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Non-autonomous cantor sets from decimal expansions: Construction, dimension, and Hölder regularity

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Abstract: We construct a class of digit-defined Cantor sets $C_{(d_k)} \subset [0,1]$ from a prescribed sequence $(d_k)_{k\geq 1}$ by using the digit d_k to determine the removal ratio at level k. When only finitely many digits are zero, the construction has a uniform separation property after a finite initial stage. This yields compact, perfect, totally disconnected, nowhere dense sets of Lebesgue measure zero. The natural probability measure μ is obtained by assigning equal mass to the two children of each cylinder; equivalently, μ is the pushforward of the fair Bernoulli measure on $\{0,1\}^{\mathbb{N}}$ under the coding map. If

$$\ell_k = \prod_{j=1}^{k} a_j, \qquad a_k = \frac{1}{2}\Big(1 - \frac{d_k}{10}\Big),$$

then

$$\dim_H C_{(d_k)} = \liminf_{k\to\infty} \frac{\log 2}{-\frac{1}{k}\sum\limits_{j=1}^{k} \log a_j}.$$

The associated Cantor function $F(x) = \mu([0,x])$ is Hölder continuous for every exponent below $\dim_H C_{(d_k)}$, and no exponent above this dimension is possible. Cartesian products are also treated: under Ahlfors regularity of the factors, Hausdorff dimensions add and the product Cantor function has optimal Hölder threshold equal to the smallest one-dimensional dimension. A sufficient and verifiable condition for Ahlfors regularity is the bounded comparability $2^{-k} \asymp \ell_k^{\alpha}$, where α is the limiting similarity dimension. The results identify how the asymptotic distribution of decimal digits controls dimension, singular measure regularity, and product geometry.

Keywords: irregular Cantor sets, decimal expansions, Hausdorff dimension, Hölder continuity, non-autonomous iterated function systems, product fractals, Ahlfors regularity

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1. Introduction

Classical Cantor sets, introduced by Cantor [1], are generated by a fixed removal rule, such as deleting the middle third of each interval. Their exact self-similarity makes them central examples in fractal geometry and geometric measure theory. The theory of iterated function systems provides a precise language for such constructions [2], and the standard references of Mandelbrot [3], Falconer [4], and Mattila [5] describe their metric, dimensional, and measure-theoretic properties. Many arithmetic and dynamical constructions, however, exhibit scale changes that are not governed by a single finite family of contractions.

Non-autonomous iterated function systems allow the contraction ratios to vary from one level to the next. This level dependence occurs naturally in Moran constructions and in fractals generated by aperiodic data; related dimension theory appears in work on non-autonomous and conformal systems [6–8]. At the same time, decimal expansions provide an elementary but highly irregular source of sequences. Their digit frequencies and Diophantine features connect fractal constructions with distribution modulo one, normality, and approximation theory [9]. Digit-dependent fractals therefore provide a concrete setting in which arithmetic irregularity and geometric regularity can be studied simultaneously.

We explore Cantor sets generated directly from decimal digits, where the digit d_k determines the proportion removed from each interval at level k, so that the resulting construction is self-similar only in the exceptional case of an eventually constant or periodic digit pattern. The main objective is to determine which geometric and analytic features remain stable in this non-autonomous setting, including topology, Lebesgue measure, Hausdorff dimension, natural measures, Hölder regularity of the Cantor function, and behavior under Cartesian products. Throughout the main results, the digit sequence satisfies the finite-zero hypothesis: there exists an integer $K_0 \in \mathbb{N}$ such that $d_k \in \{1, 2, \dots, 9\}$ for all $k \geq K_0$, or equivalently, the digit sequence contains only finitely many zeros. This assumption gives a uniform positive lower bound for the gaps created after a finite initial stage. It is sufficient for the Cantor set to be totally disconnected and for the natural measure to satisfy sharp Frostman estimates. The condition is not necessary in all possible variants of the construction; nevertheless, without some control on zero digits, the conclusions may fail, as shown in Example 2.

The status of familiar constants illustrates the role of the hypothesis. It is not known whether the decimal expansions of $\pi - 3$ or $e - 2$ contain finitely many zeros, and the expected normality of these constants would imply that zeros occur with positive frequency. Consequently, any application of the finite-zero theory to such constants must be understood as conditional on the stated digit property. The results below are therefore formulated for arbitrary prescribed sequences satisfying Hypothesis (H), independently of unproved assertions about specific constants.

The original contribution of this work is the complete analysis of this digit-level construction: the set $C_{(d_k)}$ is compact, perfect, totally disconnected, and nowhere dense, and has Lebesgue measure zero (Propositions 2 and 3); the natural measure μ, obtained by assigning mass 2^{-k} to every level-k cylinder, is the pushforward of the fair Bernoulli measure on $\{0, 1\}^{\mathbb{N}}$ under the coding map (Proposition 4); the Hausdorff dimension is given by

$$\dim_H C_{(d_k)} = \liminf_{k \to \infty} \frac{\log 2}{-\frac{1}{k} \sum_{j=1}^{k} \log a_j}, \qquad a_k = \frac{1}{2}\left(1 - \frac{d_k}{10}\right),$$

as proved in Theorem 1; the Cantor function $F(x) = \mu([0, x])$ is Hölder continuous for every exponent β satisfying $0 < \beta < \dim_H C_{(d_k)}$, and this threshold is optimal because no exponent $\beta > \dim_H C_{(d_k)}$ is possible (Theorem 2); for Cartesian products $K_n = C_{(d^{(1)})} \times \cdots \times C_{(d^{(n)})}$, Hausdorff dimensions add when the factors are Ahlfors regular, while the product Cantor function has Hölder threshold equal to the minimum of the individual thresholds (Theorems 3 and 4); finally, Ahlfors regularity follows from the uniform dimensionality condition $2^{-k} \asymp \ell_k^{\alpha}$, where $\alpha = \lim_{k \to \infty} \alpha_k$ and $\ell_k = \prod_{j=1}^{k} a_j$ (Theorem 5). The distinctive point is that the optimal dimension and regularity exponents are governed by the lower asymptotic average of $-\log a_k$. Thus the limiting geometry records not merely the set of digits that occur, but the way their partial averages fluctuate across scales.

2. One-dimensional irregular Cantor sets

2.1. Construction and basic properties

Let $(d_k)_{k \geq 1}$ be a sequence with $d_k \in \{0, 1, \dots, 9\}$. The construction is defined at the level of digit sequences, which avoids the ambiguity of decimal representations such as $0.1000\ldots = 0.0999\ldots$. When a real number $\xi = 0.d_1 d_2 d_3 \ldots$ is used to generate a sequence, the terminating decimal representation is chosen whenever two representations are available.

Assume Hypothesis (H), and let K_0 be such that $d_k \geq 1$ for all $k \geq K_0$. The finitely many initial levels produce only finitely many affine placements of the tail construction. These finite placements may share endpoints when a zero digit occurs, but they do not affect Hausdorff dimension, Lebesgue measure zero, or the Hölder thresholds, and they change only constants in the metric estimates. For clarity, the proofs are written in the normalized case

$$d_k \geq 1 \qquad \text{for all } k \geq 1. \tag{1}$$

The general finite-zero case follows by applying the normalized estimates to the tail beginning at K_0 and by taking maxima over the finite initial family of cylinders.

Define the removal ratio and the contraction factor at level k by

$$r_k = \frac{d_k}{10}, \qquad a_k = \frac{1}{2}(1 - r_k) = \frac{1}{2}\Big(1 - \frac{d_k}{10}\Big).$$

Under (1),

$$0.05 \le a_k \le 0.45, \qquad a_{\min} := \inf_{k\ge1} a_k > 0, \qquad a_{\max} := \sup_{k\ge1} a_k < 1.$$

These bounds provide both shrinking of cylinder lengths and uniform separation of distinct cylinders.

Definition 1. Set $C_0 = [0,1]$. Suppose that C_{k-1} is a finite union of closed intervals. From every interval $I \subset C_{k-1}$ of length $|I|$, remove the open middle interval of length $|I|r_k$. The two remaining closed intervals each have length

$$\frac{1}{2}(|I| - |I|r_k) = a_k|I|.$$

The union of all intervals remaining after level k is denoted by C_k. The digit-defined Cantor set is

$$C_{(d_k)} = \bigcap_{k=0}^{\infty} C_k.$$

The same construction can be expressed through level-dependent affine maps

$$\phi_{k,0}(x) = a_k x, \qquad \phi_{k,1}(x) = a_k x + (1 - a_k).$$

For a word $w = (j_1, \dots, j_k) \in \{0,1\}^k$, set

$$I_w = I_{j_1\dots j_k} = \phi_{1,j_1} \circ \phi_{2,j_2} \circ \cdots \circ \phi_{k,j_k}([0,1]).$$

The order of composition records the chronological order of the construction: the map indexed by the first digit places the first-level interval, and later maps refine it. Induction gives

$$C_k = \bigcup_{w\in\{0,1\}^k} I_w, \qquad C_{(d_k)} = \bigcap_{k\ge1} C_k.$$

Proposition 1. *Every level-k cylinder interval has length*

$$|I_w| = \prod_{t=1}^{k} a_t =: \ell_k.$$

Moreover, $\ell_k \to 0$ as $k \to \infty$.

Proof. Each map ϕ_{t,j_t} scales length by a_t. The length of a composition is therefore the product of the corresponding scaling factors. Since $a_t \le a_{\max} < 1$ in the normalized case, $\ell_k \le a_{\max}^k \to 0$. In the finite-zero case, the same conclusion holds after the finite initial segment. □

Proposition 2. *Under Hypothesis (H), the set $C_{(d_k)}$ is compact, perfect, totally disconnected, and nowhere dense.*

Proof. First consider the normalized case (1). Each C_k is a finite union of closed intervals and is therefore compact. The sets are nested, so $C_{(d_k)}$ is compact.

Let $x \in C_{(d_k)}$, and let I_k be the level-k cylinder containing x. Choose an endpoint y_k of I_k different from x. Endpoints are retained at all later stages; hence $y_k \in C_{(d_k)}$, and $|x - y_k| \le \ell_k \to 0$. Thus every point of $C_{(d_k)}$ is an accumulation point, and the set is perfect.

For total disconnectedness, take distinct points $x < y$ in $C_{(d_k)}$. Choose k so large that $\ell_k < y - x$. Then x and y cannot lie in the same level-k cylinder. Since every removal ratio is positive in the normalized case, distinct level-k cylinders are separated by open gaps. Hence a gap separates x from y, and no connected subset of $C_{(d_k)}$ can contain both points. The connected components are therefore singletons.

If $C_{(d_k)}$ contained a nonempty open interval (a, b), then $(a, b) \subset C_k$ for every k. For k sufficiently large, $\ell_k < b - a$, so no level-k cylinder can contain (a, b). This contradicts the fact that C_k is the union of level-k cylinders. Therefore $C_{(d_k)}$ has empty interior, and since it is closed, it is nowhere dense.

For a sequence satisfying Hypothesis (H), the set is a finite union of affine copies of a normalized tail construction, with possible intersections only at finitely many endpoints created by the initial zero levels. Finite unions of compact perfect nowhere dense sets with finite pairwise intersections retain compactness, perfectness, total disconnectedness, and empty interior in the present construction. The proposition follows. □

Proposition 3. *The Lebesgue measure of $C_{(d_k)}$ is*

$$\mathcal{L}(C_{(d_k)}) = \prod_{k=1}^{\infty} \Big(1 - \frac{d_k}{10}\Big).$$

In particular, under Hypothesis (H), $\mathcal{L}(C_{(d_k)}) = 0$.

Proof. At level k, each interval from level $k-1$ retains the proportion $1 - d_k/10$ of its length. Hence

$$\mathcal{L}(C_k) = \prod_{j=1}^{k} \Big(1 - \frac{d_j}{10}\Big).$$

Continuity from above for the decreasing sequence (C_k) gives the stated infinite product. Under Hypothesis (H), all sufficiently large factors are at most 0.9, so the product is zero. □

Example 1. If $d_k \equiv 1$, then $r_k = 0.1$ and $a_k = 0.45$ for every k. The set is the self-similar Cantor set obtained by removing the middle 10% at every step, and

$$\dim_H C_{(d_k)} = \frac{\log 2}{-\log 0.45} \approx 0.868.$$

The associated Cantor–Lebesgue function is Hölder continuous precisely below this dimensional threshold, with exact threshold behavior described by Theorem 2.

Example 2. Let $d_1 = 1$ and $d_k = 0$ for all $k \geq 2$, corresponding to the terminating expansion $0.1000\ldots$. At the first level, the middle 10% is removed, leaving $[0, 0.45] \cup [0.55, 1]$. At every later level, the removal ratio is zero, so no further deletion occurs. Thus

$$C_{(d_k)} = [0, 0.45] \cup [0.55, 1].$$

This set has positive Lebesgue measure and contains intervals. The example shows that the conclusions of Proposition 2 and Proposition 3 can fail when zero digits are not controlled.

2.2. Natural measure via symbolic coding

The hierarchical construction defines a canonical probability measure by assigning equal mass to the two children of every cylinder.

Let $\Sigma = \{0, 1\}^{\mathbb{N}}$ be the space of infinite binary sequences with the product topology. For a finite word $w = (j_1, \ldots, j_k)$, the symbolic cylinder is

$$[w] = \{\omega \in \Sigma : \omega_1 = j_1, \ldots, \omega_k = j_k\}.$$

Let ν be the fair Bernoulli measure on Σ, so that

$$\nu([w]) = 2^{-k}.$$

Define the coding map $\pi : \Sigma \to [0,1]$ by

$$\{\pi(\omega)\} = \bigcap_{k\geq 1} I_{\omega_1\ldots\omega_k}.$$

The intersection is a singleton because the cylinders are nested and $\ell_k \to 0$. The map π is continuous and satisfies $\pi(\Sigma) = C_{(d_k)}$.

Definition 2. The natural measure on $C_{(d_k)}$ is

$$\mu = \pi_*\nu, \qquad \mu(A) = \nu(\pi^{-1}(A))$$

for every Borel set $A \subset [0,1]$.

Proposition 4. *For every word $w \in \{0,1\}^k$,*

$$\mu(I_w) = 2^{-k}.$$

Consequently, μ is non-atomic and is the unique Borel probability measure on $C_{(d_k)}$ with these cylinder masses.

Proof. The preimage of I_w under π agrees with the symbolic cylinder $[w]$, except possibly at endpoints where two symbolic addresses can represent the same point. The set of such endpoints is countable and has ν-measure zero. Hence

$$\mu(I_w) = \nu([w]) = 2^{-k}.$$

Every point has at most two symbolic addresses, each of ν-measure zero, so μ has no atoms. The cylinder intervals generate the relative Borel σ-algebra of $C_{(d_k)}$. Any Borel probability measure with the same cylinder masses must therefore coincide with μ on a generating algebra and hence on all Borel subsets of $C_{(d_k)}$. □

The measure is self-similar only in a level-dependent sense. For $m \geq 0$, let $\mu^{(m)}$ denote the natural measure generated by the tail sequence $(d_{m+1}, d_{m+2}, \ldots)$, and for $w = (j_1, \ldots, j_k)$ write

$$S_w = \phi_{1,j_1} \circ \cdots \circ \phi_{k,j_k}.$$

Then every Borel set $A \subset [0,1]$ satisfies

$$\mu(A) = 2^{-k} \sum_{w\in\{0,1\}^k} \mu^{(k)}\big(S_w^{-1}(A \cap I_w)\big).$$

This identity records that the conditional measure inside a level-k cylinder is governed by the digit sequence beginning at level $k+1$, not by the original sequence restarted at level 1.

2.3. Hausdorff dimension

Set

$$\ell_k = \prod_{j=1}^{k} a_j, \qquad \alpha_k = \frac{\log 2}{-\frac{1}{k}\sum_{j=1}^{k}\log a_j}, \qquad \alpha = \liminf_{k\to\infty} \alpha_k.$$

Since $0 < a_j < 1$, the denominators are positive.

Theorem 1. *Under Hypothesis (H),*

$$\dim_H C_{(d_k)} = \alpha.$$

If $\lim_{k\to\infty} \alpha_k$ exists, then $\dim_H C_{(d_k)}$ equals that limit.

Proof. It is enough to prove the result in the normalized case (1); the finite initial segment in Hypothesis (H) does not change the limiting value of α and only decomposes the set into finitely many affine copies of a tail construction.

Upper bound. Let $s > \alpha$. Choose a subsequence (k_i) such that $\alpha_{k_i} \to \alpha$ and $\alpha_{k_i} < s$ for all sufficiently large i. Then

$$-\frac{1}{k_i}\sum_{j=1}^{k_i} \log a_j > \frac{\log 2}{s}, \qquad \ell_{k_i} < 2^{-k_i/s}.$$

The 2^{k_i} level-k_i cylinders cover $C_{(d_k)}$, each with diameter ℓ_{k_i}. Hence

$$\mathcal{H}^s_{\ell_{k_i}}(C_{(d_k)}) \le 2^{k_i}\ell_{k_i}^s < 1.$$

Letting $i \to \infty$ gives $\mathcal{H}^s(C_{(d_k)}) < \infty$. Therefore $\dim_H C_{(d_k)} \le s$, and the arbitrariness of $s > \alpha$ gives $\dim_H C_{(d_k)} \le \alpha$.

Lower bound. Fix $0 < \beta < \alpha$. By the definition of lim inf, there exists K such that $\alpha_k \ge \beta$ for all $k \ge K$. This is equivalent to

$$2^{-k} \le \ell_k^\beta \qquad (k \ge K).$$

The following separation estimate supplies the required Frostman bound.

Lemma 1. *Assume (1). There exists $\gamma > 0$, for instance $\gamma = 2/9$, such that any two distinct level-m cylinder intervals I and J satisfy*

$$\text{dist}(I, J) \ge \gamma\ell_m.$$

Proof. At level 1, the two intervals are $[0, a_1]$ and $[1 - a_1, 1]$, separated by $1 - 2a_1 = d_1/10 \ge 0.1$. Since $\ell_1 = a_1 \le 0.45$, this distance is at least $(2/9)\ell_1$.

Assume the claim holds at level $m - 1$. If two level-m cylinders have the same parent, then their distance equals $(d_m/10)\ell_{m-1}$. Since $d_m \ge 1$ and $\ell_m = a_m\ell_{m-1}$ with $a_m \le 0.45$,

$$\frac{(d_m/10)\ell_{m-1}}{\ell_m} = \frac{d_m/10}{a_m} \ge \frac{0.1}{0.45} = \frac{2}{9}.$$

If the two level-m cylinders have different parents, then they are contained in two distinct level-$(m - 1)$ cylinders. Their distance is at least the distance between the two parents, hence at least $\gamma\ell_{m-1} \ge \gamma\ell_m$. This completes the induction. □

Let $x \in C_{(d_k)}$ and $0 < r \le 1$. Choose k such that

$$\ell_{k+1} \le r < \ell_k.$$

By Lemma 1, an interval of length $2r < 2\ell_k$ can meet at most a constant number N_γ of level-k cylinders; one may take $N_\gamma = \lceil 2/\gamma \rceil + 2$. Therefore

$$\mu(B(x, r)) \le N_\gamma 2^{-k}.$$

For $k \ge K$,

$$\mu(B(x, r)) \le N_\gamma \ell_k^\beta \le N_\gamma a_{\min}^{-\beta} r^\beta,$$

because $\ell_k = \ell_{k+1}/a_{k+1} \le r/a_{\min}$. For the finitely many scales $k < K$, the constant can be enlarged. Thus there exists $C_\beta > 0$ such that

$$\mu(B(x, r)) \le C_\beta r^\beta$$

for all $x \in C_{(d_k)}$ and $0 < r \le 1$. Frostman's lemma [5] gives $\dim_H C_{(d_k)} \ge \beta$. Since $\beta < \alpha$ was arbitrary, $\dim_H C_{(d_k)} \ge \alpha$. The upper and lower bounds prove the theorem. □

2.4. Irregular Cantor function

Definition 3. The Cantor function associated with $C_{(d_k)}$ is

$$F : [0, 1] \to [0, 1], \qquad F(x) = \mu([0, x]).$$

Proposition 5. *The function F satisfies:*

1. $F(0) = 0$ *and* $F(1) = 1$*;*
2. F *is non-decreasing;*
3. F *is continuous on* $[0,1]$*;*
4. F *is constant on every connected component of* $[0,1] \setminus C_{(d_k)}$*;*
5. $F'(x) = 0$ *for Lebesgue-almost every* $x \in [0,1]$.

Proof. The first two assertions follow from the definition of F as a distribution function of a probability measure. Since μ has no atoms by Proposition 4, the distribution function is continuous. If (a,b) is a component of $[0,1] \setminus C_{(d_k)}$, then $\mu((a,b)) = 0$, and F is constant on (a,b). By Proposition 3, the complement of $C_{(d_k)}$ has full Lebesgue measure. A monotone function is differentiable almost everywhere, and on each complementary interval the derivative is zero. Hence $F'(x) = 0$ for Lebesgue-almost every x. □

Theorem 2. *Under Hypothesis (H), the Cantor function F is Hölder continuous of order β for every $0 < \beta < \alpha$, where $\alpha = \dim_H C_{(d_k)}$. Moreover, $F \notin C^{0,\beta}([0,1])$ for every $\beta > \alpha$.*

Proof. It suffices to prove the estimate in the normalized case (1); the finite-zero case changes only the Hölder constant on the finitely many initial cylinders. Let $0 < \beta < \alpha$. As in the lower bound of Theorem 1, there exists K such that $2^{-k} \le \ell_k^\beta$ for all $k \ge K$.

Take $x < y$ and set $\delta = y - x$. If $\delta = 0$, there is nothing to prove. Otherwise choose k such that $\ell_{k+1} \le \delta < \ell_k$. The interval $[x,y]$ has length less than ℓ_k, and by Lemma 1 it meets at most N_γ level-k cylinders. Therefore

$$|F(y) - F(x)| = \mu((x,y]) \le N_\gamma 2^{-k}.$$

For $k \ge K$,

$$|F(y) - F(x)| \le N_\gamma \ell_k^\beta \le N_\gamma a_{\min}^{-\beta} |x - y|^\beta.$$

The finitely many larger scales are absorbed by increasing the constant. Hence F is β-Hölder on $[0,1]$.

For optimality, let $x_k < y_k$ be the endpoints of any level-k cylinder. Then

$$|F(y_k) - F(x_k)| = 2^{-k}, \qquad |y_k - x_k| = \ell_k.$$

Choose a subsequence (k_i) such that $\alpha_{k_i} \to \alpha$. If F were β-Hölder for some $\beta > \alpha$, then for a constant $C > 0$,

$$2^{-k_i} \le C \ell_{k_i}^\beta.$$

Taking logarithms and dividing by $\log \ell_{k_i} < 0$ gives

$$\alpha_{k_i} = \frac{-k_i \log 2}{\sum_{j=1}^{k_i} \log a_j} \ge \beta + \frac{\log C}{\sum_{j=1}^{k_i} \log a_j}.$$

The last term tends to zero because $\ell_{k_i} \to 0$. Passing to the limit yields $\alpha \ge \beta$, a contradiction. Thus no Hölder exponent larger than α is possible. □

Remark 1. The theorem gives the sharp threshold but not necessarily exact α-Hölder continuity. Exact endpoint regularity requires uniform comparability between cylinder mass and cylinder length. This is precisely the role of the Ahlfors regularity condition in Section 3.4.

3. Cartesian Products

Let $(d_k^{(1)}), \ldots, (d_k^{(n)})$ be digit sequences satisfying Hypothesis (H). Write

$$C_i = C_{(d^{(i)})}, \qquad \mu_i = \text{the natural measure on } C_i, \qquad \alpha_i = \dim_H C_i.$$

3.1. Product set and product measure

Definition 4. The product Cantor set in $[0,1]^n$ is

$$K_n = C_1 \times C_2 \times \cdots \times C_n.$$

The natural product measure on K_n is

$$\nu_n = \mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_n.$$

The set K_n is compact and totally disconnected. Since each factor has empty interior in $[0,1]$, the product has empty interior in $[0,1]^n$. The measure ν_n is a Borel probability measure supported on K_n.

3.2. Hausdorff dimension of the product

For arbitrary sets, Hausdorff dimension is not always additive under Cartesian products. In the present setting, Ahlfors regularity supplies the necessary uniformity: it gives sharp covering estimates from above and Frostman estimates from below.

Theorem 3. *Assume that each μ_i is Ahlfors α_i-regular; that is, there exist constants $c_i, C_i > 0$ such that*

$$c_i r^{\alpha_i} \le \mu_i(B(x_i, r)) \le C_i r^{\alpha_i}$$

for all $x_i \in C_i$ and all $0 < r \le 1$. Then

$$\dim_H K_n = \sum_{i=1}^{n} \alpha_i.$$

Proof. *Upper bound.* For each factor, Ahlfors regularity implies the covering estimate $N_i(r) \le A_i r^{-\alpha_i}$, where $N_i(r)$ is the minimum number of intervals of radius r needed to cover C_i. Indeed, a maximal r-separated subset of C_i has disjoint balls of radius $r/2$; the lower Ahlfors bound gives at most a constant multiple of $r^{-\alpha_i}$ such balls. Taking products of these covers gives a cover of K_n by rectangular boxes of diameter at most $2\sqrt{n}\, r$, with cardinality at most $Ar^{-\sum_i \alpha_i}$. Therefore $\overline{\dim}_B K_n \le \sum_i \alpha_i$, and hence $\dim_H K_n \le \sum_i \alpha_i$.

Lower bound. Let $x = (x_1, \ldots, x_n) \in K_n$. The Euclidean ball $B(x, r)$ is contained in $\prod_i [x_i - r, x_i + r]$. The upper Ahlfors bounds give

$$\nu_n(B(x,r)) \le \prod_{i=1}^{n} \mu_i([x_i - r, x_i + r]) \le \prod_{i=1}^{n} C_i r^{\alpha_i} = C r^{\sum_i \alpha_i}$$

with a constant $C > 0$. Frostman's lemma then yields $\dim_H K_n \ge \sum_i \alpha_i$. The two bounds coincide. □

Remark 2. Ahlfors regularity is a sufficient condition rather than a purely formal requirement of the product construction. It prevents scale-dependent concentration of mass and ensures that the one-dimensional covering behavior persists uniformly across the product.

3.3. Product Cantor function

Define

$$F_n : [0,1]^n \to [0,1], \qquad F_n(x_1, \ldots, x_n) = \prod_{i=1}^{n} F_i(x_i),$$

where F_i is the Cantor function associated with C_i.

Theorem 4 (Hölder continuity of the product function)**.** *Let $\alpha = \min\{\alpha_1, \ldots, \alpha_n\}$. Then F_n is Hölder continuous of every order β with $0 < \beta < \alpha$. Moreover, $F_n \notin C^{0,\beta}([0,1]^n)$ for every $\beta > \alpha$.*

Proof. If $0 < \beta < \alpha$, then Theorem 2 gives constants C_i such that

$$|F_i(x_i) - F_i(y_i)| \le C_i |x_i - y_i|^{\beta}$$

for each i. For $x, y \in [0,1]^n$, a telescoping expansion of the product gives

$$|F_n(x) - F_n(y)| \le \sum_{i=1}^{n} |F_i(x_i) - F_i(y_i)| \le \Big(\sum_{i=1}^{n} C_i\Big) \|x-y\|_\infty^\beta.$$

The Euclidean and maximum norms are equivalent on $\mathbb{R}^n$, so F_n is β-Hölder.

Choose i_0 with $\alpha_{i_0} = \alpha$ and fix all other coordinates equal to 1. Since $F_i(1) = 1$, the restriction of F_n to this coordinate line is exactly F_{i_0}. By Theorem 2, F_{i_0} is not β-Hölder for any $\beta > \alpha$. Hence F_n cannot be β-Hölder for any $\beta > \alpha$. □

3.4. Ahlfors regularity under a uniform dimensionality condition

The natural measure need not be Ahlfors regular solely from the existence of $\dim_H C_{(d_k)}$. Endpoint regularity requires a uniform relation between the mass 2^{-k} of a cylinder and its length ℓ_k.

Definition 5. The digit sequence (d_k) satisfies the uniform dimensionality condition if the limit

$$\alpha = \lim_{k\to\infty} \alpha_k$$

exists and there are constants $c_0, C_0 > 0$ such that

$$c_0 \ell_k^\alpha \le 2^{-k} \le C_0 \ell_k^\alpha \qquad \text{for all } k \ge 1. \tag{2}$$

Condition (2) is equivalent to boundedness above and below of $2^{-k}/\ell_k^\alpha$. It holds for eventually periodic digit sequences with digits in $\{1, \dots, 9\}$, and more generally for sequences whose partial sums satisfy

$$\sum_{j=1}^{k} \log a_j = -\frac{k \log 2}{\alpha} + O(1).$$

Convergence of α_k alone is weaker; it allows unbounded fluctuations in the ratio $2^{-k}/\ell_k^\alpha$, which can prevent exact α-Hölder regularity.

Theorem 5 (Ahlfors regularity of μ)**.** *Assume Hypothesis (H) and uniform dimensionality with exponent α. Then μ is Ahlfors α-regular: there exist constants $c, C > 0$ such that*

$$c r^\alpha \le \mu(B(x,r)) \le C r^\alpha$$

for all $x \in C_{(d_k)}$ and $0 < r \le 1$.

Proof. Work in the normalized case (1); the finite-zero case again changes only the constants. Fix $x \in C_{(d_k)}$ and $0 < r \le 1$, and choose k with $\ell_{k+1} \le r < \ell_k$.

For the upper bound, Lemma 1 implies that $B(x,r)$ meets at most N_γ level-k cylinders. Hence

$$\mu(B(x,r)) \le N_\gamma 2^{-k} \le N_\gamma C_0 \ell_k^\alpha \le N_\gamma C_0 a_{\min}^{-\alpha} r^\alpha.$$

For the lower bound, let I be a level-$(k+1)$ cylinder containing x. Since $\operatorname{diam} I = \ell_{k+1} \le r$, the whole cylinder I is contained in $B(x,r)$. Therefore

$$\mu(B(x,r)) \ge \mu(I) = 2^{-(k+1)} \ge c_0 \ell_{k+1}^\alpha.$$

Because $\ell_{k+1} = a_{k+1}\ell_k \ge a_{\min}\ell_k > a_{\min} r$,

$$\mu(B(x,r)) \ge c_0 a_{\min}^\alpha r^\alpha.$$

The upper and lower estimates prove Ahlfors regularity. □

Theorem 6. *If each μ_i is Ahlfors α_i-regular, then ν_n is Ahlfors $\sum_{i=1}^{n} \alpha_i$-regular on K_n, with respect to the Euclidean metric.*

Proof. Let $x = (x_1, \dots, x_n) \in K_n$ and $0 < r \leq 1$. Since

$$B(x,r) \subset \prod_{i=1}^{n} [x_i - r, x_i + r],$$

the upper Ahlfors bounds yield

$$\nu_n(B(x,r)) \leq \prod_{i=1}^{n} C_i r^{\alpha_i} = C r^{\sum_i \alpha_i}.$$

For the lower bound, the product of intervals

$$\prod_{i=1}^{n} [x_i - r/\sqrt{n}, x_i + r/\sqrt{n}]$$

is contained in $B(x,r)$. Therefore

$$\nu_n(B(x,r)) \geq \prod_{i=1}^{n} c_i \left(\frac{r}{\sqrt{n}} \right)^{\alpha_i} = c r^{\sum_i \alpha_i}.$$

Thus ν_n is Ahlfors regular with exponent $\sum_i \alpha_i$. □

4. Results and discussion

4.1. Interpretation of the dimension formula

The formula in Theorem 1 shows that the Hausdorff dimension is determined by the lower asymptotic behavior of the averages

$$\frac{1}{k} \sum_{j=1}^{k} (-\log a_j).$$

Digits that produce larger removals, such as $d_k = 9$, make a_k smaller and increase this average; along subsequences where this effect is pronounced, the dimension decreases. Digits that produce smaller removals, such as $d_k = 1$, have the opposite effect. The use of the lim inf is therefore essential: the dimension is controlled by the most restrictive persistent scales, not by occasional favorable scales.

The Hölder result has the same interpretation. The Cantor function changes by exactly 2^{-k} across a level-k cylinder of length ℓ_k. Consequently, the ratio between mass and length at the scales where α_k approaches its liminf prevents any exponent above $\dim_H C_{(d_k)}$. Below that threshold, the uniform gap estimate provides enough separation to turn the Frostman bound into a global Hölder estimate.

4.2. Special digit sequences

Eventually periodic sequences. If (d_k) is eventually periodic with period p, say $d_{k+p} = d_k$ for all $k \geq N$, then the contraction factors are eventually periodic and

$$\dim_H C_{(d_k)} = \frac{p \log 2}{-\sum_{m=1}^{p} \log a_{N+m}}.$$

After grouping each full period into one step, the tail is a self-similar Cantor set with 2^p maps of common contraction ratio $\prod_{m=1}^{p} a_{N+m}$. In particular, for a constant nonzero digit d,

$$\dim_H C_{(d_k)} = \frac{\log 2}{-\log \left(\frac{1}{2}(1 - d/10) \right)}.$$

Normal digit frequencies. A simply normal base-ten sequence has each digit $0, 1, \ldots, 9$ with frequency $1/10$. Such a sequence does not satisfy Hypothesis (H), because zeros occur with positive frequency. If one formally inserts these digit frequencies into the average defining α_k, the value is

$$\frac{\log 2}{-\frac{1}{10}\sum_{d=0}^{9}\log\left(\frac{1}{2}(1-d/10)\right)} \approx 0.467.$$

This number is not asserted as a consequence of Theorem 1, since that theorem assumes finite zeros. It indicates instead why digit sequences with infinitely many zeros require a separate separation analysis.

Liouville-type sequences. Liouville-type decimals, such as $\sum_{n=1}^{\infty} 10^{-n!}$, contain very long blocks of zeros. At zero levels no interval is removed, so adjacent subcylinders may touch and the uniform gap estimate used above is unavailable at those levels. The finite-zero theory therefore does not cover such sequences. A satisfactory treatment would need hypotheses that measure how frequently positive-removal levels occur and how long zero blocks are allowed to persist.

4.3. Implications for products and regularity

The product results show that dimension additivity is not merely a formal consequence of taking Cartesian products. The key condition is regular distribution of mass across scales. When $2^{-k} \asymp \ell_k^{\alpha}$, the natural measure behaves like an α-dimensional volume on the Cantor set, and this behavior is stable under finite products. Without this comparability, the Hausdorff dimension of each factor is still given by Theorem 1, but endpoint Hölder regularity and product dimension statements may require additional scale information.

For applications, the most transparent digit classes are eventually periodic sequences and balanced aperiodic sequences with bounded fluctuations of $\sum_{j=1}^{k}\log a_j$ around its linear trend. These sequences exhibit both irregular digit order and regular mass-length scaling, making them suitable models for non-autonomous fractal measures with sharp analytic estimates.

5. Conclusion

Digit-controlled removal rules produce a concrete class of non-autonomous Cantor sets whose geometry is governed by the asymptotic behavior of the digit sequence. Under the finite-zero hypothesis, the construction yields compact, perfect, totally disconnected, nowhere dense sets of Lebesgue measure zero. The natural Bernoulli pushforward measure assigns mass 2^{-k} to each level-k cylinder, and the Hausdorff dimension is exactly

$$\liminf_{k\to\infty} \frac{\log 2}{-\frac{1}{k}\sum_{j=1}^{k}\log a_j}.$$

This liminf is also the sharp Hölder threshold for the associated Cantor function: all smaller exponents hold globally, and every larger exponent fails.

The analysis clarifies the role of scale uniformity. The finite-zero condition supplies separation, while the stronger comparability $2^{-k} \asymp \ell_k^{\alpha}$ supplies Ahlfors regularity and exact endpoint behavior. In products, this regularity ensures additivity of Hausdorff dimension and identifies the least regular coordinate as the Hölder threshold of the product Cantor function. These results give a precise link between decimal digit statistics, singular measures, and fractal regularity, and they isolate the additional difficulties created by digit sequences with infinitely many zeros.

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