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On some uncertainty principles for the multidimensional Fourier-Bessel transform

Fatima Elgadiri* and Abdellatif Akhlidj

Departement of Mathematics, Faculty of Sciences Ain Chock, University of Hassan II, Casablanca, Morocco
* Correspondence: elgadiri.fatima@gmail.com

Abstract: The multidimensional Fourier-Bessel transform is a generalization of Fourier-Bessel transform that obeys the same uncertainty principles as the classical Fourier transform. In this paper, we establish the following uncertainty principles; an $L^p - L^q$-version of Morgan's theorem, the Donoho-Stark uncertainty principles and bandlimited principles of concentration type for the multidimensional Fourier-Bessel transform.

Keywords: multidimensional fourier-Bessel transform, Morgan's theorem, Donoho-Stark uncertainty principle, bandlimited uncertainty principles

MSC: 42B10, 44A15, 44A30.

1. Introduction

I In 1934, Morgan [1] proved a famous result called Morgan's theorem. It states that, for all $\alpha > 0$, $\beta > 0$, $\delta > 2$ and $\gamma = \frac{\delta}{\delta-1}$, then for all measurable function f on $\mathbb{R}$, the conditions:

$$e^{\alpha|x|^\delta} f(x) \in L^\infty(\mathbb{R}) \text{ and } e^{\beta|x|^\gamma} \hat{f}(x) \in L^\infty(\mathbb{R}),$$

imply $f = 0$ almost everywhere if and only if

$$(\alpha\delta)^{\frac{1}{\delta}} (\beta\gamma)^{\frac{1}{\gamma}} > \left(sin\left(\frac{\pi}{2}(\gamma - 1) \right) \right)^{\frac{1}{\gamma}},$$

where $\hat{f}$ is the classical Fourier transform of f on $\mathbb{R}$.

Recently, in 2003, Farah and Mokni [2] have generalized Morgan's theorem to an $L^p - L^q-$version, where $1 \leq p, q \leq +\infty$, as follows:

Let $b > 0$, $a > 0$, γ, η positive real numbers satisfying $\gamma > 2$, and $\frac{1}{\gamma} + \frac{1}{\eta} = 1$, then for all measurable function f on $\mathbb{R}$, the conditions:

$$e^{a|x|^\gamma} f(x) \in L^p(\mathbb{R}) \quad \text{and} \quad e^{b|y|^\eta} \hat{f}(y) \in L^q(\mathbb{R}),$$

imply $f = 0$ almost everywhere if and only if

$$(a\gamma)^{\frac{1}{\gamma}} (b\eta)^{\frac{1}{\eta}} > (sin(\frac{\pi}{2}(\eta - 1)))^{\frac{1}{\eta}}.$$

They also extended this theorem to the euclidean space $\mathbb{R}^n$, to the Heisenberg group and to noncompact real symmetric spaces [2,3].

In 1989, Donoho and Stark [4] established an uncertainty principle for finite cyclic groups $\mathbb{Z}_n$. For a nonzero function f defined on $\mathbb{Z}_n$ which its support is denoted by $supp(f) = \{x \in \mathbb{Z}_n : f(x) \neq 0\}$. Then,

$$|supp(f)| |supp(\hat{f})| \geq n. \tag{1}$$

Later on, the Donoho-Stark uncertainty principle has been generalized by Smith [5] to finite abelian groups, to abelian groups by Özaydm and Przebinda [6], and also to compact groups by Alagic and Russell [7].

The Donoho–Stark uncertainty principles relies on the concept of $\epsilon-$concentration of a function (signal) on a measurable set $U \subseteq \mathbb{R}^n$, in both the space and frequency domains. Given $\epsilon \geq 0$, a function $f \in L^2(\mathbb{R}^n)$ is $\epsilon-$concentrated ($\epsilon-$timelimited) on a measurable set $U \subseteq \mathbb{R}^n$ if

$$\left(\int_{\mathbb{R}^n \setminus U} |f(x)|^2 \, dx \right)^{\frac{1}{2}} \leq \epsilon \, \|f\|_2 \,. \tag{2}$$

The theorem of Donoho and Stark states that, for $f \in L^2(\mathbb{R}^n)$, $f \neq 0$, is ϵ_T-concentrated on $T \subseteq \mathbb{R}^n$, and its Fourier transform $\hat{f}$ is ϵ_S-concentrated (ϵ_S-bandlimited) on $S \subseteq \mathbb{R}^n$, then

$$|T||S| \geq (1 - \epsilon_T - \epsilon_S)^2. \tag{3}$$

This paper is organized in the following way. We begin by giving an overview and some background information on harmonic analysis in relation to the Multidimensional Fourier-Bessel transform. Next, we deal with an $L^p - L^q-$version of Morgan's theorem for the multidimensional Fourier-Bessel transform. Finally, we establish Donoho-Stark and bandlimited uncertainty principles for the multidimensional Fourier-Bessel transform.

2. Harmonic analysis associated with the multidimensional Fourier-Bessel operator

In this section we give an overview of the harmonic analysis associated with the multidimensional Fourier-Bessel operator, one can see [8–11]. Consider the differential operator,

$$B_\alpha f(x) = \sum_{j=1}^d \frac{\partial^2 f}{\partial x_j^2}(x) + \frac{2\alpha_j + 1}{x_j} \frac{\partial f}{\partial x_j}(x), \tag{4}$$

where $x = (x_1, x_2, ..., x_d) \in \mathbb{R}_+^d$; $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$ such that, $\alpha_j > -\frac{1}{2}$ for all $j = 1, ..., d$.

We define the space $L_\alpha^p(\mathbb{R}_+^d) = L^p(\mathbb{R}_+^d, x^{2\alpha+1} dx)$, $1 \leq p \leq +\infty$, as the class of measurable functions f on $\mathbb{R}_+^d$ for which $\|f\|_{p,\alpha} < +\infty$, where

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}_+^d} |f(x)|^p x^{2\alpha+1} dx \right)^{\frac{1}{p}} \quad \text{if} \quad p < +\infty, \tag{5}$$

where $x^{2\alpha+1} = x_1^{2\alpha_1+1} x_2^{2\alpha_2+1} ... x_d^{2\alpha_d+1}$, $\quad dx = dx_1 dx_2 ... dx_d$, and

$$\|f\|_{\infty,\alpha} = \|f\|_\infty = ess \sup_{x \in \mathbb{R}_+^d} |f(x)|. \tag{6}$$

We denote by the function

$$\omega_{\alpha,\lambda}(x) = j_{\alpha_1}(\lambda_1 x_1) j_{\alpha_2}(\lambda_2 x_2) ... j_{\alpha_d}(\lambda_d x_d), \tag{7}$$

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_d) \in \mathbb{C}^d$ and $x = (x_1, x_2, ..., x_d) \in \mathbb{R}_+^d$, the Fourier-Bessel kernel on $\mathbb{C}^d \times \mathbb{R}^d$ of the Fourier-Bessel operator B_α.

Where $j_{\alpha_i}(x)$ is the Bessel function of the first kind and $\Gamma(x)$ is the gamma function defined by,

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt.$$

$$j_{\alpha_i}(z) = \Gamma(\alpha_i + 1) \sum_{n=0}^\infty \frac{(-1)^n}{n! \Gamma(n + \alpha_i + 1)} (\frac{z}{2})^{2n}, \quad z \in \mathbb{C}.$$

From the properties of the function j_{α_i} the function $(\lambda, x) \mapsto \omega_{\alpha,\lambda}(x)$ satisfies the following properties:

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Proposition 1. *For all $\lambda \in \mathbb{C}^d$ and $x \in \mathbb{R}^d$ we have:*

$$|\omega_{\alpha,\lambda}(x)| \le e^{\|Im\lambda\|.\|x\|}. \tag{8}$$

In particular, for all $\lambda \in \mathbb{C}^d$ and $x \in \mathbb{R}^d$ we have:

$$|\omega_{\alpha,\lambda}(x)| \le 1. \tag{9}$$

Definition 1. The multidimensional Fourier-Bessel transform of order the multi-index α is defined for a multivariate function $f \in L^1_\alpha(\mathbb{R}^d_+)$ by:

$$\mathcal{F}_\alpha(f)(\lambda) = \int_{\mathbb{R}^d_+} f(x)\omega_{\alpha,\lambda}(x)x^{2\alpha+1}dx \quad \text{for all} \quad \lambda \in \mathbb{R}^d_+. \tag{10}$$

Proposition 2. [9,10]

i) *The multidimensional Fourier-Bessel transform $\mathcal{F}_\alpha$ maps continously and injectively from $L^1_\alpha(\mathbb{R}^d_+)$ into the space $C_0(\mathbb{R}^d_+)$(continous function on $\mathbb{R}^d_+$ vanishing at infinity).*

ii) *If both f and $\mathcal{F}_\alpha(f)$ are in $L^1_\alpha(\mathbb{R}^d_+)$, then*

$$f(x) = \int_{\mathbb{R}^d_+} \mathcal{F}_\alpha(f)(\lambda)\omega_{\alpha,\lambda}(x)d\mu_\alpha(\lambda), \tag{11}$$

where

$$d\mu_\alpha(\lambda) = \prod_{i=1}^d \left(2^{\alpha_i}\Gamma(\alpha_i+1)^{-2}\lambda_i^{2\alpha_i+1}d\lambda_i \right). $$

iii) *For every $f \in L^1_\alpha(\mathbb{R}^d_+) \cap L^2_\alpha(\mathbb{R}^d_+)$, we have*

$$\int_{\mathbb{R}^d_+} |\mathcal{F}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda) = \int_{\mathbb{R}^d_+} |f(x)|^2 x^{2\alpha+1}dx. \tag{12}$$

iv) *For $f \in L^1_\alpha(\mathbb{R}^d_+)$,*

$$\|\mathcal{F}_\alpha(f)\|_{\infty,\alpha} \le \|f\|_{1,\alpha}. \tag{13}$$

v) *The multidimensional Fourier-Bessel transform $\mathcal{F}_\alpha$ extends uniquely to an isometric isomorphism from $L^2_\alpha(\mathbb{R}^d_+)$ onto $L^2_\alpha(\mathbb{R}^d_+, d\mu_\alpha)$. The inverse transform is given by:*

$$\mathcal{F}_\alpha^{-1}(g)(x) = \int_{\mathbb{R}^d_+} g(\lambda)\omega_{\alpha,\lambda}(x)d\mu_\alpha(\lambda), $$

where the integral converges in $L^2_\alpha(\mathbb{R}^d_+)$.

By using relations (11) and (12) with Marcinkiewicz's interpolation theorem [12], we deduce that for every $1 \le p \le 2$, and for every $f \in L^p_\alpha(\mathbb{R}^d_+)$, the function $\mathcal{F}_\alpha(f)$ belongs to the space $L^q_\alpha(\mathbb{R}^d_+, d\mu_\alpha(x))$, with $q = \frac{p}{p-1}$, and we have,

$$\|\mathcal{F}_\alpha(f)\|_{q,\alpha} \le \|f\|_{p,\alpha}. \tag{14}$$

3. An $L^p - L^q-$version of Morgan's theorem for the multidimensional Fourier-Bessel transform

In this section, we establish an analogue of $L^p - L^q$-version of Morgan's theorem for the multidimensional Fourier-Bessel transform. We start by getting the following lemma of Phragmén-Lindelöf type that follows using the method described in [1,2,13]. We need this result to prove the main result of this paper.

Lemma 1. *(Phragmén-Lindelöf) We assume that $\rho \in]1;2[$, $q \in [1,+\infty]$, $\sigma > 0$, and $B > \sigma sin(\frac{\pi}{2}(\rho-1))$.If g is an entire funstion on $\mathbb{C}$ satisfying :*

$$|g(x+iy)| \le const.e^{\sigma|y|^\rho}, \tag{15}$$

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and

$$e^{B|x|^\rho} g_{/\mathbb{R}} \in L_\alpha^q \left(\mathbb{R}, |r|^{2\alpha+1} dr \right), \tag{16}$$

for all $x, y \in \mathbb{R}$, then $g = 0$.

Proof. See [2]. $\square$

It's clear that we have the following lemma by using (8), Hölder's inequality and the theorem of differentiation under the integral sign.

Lemma 2. *Let $p \in [1, +\infty], \gamma > 2$ and f a measurable function on $\mathbb{R}_+^d$ such that for all $a > 0$:*

$$e^{a\|x\|^\gamma} f(x) \in L_\alpha^p(\mathbb{R}_+^d), \tag{17}$$

then the function defined on $\mathbb{C}^d$ by:

$$\mathcal{F}_\alpha(\lambda) = \int_{\mathbb{R}_+^d} f(x) \omega_{\alpha,\lambda}(x) x^{2\alpha+1} dx \quad for\ all \quad x \in \mathbb{R}_+^d, \tag{18}$$

is well defined and entire on $\mathbb{C}^d$; moreover, we have:

$$\forall\ \xi, \zeta \in \mathbb{R}_+^d \quad |\mathcal{F}_\alpha(f)(\xi + i\zeta)| \leq \int_{\mathbb{R}_+^d} |f(x)| e^{\|x\|\|\zeta\|} x^{2\alpha+1} dx. \tag{19}$$

The following Morgan-type theorem may be viewed as the analogue of the classical Morgan uncertainty principle for the Euclidean Fourier transform. In the Fourier–Bessel setting, the weighted measure $x^{2\alpha+1} dx$ and the oscillatory kernel $\omega_{\alpha,\lambda}(x)$ replace the role of the Euclidean structure. The proof follows the classical Phragmén–Lindelöf strategy but requires a careful adaptation to the growth properties of $\mathcal{F}_\alpha(f)$ as an entire function on $\mathbb{C}^d$.

Theorem 1. *Let $1 \leq p, q \leq \infty$, $b > 0$ $a > 0$, γ, η positive real numbers satisfying $\gamma > 2$ and $\frac{1}{\gamma} + \frac{1}{\eta} = 1$, then for all measurable function f on $\mathbb{R}_+^d$, the conditions:*

$$e^{a\|x\|^\gamma} f(x) \in L_\alpha^p(\mathbb{R}_+^d), \tag{20}$$

and

$$e^{b\|x\|^\eta} \mathcal{F}_\alpha f(x) \in L_\alpha^q(\mathbb{R}_+^d), \tag{21}$$

imply $f = 0$ almost everywhere if

$$(a\gamma)^{\frac{1}{\gamma}} (b\eta)^{\frac{1}{\eta}} > \left(sin \left(\frac{\pi}{2}(\eta - 1) \right) \right)^{\frac{1}{\eta}}. \tag{22}$$

Proof. Let f be a measurable function on $\mathbb{R}_+^d$ satisfying (20), we want to prove that the multidimensional Fourier-Bessel transform satisfies the hypotheses (15) and (16) of Lemma 1 and hence we deduce that $f = 0$ almost everywhere. We apply Hölder's inequality to (19) with: $\frac{1}{p} + \frac{1}{p'} = 1$

$$|\mathcal{F}_\alpha(f)(\xi + i\zeta)| \leq \left(\int_{\mathbb{R}_+^d} |f(x)| e^{a\|x\|^\gamma} x^{2\alpha+1} dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}_+^d} e^{-a\|x\|^\gamma} e^{\|x\|\|\zeta\|} x^{2\alpha+1} dx \right)^{\frac{1}{p'}}$$

$$\leq const \left(\int_{\mathbb{R}_+^d} e^{-ap'\|x\|^\gamma} e^{p'\|x\|\|\zeta\|} x^{2\alpha+1} dx \right)^{\frac{1}{p'}},$$

Let $I = \int_{\mathbb{R}_+^d} e^{-ap'\|x\|^\gamma} e^{p'\|x\|\|\zeta\|} x^{2\alpha+1} dx$, and let $C > 0$, we have the convexity inequality:

$$|ty| \leq \left(\frac{1}{\gamma} \right) |t|^\gamma + \left(\frac{1}{\eta} \right) |y|^\eta,$$

with $t = C \|x\|$ and $y = \left(\frac{1}{C}\right) \|\zeta\|$, we obtain:

$$\|x\| \|\zeta\| \leq \frac{C^{\gamma}}{\gamma} \|x\|^{\gamma} + \frac{1}{\eta C^{\eta}} \|\zeta\|^{\eta},$$

then we obtain

$$I \leq \int_{\mathbb{R}^d_+} e^{-ap'\|x\|^{\gamma}} e^{p'\frac{C^{\gamma}}{\gamma}\|x\|^{\gamma} + p'\frac{\|\zeta\|^{\eta}}{\eta C^{\eta}}} x^{2\alpha+1} dx$$

$$\leq e^{p'\frac{\|\zeta\|^{\eta}}{\eta C^{\eta}}} \int_{\mathbb{R}^d_+} e^{-p'(a-\frac{C^{\gamma}}{\gamma})\|x\|^{\gamma}} x^{2\alpha+1} dx.$$

Now, condition (22) implies

$$(a\gamma)^{1/\gamma} > (b\eta)^{-1/\eta} \left(\sin\left(\frac{\pi}{2}(\eta-1)\right)\right)^{1/\eta},$$

so the open interval

$$\left((b\eta)^{-1/\eta} \left(\sin\left(\frac{\pi}{2}(\eta-1)\right)\right)^{1/\eta}, (a\gamma)^{1/\gamma}\right),$$

is nonempty. Choose any C in this interval. Then:

    * $a - C^{\gamma}/\gamma > 0$, so the integral in I converges;

    * $b > \frac{1}{\eta C^{\eta}} \sin\left(\frac{\pi}{2}(\eta-1)\right)$, which ensures that the growth condition required in Lemma 1 is satisfied.

Consequently,

$$|\mathcal{F}_{\alpha} f(\xi + i\zeta)| \leq \text{const} \cdot e^{\frac{p'}{\eta C^{\eta}}\|\zeta\|^{\eta}},$$

and since $e^{b\|\cdot\|^{\eta}} \mathcal{F}_{\alpha} f \in L^q_{\alpha}(\mathbb{R}^d_+)$, we may apply Lemma (Phragmén–Lindelöf) with $\rho = \eta$, $\sigma = \frac{p'}{\eta C^{\eta}}$, and $B = bp'$, noting that $B > \sigma \sin\left(\frac{\pi}{2}(\eta-1)\right)$ by construction. It follows that $\mathcal{F}_{\alpha} f = 0$, and by the injectivity of $\mathcal{F}_{\alpha}$, we conclude $f = 0$ a.e. $\square$

Now we are going to study the sharpness of the condition

$$(a\gamma)^{\frac{1}{\gamma}} (b\eta)^{\frac{1}{\eta}} > \left(\sin\left(\frac{\pi}{2}(\eta-1)\right)\right)^{\frac{1}{\eta}}.$$

We start by the following proposition.

Proposition 3. *Let $a, b > 0$ and γ, η positive real numbers satisfying $\gamma > 2$ and $\eta = \frac{\gamma}{\gamma-1}$. If $(a\gamma)^{\frac{1}{\gamma}}(b\eta)^{\frac{1}{\eta}} = \left(\sin\left(\frac{\pi}{2}(\eta-1)\right)\right)^{\frac{1}{\eta}}$, then for all $m \in \mathbb{N}$ there exists a non-zero measurable function f on $\mathbb{R}^d_+$ satisfying the conditions:*

$$\left\|(1+\|x\|)^{-m} e^{a\|x\|^{\gamma}} f\right\|_{\infty,\alpha} \quad \text{and} \quad \left\|(1+\|x\|)^{-m'} e^{b\|x\|^{\eta}} \mathcal{F}_{\alpha}(f)\right\|_{\infty,\alpha},$$

where

$$m' = \frac{2m + (2-\gamma)(2\beta + d + 2) - 2}{2\gamma - 2},$$

and a, b, η, γ are given by the relation

$$(a\gamma)^{\frac{1}{\gamma}}(b\eta)^{\frac{1}{\eta}} = \left(\sin\left(\frac{\pi}{2}(\eta-1)\right)\right)^{\frac{1}{\eta}}.$$

Proof. We consider the following function as in [1],

$$f(x) = -i \int_C z^{\nu} e^{z^q - qA\|x\|^2 z} dz$$

where $q = \frac{\gamma}{\gamma-2}$, $A^\gamma = \frac{1}{4}\left((\gamma-2)a\right)^2$, $\nu = \frac{2m+4-\gamma}{2(\gamma-2)}$ and C the path which lies in the half plane $Re(z) > 0$, and goes to infinity, in the direction $\theta = arg(z) = \pm\theta_0$; $\frac{\pi}{2q} < \theta_0 < \frac{\pi}{2}$. By adapting the method of Morgan in [1], we obtain

$$f(x) \sim (\gamma-2)\left(\frac{(\gamma-2)}{2}a\right)^{\frac{m}{\gamma}}\sqrt{\frac{\pi}{\gamma}}x^m e^{-a\|x\|^\gamma}.$$

On the other hand

$$(\forall y \in \mathbb{R}_+^d): \ \mathcal{F}_\alpha(f)(y) = \int_{\mathbb{R}_+^d} f(x)\omega_{\alpha,y}(x)x^{2\alpha+1}dx$$
$$= -i\int_{\mathbb{R}_+^d}\int_C z^\nu e^{z^q - qA\|x\|^2 z}\omega_{\alpha,y}(x)x^{2\alpha+1}dxdz.$$

Using Fubini's theorem, we obtain:

$$(\forall y \in \mathbb{R}_+^d): \ \mathcal{F}_\alpha(f)(y) = -i\int_C z^\nu e^{z^q}\mathcal{F}_\alpha\left(e^{-qAz\|x\|^2}\right)dz.$$

From the fact that (see [8])

$$\mathcal{F}_\alpha\left(e^{-\frac{\|x\|^2}{4s}}\right)(x) = s^{\frac{d}{2}}(4s)^\alpha\left(\prod_{i=1}^d \Gamma(\alpha_i+1)\right)e^{-s\|x\|^2}. \tag{23}$$

We obtain,

$$\mathcal{F}_\alpha(f)(y) = -i\frac{\prod_{k=1}^d \Gamma(\alpha_k+1)}{2^d(qA)^{\alpha+d/2}}\int_C z^\nu e^{z^q}z^{-(\alpha+d/2)}e^{-\frac{\|x\|^2}{4Aqz}}dz$$
$$= -i\frac{\prod_{k=1}^d \Gamma(\alpha_k+1)}{2^d(qA)^{\alpha+d/2}}\int_C z^{\nu-(\alpha+d/2)}e^{z^q-\frac{\|x\|^2}{4Aqz}}dz.$$

$\square$

Using the saddle-point method in calculating the integral, we obtain that:

$$\forall y \in \mathbb{R}_+^d, \quad \mathcal{F}_\alpha(f)(y) = O\left(y^{m'}e^{-b\|y\|^\eta}\right),$$

with

$$m' = \frac{2m + (2-\gamma)(2\alpha+d) - 2}{2\gamma-2}.$$

Proposition 4. *Let $p,q \in [1,+\infty]$, $a,b > 0$ and γ,η positive real numbers satisfying $\gamma > 2$ and $\eta = \frac{\gamma}{\gamma-1}$. If $(a\gamma)^{\frac{1}{\gamma}}(b\eta)^{\frac{1}{\eta}} \leq \left(\sin\left(\frac{\pi}{2}(\eta-1)\right)\right)^{\frac{1}{\eta}}$, then there are infinitely many measurable functions f on $\mathbb{R}_+^d$ satisfying the conditions:*

$$e^{a\|x\|^\gamma}f \in L_\alpha^p(\mathbb{R}_+^d) \quad \text{and} \quad e^{b\|x\|^\eta}\mathcal{F}_\alpha(f) \in L_\alpha^q(\mathbb{R}_+^d).$$

Proof. We take

$$m' = \frac{2m + (2-\gamma)(2\alpha+d) - 2}{2\gamma-2}.$$

By Proposition 3, there exists a non zero measurable function f on $\mathbb{R}_+^d$, such that

$$\left\|(1+\|x\|)^{-m}e^{a\|x\|^\gamma}f\right\|_{\infty,\alpha} \quad \text{and} \quad \left\|(1+\|y\|)^{-m'}e^{b\|y\|^\eta}\mathcal{F}_\alpha(f)\right\|_{\infty,\alpha}.$$

Then, for all $x,y \in \mathbb{R}_+^d$, the function f satisfies the inequalities

$$e^{a\|x\|^\gamma}|f(x)| \leq const\,(1+\|x\|)^m \quad \text{and} \quad e^{b\|y\|^\eta}|\mathcal{F}_\alpha(f)(y)| \leq const\,(1+\|y\|)^{m'}.$$

The conditions

$$m < -\frac{2\alpha + d}{p} \qquad \text{and} \qquad m' < -\frac{2\alpha + d}{q},$$

implies that

$$\left\| e^{a\|x\|^\gamma} f \right\|_{p,\alpha} < +\infty \qquad \text{and} \qquad \left\| e^{b\|y\|^\eta} \mathcal{F}_\alpha(f) \right\|_{q,\alpha} < +\infty,$$

which proves that there exists infinitely many measurable functions on $\mathbb{R}_+^d$ satisfying the conditions when $(a\gamma)^{\frac{1}{\gamma}}(b\eta)^{\frac{1}{\eta}} \le \left(sin \left(\frac{\pi}{2}(\eta - 1) \right) \right)^{\frac{1}{\eta}}$. $\square$

4. Donoho-Stark uncertainty principle for the multidimensional Fourier-Bessel transform

In [4], Donoho and Stark demonstrate that it is not required to assume that the support and spectrum (transform support) are concentrated on intervals, and that intervals can be substituted by measurable sets, with the length of the interval naturally replaced by the set's measure.

In the section, we establish the analogue of Donoho-Stark uncertainty principle for the multidimensional Fourier-Bessel transform based on the techniques of Donoho and Stark [4]. First, we revist the concept of ϵ−concentration of a signal on a measurable set $E \subseteq \mathbb{R}_+^d$. We show the uncertainty inequality of Donoho-Stark type on $L_\alpha^1(\mathbb{R}_+^d) \cap L_\alpha^p(\mathbb{R}_+^d)$ for $1 < p \le 2$. If F is a set of finite measure of $\mathbb{R}_+^d$, we put: $|F| = \int_F x^{2\alpha+1}dx$. Note that $0 < \epsilon_E, \epsilon_F < 1$.

A function $f \in L_\alpha^p(\mathbb{R}_+^d)$ is ϵ_E−concentrated on a measurable set $E \subseteq \mathbb{R}_+^d$. If there is a measurable function g vanishing outside E, such that

$$\|f - g\|_{p,\alpha} \le \epsilon_E \|f\|_{p,\alpha}. \tag{24}$$

Therefore, if we introduce a projection operator P_E as,

$$(P_E f)(x) = \begin{cases} f(x) & , \quad x \in E, \\ 0 & , \quad x \in \mathbb{R}_+^d \setminus E, \end{cases} \tag{25}$$

$$P_E f = \chi_E f.$$

With χ_E the characteristic function of E. Then, f is ϵ_E−concentrated on E if and only if,

$$\|f - P_E f\|_{p,\alpha} \le \epsilon_E \|f\|_{p,\alpha}. \tag{26}$$

We define a projection operator Q_E as follows,

$$\mathcal{F}_\alpha(Q_E f) = P_E(\mathcal{F}_\alpha(f)). \tag{27}$$

Then, $\mathcal{F}_\alpha(f)$ is ϵ_F−concentrated on F if and only if,

$$\|\mathcal{F}_\alpha(f) - \mathcal{F}_\alpha(Q_F f)\|_{q,\alpha} \le \epsilon_F \|\mathcal{F}_\alpha(f)\|_{q,\alpha}. \tag{28}$$

Lemma 3. *If $|F| < \infty$ and $f \in L_\alpha^1(\mathbb{R}_+^d) \cap L_\alpha^p(\mathbb{R}_+^d)$ for $1 \le p \le 2$, then we have*

$$Q_F(f)(x) = \int_F \mathcal{F}_\alpha(f)(\lambda)\omega_{\alpha,\lambda}(x)d\mu_\alpha(\lambda). \tag{29}$$

Proof. Let $f \in L_\alpha^1(\mathbb{R}_+^d) \cap L_\alpha^p(\mathbb{R}_+^d)$ for $1 \le p \le 2$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$\begin{aligned} \|\mathcal{F}_\alpha(Q_F(f))\|_{1,\alpha} &= \int_F |\mathcal{F}_\alpha(f)(x)| \, x^{2\alpha+1}dx \\ &\le |F|^{\frac{1}{p}} \|\mathcal{F}_\alpha(f)\|_{q,\alpha} \\ &\le |F|^{\frac{1}{p}} \|f\|_{p,\alpha}, \end{aligned}$$

and we have also,

$$\|\mathcal{F}_\alpha\left(Q_F(f)\right)\|_2 = \left(\int_F |\mathcal{F}_\alpha(f)(x)|^2\, x^{2\alpha+1}dx\right)^{\frac{1}{2}}$$
$$\leq |F|^{\frac{q-2}{2q}}\, \|\mathcal{F}_\alpha(f)\|_{q,\alpha}$$
$$\leq |F|^{\frac{q-2}{2q}}\, \|f\|_{p,\alpha}.$$

Then, $\mathcal{F}_\alpha\left(Q_F(f)\right) \in L_\alpha^1(\mathbb{R}_+^d) \cap L_\alpha^2(\mathbb{R}_+^d)$ and by (11), we obtain,

$$Q_F(f) = \mathcal{F}_\alpha^{-1}\left[P_F \mathcal{F}_\alpha(f)\right].$$

Which completes the proof. □

Lemma 4. *Let $f \in L_\alpha^1(\mathbb{R}_+^d) \cap L_\alpha^p(\mathbb{R}_+^d)$ for $1 \leq p \leq 2$, then we have*

$$\|\mathcal{F}_\alpha\left(Q_F(f)\right)\|_{q,\alpha} \leq \|f\|_{p,\alpha}. \tag{30}$$

Proof. Let $f \in L_\alpha^1(\mathbb{R}_+^d) \cap L_\alpha^p(\mathbb{R}_+^d)$ for $1 \leq p \leq 2$, and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\mathcal{F}_\alpha\left(Q_F(f)\right)\|_{q,\alpha} = \left(\int_F |\mathcal{F}_\alpha(f)(\xi)|^q\, \xi^{2\alpha+1}d\xi\right)^{\frac{1}{q}}$$
$$\leq \|\mathcal{F}_\alpha(f)\|_{q,\alpha}$$
$$\leq \|f\|_{p,\alpha}.$$

Which completes the proof. □

Lemma 5. *Let $f \in L_\alpha^1(\mathbb{R}_+^d) \cap L_\alpha^p(\mathbb{R}_+^d)$ for $1 \leq p \leq 2$, then we have*

$$\|\mathcal{F}_\alpha\left(Q_F P_E f\right)\|_{q,\alpha} \leq |E|^{\frac{1}{q}}|F|^{\frac{1}{q}}\, \|f\|_{p,\alpha}. \tag{31}$$

Proof. If $|E| = \infty$ or $|F| = \infty$, the inequality is clear. Assume that $|E| < \infty$ and $|F| < \infty$. Let $f \in L_\alpha^1(\mathbb{R}_+^d) \cap L_\alpha^p(\mathbb{R}_+^d)$ for $1 \leq p \leq 2$, and $\frac{1}{q} + \frac{1}{p} = 1$. We have,

$$\mathcal{F}_\alpha\left(Q_F P_E f\right) = \chi_F \mathcal{F}_\alpha\left(P_E f\right),$$

thus,

$$\|\mathcal{F}_\alpha\left(Q_F P_E f\right)\|_{q,\alpha} = \left(\int_F |\mathcal{F}_\alpha\left(P_E f\right)(x)|^q\, x^{2\alpha+1}dx\right)^{\frac{1}{q}}.$$

Since,

$$\mathcal{F}_\alpha\left(P_E f\right)(\lambda) = \int_E f(x)\omega_{\alpha,\lambda}(x)x^{2\alpha+1}dx.$$

By Hölder's inequality, we can get,

$$|\mathcal{F}_\alpha\left(P_E f\right)| \leq \left(\int_E |\omega_{\alpha,\lambda}(x)|^q\, x^{2\alpha+1}dx\right)^{\frac{1}{q}} \left(\int_E |f(x)|^p\, x^{2\alpha+1}dx\right)^{\frac{1}{p}}$$
$$\leq |E|^{\frac{1}{q}}\, \|f\|_{p,\alpha}.$$

Therefore,

$$\|\mathcal{F}_\alpha\left(Q_F P_E f\right)\|_{q,\alpha} \leq |E|^{\frac{1}{q}}\, \|f\|_{p,\alpha} \left(\int_F |\chi_F|^q\, x^{2\alpha+1}dx\right)^{\frac{1}{q}} \leq |E|^{\frac{1}{q}}|F|^{\frac{1}{q}}\, \|f\|_{p,\alpha}.$$

This completes the proof. □

The classical Donoho–Stark uncertainty principle states that a non-zero signal cannot be simultaneously well-concentrated in both time and frequency domains: if f is ε_T-concentrated on a set T and $\hat{f}$ (Fourier transform) is ε_S-concentrated on S, then $|T||S| \geq (1 - \varepsilon_T - \varepsilon_S)^2$. In our framework, the Lebesgue measure is replaced by the Fourier-Bessel weight $x^{2\alpha+1}dx$, and the Fourier transform is replaced by $\mathcal{F}_\alpha$. The projections P_E and Q_F are defined analogously via multiplication and transform-domain restriction. The following theorem shows that the same qualitative phenomenon holds: strong concentration in both domains forces the function to be small unless the weighted measures $|E|$ and $|F|$ are sufficiently large.

Theorem 2. *Let E and F be measurable subsets of $\mathbb{R}_+^d$ and $f \in L_\alpha^1(\mathbb{R}_+^d) \cap L_\alpha^p(\mathbb{R}_+^d)$ for $1 \leq p \leq 2$, and $\frac{1}{p} + \frac{1}{q} = 1$. If f is ϵ_E-concentrated on E in $L_\alpha^p(\mathbb{R}_+^d)-$norm, then*

$$\|\mathcal{F}_\alpha(f)\|_{q,\alpha} \leq \frac{\epsilon_E + |E|^{\frac{1}{q}}|F|^{\frac{1}{q}}}{1 - \epsilon_F} \|f\|_{p,\alpha}. \tag{32}$$

Proof. Let $f \in L_\alpha^1(\mathbb{R}_+^d) \cap L_\alpha^p(\mathbb{R}_+^d)$ for $1 \leq p \leq 2$, and $\frac{1}{p} + \frac{1}{q} = 1$. Applying (14), (31) and the triangle inequality,

$$\begin{aligned}
\|\mathcal{F}_\alpha(f) - \mathcal{F}_\alpha(Q_F P_E f)\|_{q,\alpha} &\leq \|\mathcal{F}_\alpha(f) - \mathcal{F}_\alpha(Q_F f)\|_{q,\alpha} + \|\mathcal{F}_\alpha(Q_F f) - \mathcal{F}_\alpha(Q_F P_E f)\|_{q,\alpha} \\
&\leq \epsilon_F \|\mathcal{F}_\alpha(f)\|_{q,\alpha} + \|\mathcal{F}_\alpha(Q_F f - Q_F P_E f)\|_{q,\alpha} \\
&\leq \epsilon_F \|\mathcal{F}_\alpha(f)\|_{q,\alpha} + \|f - P_E f\|_{p,\alpha} \\
&\leq \epsilon_F \|\mathcal{F}_\alpha(f)\|_{q,\alpha} + \epsilon_E \|f\|_{p,\alpha}.
\end{aligned}$$

Now, we apply the Lemma 5 and the triangle inequality again to get,

$$\|\mathcal{F}_\alpha(f)\|_{q,\alpha} \leq \|\mathcal{F}_\alpha(f) - \mathcal{F}_\alpha(Q_F P_E f)\|_{q,\alpha} + \|\mathcal{F}_\alpha(Q_F P_E f)\|_{q,\alpha}.$$

This completes the proof. □

Corollary 1. *For $q = 2$, the expression (32) becomes,*

$$|E||F| \geq (1 - \epsilon_E - \epsilon_F)^2. \tag{33}$$

Theorem 3. *Let E and F be measurable subsets of $\mathbb{R}_+^d$ and $\mathcal{F}_\alpha(f) \in L_\alpha^1(\mathbb{R}_+^d) \cap L_\alpha^p(\mathbb{R}_+^d)$ for $1 \leq p \leq 2$, and $\frac{1}{p} + \frac{1}{q} = 1$. If f is ϵ_E-concentrated on E in $L_\alpha^1(\mathbb{R}_+^d)-$norm, and $\mathcal{F}_\alpha(f)$ is ϵ_F-concentrated on F in $L_\alpha^q(\mathbb{R}_+^d)-$norm, then*

$$\|\mathcal{F}_\alpha(f)\|_{q,\alpha} \leq \frac{|E|^{\frac{1}{q}}|F|^{\frac{1}{q}}}{(1 - \epsilon_E)(1 - \epsilon_F)} \|f\|_{p,\alpha}. \tag{34}$$

Proof. Let $f \in L_\alpha^1(\mathbb{R}_+^d) \cap L_\alpha^p(\mathbb{R}_+^d)$ for $1 \leq p \leq 2$, and $\frac{1}{p} + \frac{1}{q} = 1$. Since $\mathcal{F}_\alpha(f)$ is ϵ_F-concentrated on F in $L_\alpha^q(\mathbb{R}_+^d)-$norm, then by (13) and the triangle inequality, we get

$$\begin{aligned}
\|\mathcal{F}_\alpha(f)\|_{q,\alpha} &\leq \|\mathcal{F}_\alpha(f) - \mathcal{F}_\alpha(Q_F f)\|_{q,\alpha} + \|\mathcal{F}_\alpha(Q_F f)\|_{q,\alpha} \\
&\leq \epsilon_F \|\mathcal{F}_\alpha(f)\|_{q,\alpha} + \left(\int_F |\mathcal{F}_\alpha(f)(x)|^q \, x^{2\alpha+1}dx \right)^{\frac{1}{q}} \\
&\leq \epsilon_F \|\mathcal{F}_\alpha(f)\|_{q,\alpha} + |F|^{\frac{1}{q}} \|\mathcal{F}_\alpha(f)\|_\infty \\
&\leq \epsilon_F \|\mathcal{F}_\alpha(f)\|_{q,\alpha} + |F|^{\frac{1}{q}} \|f\|_{1,\alpha}.
\end{aligned}$$

This will lead to

$$\|\mathcal{F}_\alpha(f)\|_{q,\alpha} \leq \frac{|F|^{\frac{1}{q}}}{1 - \epsilon_F} \|f\|_{1,\alpha}. \tag{35}$$

Since f is ϵ_E−concentrated on E in $L^1_\alpha(\mathbb{R}^d_+)$−norm, we can immediately obtain that,

$$\begin{aligned}
\|f\|_{1,\alpha} &\leq \|f - P_E f\|_{1,\alpha} + \|P_E f\|_{1,\alpha} \\
&\leq \epsilon_E \|f\|_{1,\alpha} + \int_E |f(x)| \, x^{2\alpha+1} dx \\
&\leq \epsilon_E \|f\|_{1,\alpha} + |E|^{\frac{1}{q}} \|f\|_{p,\alpha} \,,
\end{aligned}$$

then, we get

$$\|f\|_{1,\alpha} \leq \frac{|E|^{\frac{1}{q}}}{1 - \epsilon_E} \|f\|_{p,\alpha} . \tag{36}$$

By inserting (36) into (35), we obtain (34). □

This theorem generalizes the result obtained by Donoho-Stark in [10]. In the particular case, when $p = 2$, we obtain the following corollary.

Corollary 2. *For $q = 2$, we have*

$$\sqrt{|E||F|} \geq (1 - \epsilon_E)(1 - \epsilon_F). \tag{37}$$

Now we generalize the Theorem 3 on the space $L^{p_1}_\alpha(\mathbb{R}^d_+) \cap L^{p_2}_\alpha(\mathbb{R}^d_+)$. In fact, when the space $L^p_\alpha(\mathbb{R}^d_+)$ is substituted by $L^{p_1}_\alpha(\mathbb{R}^d_+) \cap L^{p_2}_\alpha(\mathbb{R}^d_+)$, we find the following result.

Theorem 4. *Let E and F be measurable subsets of $\mathbb{R}^d_+$ and $f \in L^1_\alpha(\mathbb{R}^d_+) \cap L^{p_1}_\alpha(\mathbb{R}^d_+) \cap L^{p_2}_\alpha(\mathbb{R}^d_+)$ for $1 < p_1 < p_2 \leq 2$, and $\frac{1}{p_1} + \frac{1}{q_1} = 1$, $\frac{1}{p_2} + \frac{1}{q_2} = 1$. If f is ϵ_E−concentrated on E in $L^{p_1}_\alpha(\mathbb{R}^d_+)$−norm and $\mathcal{F}_\alpha(f)$ is ϵ_F−concentrated on F in $L^{q_2}_\alpha(\mathbb{R}^d_+)$−norm then*

$$\|\mathcal{F}_\alpha(f)\|_{q_2,\alpha} \leq \frac{|E|^{\frac{p_2-p_1}{p_2 p_1}} |F|^{\frac{q_1-q_2}{q_1 q_2}}}{(1 - \epsilon_E)(1 - \epsilon_F)} \|f\|_{p_2,\alpha} . \tag{38}$$

Proof. Let E and F be measurable subsets of $\mathbb{R}^d_+$ and $f \in L^1_\alpha(\mathbb{R}^d_+) \cap L^{p_1}_\alpha(\mathbb{R}^d_+) \cap L^{p_2}_\alpha(\mathbb{R}^d_+)$ for $1 < p_1 < p_2 \leq 2$, and $\frac{1}{p_1} + \frac{1}{q_1} = 1$, $\frac{1}{p_2} + \frac{1}{q_2} = 1$. Since $\mathcal{F}_\alpha(f)$ is ϵ_F−concentrated on F in $L^{q_2}_\alpha(\mathbb{R}^d_+)$−norm, then

$$\begin{aligned}
\|\mathcal{F}_\alpha(f)\|_{q_2,\alpha} &\leq \epsilon_F \|\mathcal{F}_\alpha(f)\|_{q_2,\alpha} + \left(\int_F |\mathcal{F}_\alpha(f)(x)|^{q_2} \, x^{2\alpha+1} dx \right)^{\frac{1}{q_2}} \\
&\leq \epsilon_F \|\mathcal{F}_\alpha(f)\|_{q_2,\alpha} + |F|^{\frac{q_1-q_2}{q_1 q_2}} \|f\|_{q_1,\alpha} \\
&\leq \frac{|F|^{\frac{q_1-q_2}{q_1 q_2}}}{1 - \epsilon_F} \|f\|_{q_1,\alpha} \,,
\end{aligned}$$

thus, by (14), we obtain,

$$\|\mathcal{F}_\alpha(f)\|_{q_2,\alpha} \leq \frac{|F|^{\frac{q_1-q_2}{q_1 q_2}}}{1 - \epsilon_F} \|f\|_{p_1,\alpha} . \tag{39}$$

On the other hand, since f is ϵ_E−concentrated on E in $L^{p_1}_\alpha(\mathbb{R}^d_+)$−norm, we can immediately obtain that,

$$\begin{aligned}
\|f\|_{p_1,\alpha} &\leq \epsilon_E \|f\|_{p_1,\alpha} + \left(\int_E |f(x)|^{p_1} \, x^{2\alpha+1} dx \right)^{\frac{1}{p_1}} \\
&\leq \epsilon_E \|f\|_{p_1,\alpha} + |E|^{\frac{p_2-p_1}{p_1 p_2}} \|f\|_{p_2,\alpha} \,,
\end{aligned}$$

then, we get

$$\|f\|_{p_1,\alpha} \leq \frac{|E|^{\frac{p_2-p_1}{p_1 p_2}}}{1 - \epsilon_E} \|f\|_{p_2,\alpha} . \tag{40}$$

By combining (40) and (39), we immediately get the desired result.   □

Corollary 3. *If $p_2 = 2$, the formula (38) becomes,*

$$|E|^{\frac{2-p}{2p}}|F|^{\frac{q-2}{2q}} \geq (1 - \epsilon_E)(1 - \epsilon_F),\tag{41}$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

5. Bandlimited principles for the multidimensional Fourier-Bessel transform

Based on the works of Donoho and Stark in [4] and Soltani in [14], we show bandlimited principles of concentration type on $L_\alpha^p(\mathbb{R}_+^d)$ for $1 < p \leq 2$ and generalize this result on $L_\alpha^p(\mathbb{R}_+^d) \cap L_\alpha^r(\mathbb{R}_+^d)$ for $1 < p < r \leq 2$.

Definition 2. For $1 \leq r \leq 2$, we denote by $\mathcal{B}^r(T)$ the set of the functions $h \in L_\alpha^1(\mathbb{R}_+^d) \cap L_\alpha^r(\mathbb{R}_+^d)$ that are bandlimited to T. It means that every $h \in \mathcal{B}^r(T)$ holds $Q_T h = h$. Moreover, it is said that f is ϵ_T-bandlimited to T in $L_\alpha^r(\mathbb{R}_+^d)-$norm if there exists a function $h \in \mathcal{B}^r(T)$ such that,

$$\|f - g\|_{r,\alpha} \leq \epsilon_T \|f\|_{r,\alpha}.\tag{42}$$

The space $\mathcal{B}^r(T)$ satisfies the following properties.

Lemma 6. *Let E and T be measurable subsets of $\mathbb{R}_+^d$. For $h \in \mathcal{B}^r(T)$ with $1 \leq r \leq 2$, we have,*

$$\|P_E h\|_{r,\alpha} \leq |E|^{\frac{1}{r}}|T|^{\frac{1}{r}} \|h\|_{r,\alpha},\tag{43}$$

with $\frac{1}{r} + \frac{1}{r'} = 1$.

Proof. If $|E| = \infty$ or $|T| = \infty$, the inequality is clear. Assume that $|E| < \infty$ and $|T| < \infty$. For $h \in \mathcal{B}^r(T)$ with $\frac{1}{r} + \frac{1}{r'} = 1$. From Lemma 3, we have

$$Q_T f(t) = \int_T \mathcal{F}_\alpha(f)(\lambda)\omega_{\alpha,\lambda}(t)d\mu_\alpha(\lambda).$$

Using Hölder's inequality together with Hausdroff-Young inequality, we get,

$$|h(t)| \leq |T|^{\frac{1}{r}} \|\mathcal{F}_\alpha(h)\|_{r',\alpha}$$
$$\leq |T|^{\frac{1}{r}} \|h\|_{r,\alpha},$$

hence, we have

$$\|P_E h\|_{r,\alpha} = \left(\int_E |h(t)|^r dt\right)^{\frac{1}{r}}$$
$$\leq |T|^{\frac{1}{r}}|E|^{\frac{1}{r}} \|h\|_{r,\alpha},$$

which completes the proof.   □

The following concentration principles for bandlimited functions generalize the results of Donoho–Stark and Soltani in the Euclidean setting. Here, bandlimitedness is defined with respect to the support of the Fourier–Bessel transform, and the inequalities reflect the interaction between time-concentration and weighted frequency-localization. In classical signal processing, a bandlimited function cannot be arbitrarily concentrated in time; this is the essence of the "energy concentration" or "prolate spheroidal" problem studied by Slepian, Landau, and Pollak [15–19]. Donoho and Stark and later Soltani extended this idea to approximate concentration and bandlimiting. Here, we adapt these ideas to the Fourier–Bessel setting: a function that is nearly bandlimited to a set T (in the sense that it is close in L_α^r-norm to a function whose transform vanishes outside T) cannot be too concentrated on a small-measure set $E \subset \mathbb{R}_+^d$ unless $|E||T|$ is large.

Theorem 5. *Let E and T be measurable subsets of $\mathbb{R}^d_+$ and $f \in L^1_\alpha(\mathbb{R}^d_+) \cap L^r_\alpha(\mathbb{R}^d_+)$ for $1 \le r \le 2$, and $\frac{1}{r} + \frac{1}{r'} = 1$. If f is ϵ_T−bandlimited to T in $L^r_\alpha(\mathbb{R}^d_+)$−norm, then*

$$\|P_E f\|_{r,\alpha} \le \left(|T|^{\frac{1}{r}} |E|^{\frac{1}{r}} (1 + \epsilon_T) + \epsilon_T \right) \|f\|_{r,\alpha}. \tag{44}$$

Proof. Since the function f is ϵ_E−bandlimited to E in $L^r_\alpha(\mathbb{R}^d_+)$−norm, then there exists $h \in \mathcal{B}^r(E)$, satisfying,

$$\|P_E f\|_{r,\alpha} \le \|P_E h\|_{r,\alpha} + \|P_E(f - h)\|_{r,\alpha}$$
$$= \|P_E h\|_{r,\alpha} + \left(\int_E |(f-h)(t)|^r \, dt \right)^{\frac{1}{r}}$$
$$\le \|P_E h\|_{r,\alpha} + \left(\int_{\mathbb{R}^d_+} |(f-h)(t)|^r \, dt \right)^{\frac{1}{r}}.$$

Then, we obtain

$$\|P_E f\|_{r,\alpha} \le \|P_E h\|_{r,\alpha} + \epsilon_T \|f\|_{r,\alpha}. \tag{45}$$

It is evident to see,

$$\|h\|_{r,\alpha} \le (1 + \epsilon_T) \|f\|_{r,\alpha}. \tag{46}$$

By combining (43) and (46) and inserting them into (45), then we obtain the desired result.   □

In the following we state an $L^p_\alpha(\mathbb{R}^d_+) \cap L^r_\alpha(\mathbb{R}^d_+)$ bandlimited uncertainty principles of concentration type.

Theorem 6. *Let E and F be measurable subsets of $\mathbb{R}^d_+$ and $f \in L^p_\alpha(\mathbb{R}^d_+) \cap L^r_\alpha(\mathbb{R}^d_+)$, with $1 \le p \le r \le 2$. If f is ϵ_E−concentrated to E in $L^p_\alpha(\mathbb{R}^d_+)$−norm and ϵ_F−bandlimited to F in $L^r_\alpha(\mathbb{R}^d_+)$−norm, then*

$$\|f\|_{p,\alpha} \le \frac{|E|^{\frac{r-p}{rp}}}{1 - \epsilon_F} \left[|E|^{\frac{1}{r}} |F|^{\frac{1}{r}} (1 + \epsilon_F) + \epsilon_F \right] \|f\|_{r,\alpha}, \tag{47}$$

where $\frac{1}{s} + \frac{1}{r} = 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Assume that $|E| < \infty$ and $|F| < \infty$. Let $f \in L^p_\alpha(\mathbb{R}^d_+) \cap L^r_\alpha(\mathbb{R}^d_+)$ with $1 \le p \le r \le 2$. Since f is ϵ_E−concentrated to E in $L^p_\alpha(\mathbb{R}^d_+)$−norm, then by Hölder's inequality, we deduce that,

$$\|f\|_{p,\alpha} \le \|f - P_E f\|_{p,\alpha} + \|P_E f\|_{p,\alpha}$$
$$\le \epsilon_E \|f\|_{p,\alpha} + \|P_E f\|_{p,\alpha}$$
$$\le \epsilon_E \|f\|_{p,\alpha} + |E|^{\frac{r-p}{pr}} \|P_E f\|_{r,\alpha}.$$

Then,

$$\|f\|_{p,\alpha} \le \frac{|E|^{\frac{r-p}{pr}}}{1 - \epsilon_E} \|P_E f\|_{r,\alpha}. \tag{48}$$

Since f is ϵ_F−bandlimited to F in $L^r_\alpha(\mathbb{R}^d_+)$−norm, by definition there is a $g \in \mathcal{B}^r(F)$ with $\|f - g\|_{r,\alpha} \le \epsilon_F \|f\|_{r,\alpha}$. Then we get,

$$\|P_E f\|_{r,\alpha} \le \|P_E g\|_{r,\alpha} + \|P_E(f - g)\|_{r,\alpha}$$
$$\le \|P_E g\|_{r,\alpha} + \epsilon_F \|f\|_{r,\alpha}.$$

We have $g \in \mathcal{B}^r(F)$, then $g(x) = \mathcal{F}^{-1}_\alpha (P_F \mathcal{F}_\alpha(g))$.

$$|g(x)| \le |F|^{\frac{1}{r}} \|\mathcal{F}_\alpha(g)\|_{s,\alpha}$$
$$\le |F|^{\frac{1}{r}} \|g\|_{r,\alpha},$$

hence, we get,

$$\|P_E g\|_{r,\alpha} = \left(\int_E |g(x)|^r \, x^{2\alpha+1} dx \right)^{\frac{1}{r}} \leq |F|^{\frac{1}{r}} |E|^{\frac{1}{r}} \|g\|_{r,\alpha}.$$

By (44) and the fact that $\|g\|_{r,\alpha} \leq (1+\epsilon_F) \|f\|_{r,\alpha}$. We get,

$$\|P_E g\|_{r,\alpha} \leq |F|^{\frac{1}{r}} |E|^{\frac{1}{r}} (1+\epsilon_F) \|f\|_{r,\alpha}.$$

$$\|P_E f\|_{r,\alpha} \leq \left[|F|^{\frac{1}{r}} |E|^{\frac{1}{r}} (1+\epsilon_F) + \epsilon_F \right] \|f\|_{r,\alpha}.$$

Which completes the proof. □

Corollary 4. *Let E and F be measurable subsets of $\mathbb{R}_+^d$ and $f \in L_\alpha^p(\mathbb{R}_+^d)$, with $1 \leq p \leq 2$. If f is ϵ_E-concentrated to E and ϵ_F-bandlimited to F in $L_\alpha^p(\mathbb{R}_+^d)-$norm, then*

$$|E|^{\frac{1}{p}} |F|^{\frac{1}{p}} \geq \frac{1-\epsilon_F-\epsilon_E}{1+\epsilon_F}. \tag{49}$$

References

[1] Morgan, G. (1934). A note on Fourier transforms. *Journal of the London Mathematical Society, 1*(3), 187-192.

[2] Ben Farah, S. (2003). Uncertainty principle and $L^p - L^q$-version of Morgan's theorem on some groups. *Russian Journal of Mathematical Physics, 10*, 245-260.

[3] Ben Farah, S., & Mokni, K. (2005). Uncertainty principle and $L^p - L^q$-version of Morgan's theorem on semi-simple lie groups. *Integral Transforms and Special Functions, 16*(4), 281-289.

[4] Donoho, D. L., & Stark, P. B. (1989). Uncertainty principles and signal recovery. *SIAM Journal on Applied Mathematics, 49*(3), 906-931.

[5] Smith, K. T. (1990). The uncertainty principle on groups. *SIAM Journal on Applied Mathematics, 50*(3), 876-882.

[6] Özaydin, M., & Przebinda, T. (2004). An entropy-based uncertainty principle for a locally compact abelian group. *Journal of Functional Analysis, 215*(1), 241-252.

[7] Alagic, G., & Russell, A. (2008). Uncertainty principles for compact groups. *Illinois Journal of Mathematics, 52*(4), 1315-1324.

[8] Chettaoui, C., & Othmani, Y. (2014). Real Paley-Wiener theorems for the multivariable Bessel transform. *International Journal of Open Problems in Complex Analysis, 6*(1), 90-110.

[9] Watson, G. N. (1944). *A Treatise on the Theory of Bessel Functions.* Cambridge University Press.

[10] Abilov, V. A., & Kerimov, M. K. (2012). Estimates for the Fourier-Bessel transforms of multivariate functions. *Computational Mathematics and Mathematical Physics, 52*(6), 836-845.

[11] Selmi, B., & Khelifi, C. (2020). Estimate of the Fourier-Bessel Multipliers for the Poly-axially Operator. *Acta Mathematica Sinica, 36*(7), 797-811.

[12] Stein, E. M. (1956). Interpolation of linear operators. *Transactions of the American Mathematical Society, 83*(2), 482-492.

[13] Mejjaoli, H., & Trimèche, K. (2006). An L^p-L^q-version of Morgan's theorem for the Dunkl–Bessel transform. *Integral Transforms and Special Functions, 17*(8), 591-602.

[14] Soltani, F. (2014). Donoho-Stark uncertainty principle associated with a singular second-order differential operator. *International Journal of Analysis and Applications, 4*(1), 1-10.

[15] Slepian, D., & Pollak, H. O. (1961). Prolate spheroidal wave functions, Fourier analysis and uncertainty—I. *Bell System Technical Journal, 40*(1), 43-63.

[16] Landau, H. J., & Pollak, H. O. (1961). Prolate spheroidal wave functions, Fourier analysis and uncertainty—II. *Bell System Technical Journal, 40*(1), 65-84.

[17] Landau, H. J., & Pollak, H. O. (1962). Prolate spheroidal wave functions, fourier analysis and uncertainty—III: the dimension of the space of essentially time-and band-limited signals. *Bell System Technical Journal, 41*(4), 1295-1336.

[18] Slepian, D. (1964). Prolate spheroidal wave functions, Fourier analysis and uncertainty—IV: extensions to many dimensions; generalized prolate spheroidal functions. *Bell System Technical Journal, 43*(6), 3009-3057.

[19] Slepian, D. (1978). Prolate spheroidal wave functions, Fourier analysis, and uncertainty—V: The discrete case. *Bell System Technical Journal, 57*(5), 1371-1430.