 \quad \textit{on an interval,}$$

then all coefficients $c_{j,\ell}$ vanish.

Proof. This is a standard consequence of the linear independence of exponential polynomials. One may differentiate sufficiently many times to obtain a linear system whose Wronskian determinant is nonzero due to the distinctness of the μ_j. We refer to [1] for a detailed treatment. □

3. Chains in $C(K)$

We now treat commutative C^*-algebras of the form $C(K)$.

Theorem 1. *Let K be an infinite compact Hausdorff space admitting a countable point-separating family $\{u_n\} \subset C(K, \mathbb{R})$. Then $C(K, \mathbb{C})$ admits strictly ascending and strictly descending chains of dense subalgebras of length $\mathfrak{c}$.*

Proof. Let $\Lambda \subset (0, \infty)$ be a $\mathbb{Q}$-linearly independent set of cardinality $\mathfrak{c}$. For $\lambda \in \Lambda$ and $n \in \mathbb{N}$, define

$$v_{\lambda,n}(x) = e^{\lambda u_n(x)}.$$

Let G consist of $\{1\}$, all functions u_n, and all $v_{\lambda,n}$. Any nontrivial polynomial relation among finitely many elements of G yields, upon restriction to suitable subsets of K, an identity of the form treated in Lemma 1, forcing all coefficients to vanish. Hence G is algebraically independent.

Let A be the algebra generated by G. It is self-adjoint, contains the constants, and separates points of K. By the Stone–Weierstrass theorem, A is dense in $C(K, \mathbb{C})$.

To obtain a strictly ascending chain, enumerate Λ as $\{\lambda_\alpha : \alpha < \mathfrak{c}\}$ and let A_α be the algebra generated by $\{u_n\}$ together with $\{v_{\lambda_\beta,n} : \beta < \alpha\}$. Algebraic independence ensures $A_\alpha \subsetneq A_\beta$ for $\alpha < \beta$, and each A_α remains dense.

For the descending chain, define B_α by removing the generators indexed by $\{\lambda_\beta : \beta < \alpha\}$. Density follows from the presence of the u_n, while strictness again follows from algebraic independence. □

Corollary 1. *Every infinite compact metrizable space K satisfies the hypotheses of Theorem 1.*

Proof. Separability of $C(K)$ implies the existence of a countable point-separating family; see [2]. □

4. Separable Banach algebras

We now turn to the noncommutative setting.

Theorem 2. *Every separable infinite-dimensional Banach algebra admits a strictly ascending chain of dense subalgebras of length $\mathfrak{c}$.*

Proof. Let E_0 be a fixed countable dense subalgebra. Since the ambient algebra has cardinality $\mathfrak{c}$, we may construct by transfinite recursion a family $\{x_\alpha : \alpha < \mathfrak{c}\}$ such that x_α does not belong to the algebra generated by $E_0 \cup \{x_\beta : \beta < \alpha\}$.

Define E_α to be the algebra generated by $E_0 \cup \{x_\beta : \beta < \alpha\}$. Each E_α is dense since it contains E_0, and strict inclusion follows from the choice of the x_α. □

Theorem 3. *If a separable Banach algebra contains an algebraically independent family of cardinality $\mathfrak{c}$, then it admits a strictly descending chain of dense subalgebras of length $\mathfrak{c}$.*

Proof. Let $\{y_\alpha : \alpha < \mathfrak{c}\}$ be algebraically independent and fix a countable dense subalgebra E_0. For each $\alpha < \mathfrak{c}$, define D_α as the algebra generated by $E_0 \cup \{y_\beta : \beta \geq \alpha\}$.

Density follows from the inclusion of E_0. If $\alpha < \beta$, then $y_\alpha \in D_\alpha \setminus D_\beta$, since otherwise algebraic independence would be violated. Hence the chain is strictly descending. □

Remark 1. The independence hypothesis in the preceding theorem is sufficient but not known to be necessary. It would be of interest to characterize precisely those separable Banach algebras admitting descending chains of maximal length.

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References

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