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Stieltjes function and q-Laguerre-Hahn character

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Abstract: In this work, we prove that the formal Stieltjes of q-Laguerre -hahn forms is a solution of many q-Ricatti equations. As a consequence , we show that the class of those forms depends on $k \in \mathbb{N}$. Some examples are highlighted.

Keywords: q-difference equation, Stieltjes Function, H_q-semi-classical, q-Laguerre-Hahn polynomials

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1. Introduction

The q-Laguerre-Hahn regular form u_q (linear functional) its formal Stieltjes function $S(z, u_q) = -\sum_{k\geq 0} \frac{(u)_k}{z^{k+1}}$, where $(u_q)_n := \langle u_q, x^n \rangle$, $n \geq 0$, fulfills the following q-Riccati equation [1–3]

$$\phi(q^{-1}z)H_{q^{-1}}(S(z,u_q)) = B(z)S(z,u_q)S(q^{-1}z,u_q) + C(z)S(z,u_q) + D(z), \tag{1}$$

where ϕ, B, C, D are a polynomials (ϕ monic) and H_q the q-derivative operator.

The monic orthogonal polynomials sequence associated to u_q is said to be H_q-Laguerre-Hahn orthogonal polynomials.

When the Eq. (1) can not be simplifying by any polynomial the class of u_q is defined by $s(u_q) := \max(\deg B - 2, \deg C - 1, \deg D)$ [3].

In the case where $B = 0$, we deal with the H_q-semi-classical forms [4] in particular the symmetrical H_q-semi-classical forms of class one are treated in [5].

Moreover, the symmetrical H_q-Laguerre-Hahn of class zero and one are exhaustively studied in [6,7] and some examples of non symmetrical H_q-Laguerre-Hahn of class two are given in [8] via Cristoffel an Geronimus transformations.

Notice that the first order q-difference equation for Stieltjes function is studied in [1] for the H_q-classical forms.

In this work, we prove that when the form u_q is q-Laguerre-Hahn form of class $s(u_q)$ satisfying (1), its Stieltjes function $S(z, u_q)$ satisfies also the q^{k+1}-Riccati equation, $k \in \mathbb{N}$:

$$\begin{aligned}(q^{-k-1}-1)zR_{k+1}(z)H_{q^{-k-1}}\left(S(z,u_q)\right) = &- B_{k+1}(z)S(z,u_q)S(q^{-k-1}z,u_q) - \left(R_{k+1}(z)+C_{k+1}(z)\right)S(z,u_q)\\ &- D_{k+1}(z),\end{aligned} \tag{2}$$

where $B_{k+1}, R_{k+1}, C_{k+1}$ and D_{k+1} are a polynomials for $k \in \mathbb{N}$ (see Proposition 1 below), which means that the same form u_q is also q^{k+1}-Laguerre-Hahn (respectively $H_{q^{k+1}}$-semi-classical) form relatively to the operator operator $H_{q^{k+1}}$. Consequently, the class of the form u_q depends on the integer k with respect to the operator $H_{q^{k+1}}$. Of course, we have $s(u_q) = s(u_{qk+1})$ since q is replaced by q^{k+1}.

Some examples are highlighted, when $u_q := \mathcal{T}_q$ (respectively $u_q := \mathcal{U}_q$), where $\mathcal{T}_q$ (respectively $\mathcal{U}_q$) is the q-Chebyshev form of the first kind (respectively the second kind) which is H_q-classical form [9], we establish the q-difference Eq. (2) satisfied by $S(z, \mathcal{T}_q)$ (respectively $S(z, \mathcal{U}_q)$) and we prove that $\mathcal{T}_q$ (respectively $\mathcal{U}_q$) it is a

$H_{q^{k+1}}$-semi-classical form of class $2k$, $k \geq 0$. In the case where $u_q := u_q(\omega)$ where $u_q(\omega)$ is the symmetric Brenk type form which is H_q-semi-classical form of class $s(u_q(\omega)) = 1$ [8], on one hand, we determine the Stieltjes function $S(z, u_q(\omega))$, in particular, we show that $S(z, u_q(0))$ it is a solution of an q-difference equation involving the Jacobi triple product identity [10,11], also the polynomials B_{k+1}, R_{k+1}, C_{k+1} and D_{k+1} are calculated. As a consequence, we prove that the form $u_q(\omega)$ is also $H_{q^{k+1}}$-semi-classical form of class $2k+1$, $k \geq 0$. On the other hand, we establish the q^{k+1}-difference Eq. (2) satisfied by $S(z, u_q^{(1)}(\omega))$, which allowed us to show that the first associated form $u_q^{(1)}(\omega)$ it is q^{k+1}-Laguerre-form of class $2k+1$, $k \geq 0$.

2. Preliminaries and fundamental results

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle w, f \rangle$ the effect of $w \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(w)_n := \langle w, x^n \rangle$, $n \geq 0$, the moments of w. Let us introduce some useful operations in $\mathcal{P}'$. For any linear form w, any polynomial g and any $a \in \mathbb{C} - \{0\}$, $\in \mathbb{C}$, we let gw, $h_a w$ and $H_q w$ be the forms (linear functionals) defined by duality

$$\langle gw, f \rangle := \langle w, gf \rangle, \ \langle h_a w, f \rangle := \langle w, h_a f \rangle, \ \langle H_q w, f \rangle ;= -\langle w, H_q f \rangle, \ f \in \mathcal{P},$$

where

$$H_q(f)(x) = \frac{f(qx) - f(x)}{(q-1)x}, x \neq 0, \ q \in \tilde{\mathbb{C}} := \mathbb{C} - \left(\{0\} \cup \bigsqcup_{n \geq 0} \{z \in \mathbb{C}, \ z^n = 1\} \right),$$
$$H_q(f)(0) = f'(0).$$

We also define the right-multiplication of a form by a polynomial with

$$(wf)(x) := \left\langle w, \frac{xf(x) - \xi f(\xi)}{x - \xi} \right\rangle, \ w \in \mathcal{P}', \ f \in \mathcal{P}.$$

This allows to define the product of two forms:

$$\langle uw, f \rangle := \langle u, wf \rangle, \ u, w \in \mathcal{P}', \ f \in \mathcal{P}.$$

The formal Stieltjes function of $w \in \mathcal{P}'$ is defined by

$$S(z, w) = -\sum_{n \geq 0} \frac{(w)_n}{z^{n+1}}, \ z \in \mathbb{C} - \{0\}.$$

Next, we define $H_q(S(z, w))$ as following

$$H_q(S(z, w)) = \frac{S(qz, w) - S(z, w)}{(q-1)z}, \ z \in \mathbb{C} - \{0\}.$$

We call polynomial sequence (PS), the sequence of polynomials $\{P_n\}_{n \geq 0}$ when $\deg P_n = n$, $n \geq 0$. Then any polynomial P_n can be supposed monic and the sequence becomes a monic polynomial sequence (MPS). The MPS $\{P_n\}_{n \geq 0}$ is orthogonal (MOPS) with respect to $w \in \mathcal{P}'$ when the following conditions hold $\langle w, P_m P_n \rangle = r_n \delta_{n,m}$, $n, m \geq 0$, $r_n \neq 0$, $n \geq 0$. In this case the form w is said regular. The form w is called normalized if $(w)_0 = 1$. In this paper, we suppose that the forms are normalized. Thus, $\{P_n\}_{n \geq 0}$ satisfies the standard recurrence relation

$$\begin{cases} P_0(x) = 1 \quad , \quad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1}(x) - \gamma_{n+1} P_n(x), \ n \geq 0; \ \gamma_{n+1} \neq 0, \ n \geq 0. \end{cases} \tag{3}$$

The regular form w is said to be symmetric when $(w)_{2n+1} = 0$, $n \geq 0$, or equivalently $\beta_n = 0$, $n \geq 0$ in (3) [12].

Given a regular form w and the corresponding MOPS $\{P_n\}_{n\geq 0}$ satisfying (3), we define the first associated sequence $\{P_n^{(1)}\}_{n\geq 0}$ by [12]

$$P_n^{(1)}(x) = \left\langle w, \frac{P_{n+1}(x) - P_{n+1}(\xi)}{x-\xi} \right\rangle = (w\theta_0 P_{n+1})(x). \tag{4}$$

We know that $\{P_n^{(1)}\}_{n\geq 0}$ is a MOPS with respect to $w^{(1)}$ satisfying (3) with [12]

$$\beta_n^{(1)} = \beta_{n+1}, \ \gamma_{n+1}^{(1)} = \gamma_{n+2}, \ n \geq 0. \tag{5}$$

Definition 1. [3] A form w is called q-Laguerre-Hahn when it is regular and its formal Stieltjes function $S(.,w)$ satisfies the follwoing q-Riccati equation

$$(h_{q^{-1}}\phi)(z)H_{q^{-1}}(S(z,w)) = B(z)S(z,w)S(q^{-1}z,w) + C(z)S(z,w) + D(z), \tag{6}$$

with ϕ, B, C and D are a polynomials (ϕ monic).

The class of the q-Laguerre-Hahn satisfying ((6)) is $s(w) = \max(\deg(B)-2, \deg(C)-1, \deg(D))$ if and only if [3]

$$\prod_{c\in Z_\phi} \Big\{ \mid B(cq) \mid + \mid C(cq) \mid + \mid D(cq) \mid \Big\} > 0, \tag{7}$$

where Z_ϕ is the set of roots of ϕ.

Remark 1. 1. When the conditions (7) are realized, Eq. (6) can not be simplified by any polynomial [3].
2. When $B=0$, we deal with the H_q-semi-classical forms [4].
3. The case, when $B=0$, $\deg(\phi)\leq 2$, $\deg(C)\leq 1$, $\deg(D)\leq 0$, corresponds to the H_q-classical forms [13].
4. By extension, the integer $s(w)$ is also the class of the MOPS $\{P_n\}_{n\geq 0}$ with respect to w.

In what follows, we assume that $q \in \tilde{\mathbb{C}}$.
We are going to use the following notations and results [14].
For $a, a_1, ..., a_n$:

$$(a,q)_n := \begin{cases} 1, & n=0, \\ \prod_{\nu=0}^{n-1}(1-aq^\nu), & n\geq 1, \end{cases} \tag{8}$$

$$(a,q)_\infty = \prod_{\nu\geq 0}(1-aq^\nu), \mid q \mid < 1, \tag{9}$$

$$(a_1, ..., a_n; q)_\infty = \prod_{j=1}^{n}(a_j,q)_\infty, \mid q \mid < 1. \tag{10}$$

3. The difference-equations satisfied by the Stieltjes function of q-Laguerre-Hahn forms

In the sequel, we suppose that u_q is q-Laguerre-Hahn form of class $s(u_q)$ satisfying the q-Ricatti equation [3]

$$\phi(q^{-1}z)H_{q^{-1}}(S(z,u_q)) = B(z)S(z,u_q)S(q^{-1}z,u_q) + C(z)S(z,u_q) + D(z). \tag{11}$$

Lemma 1. *The Eq. (11) it is equivalent to*

$$B_1(z)S(z,u_q)S(q^{-1}z,u_q) + R_1(z)S(q^{-1}z,u_q) + C_1(z)S(z,u_q) + D_1(z) = 0, \tag{12}$$

where

$$B_1(x) = (q^{-1}-1)xB(x),\ R_1(x) = -\phi(q^{-1}x),$$
$$C_1(x) = \phi(q^{-1}x) + (q^{-1}-1)xC(x),\ D_1(x) = (q^{-1}-1)xD(x). \tag{13}$$

Moreover, we have

$$(R_1, D_1),\ (C_1, D_1),\ (B_1, R_1),\ (B_1, C_1) \neq (0,0), \tag{14}$$

$$B_1D_1 - R_1C_1 \neq 0. \tag{15}$$

Proof. As, we have $H_{q^{-1}}(S(z,u_q)) = \frac{S(q^{-1}z,u_q)-S(z,u_q)}{(q^{-1}-1)z}$, Eq. (11) can be written as following

$$(q^{-1}-1)zB(z)S(z,u_q)S(q^{-1}z,u_q) - \phi(q^{-1}z)S(q^{-1}z,u_q) + \left(\phi(q^{-1}z) + (q^{-1}-1)zC(z)\right)S(z,u_q) + (q^{-1}-1)zD(z) = 0.$$

Which provides (12) with (13).

Since the $S(z,u_q)$ is not a rational function [12], we deduce the relations (14). Suppose that $B_1D_1 - R_1C_1 = 0$. Multiplying both sides identities of (12) by the polynomial B_1, we obtain

$$B_1^2(z)S(z,u_q)S(q^{-1}z,u_q) + B_1(z)R_1(z)S(q^{-1}z,u_q) + B_1(z)C_1(z)S(z,u_q) + B_1(z)D_1(z) = 0,$$

but $B_1D_1 = R_1C_1$, we get

$$B_1^2(z)S(z,u_q)S(q^{-1}z,u_q) + B_1(z)R_1(z)S(q^{-1}z,u_q) + B_1(z)C_1(z)S(z,u_q) + C_1(z)R_1(z) = 0.$$

Which is equivalent to

$$\left(B_1(z)S(q^{-1}z,u_q) + C_1(z)\right)\left(B_1(z)S(z,u_q) + R_1(z)\right) = 0.$$

Therefore $B_1(z)S(q^{-1}z,u_q) + R_1(z) = 0$ or $B_1(z)S(z,u_q) + C_1(z) = 0$, thus $S(z,u_q)$ is a rational function. Which is absurd, then $B_1D_1 - R_1C_1 \neq 0$. Hence (15). □

Remark 2. Let $M(z)$ be the matrix defined as following

$$M(z) = \begin{pmatrix} -R_1(z) & B_1(z) \\ -D_1(z) & C_1(z) \end{pmatrix}. \tag{16}$$

Proposition 1. *The Stieltjes function $S(z,u_q)$ satisfies the following q^{-k-1}-difference equation*

$$B_{k+1}(z)S(q^{-k-1}z,u_q)S(z,u_q) + R_{k+1}(z)S(q^{-k-1}z,u_q) + C_{k+1}(z)S(z,u_q) + D_{k+1}(z) = 0,\ k \in \mathbb{N}, \tag{17}$$

where

$$\begin{pmatrix} B_{k+1}(z) \\ C_{k+1}(z) \end{pmatrix} = M(q^{-k}z)\begin{pmatrix} B_k(z) \\ C_k(z) \end{pmatrix},\ k \geq 0,$$
$$\begin{pmatrix} R_{k+1}(z) \\ D_{k+1}(z) \end{pmatrix} = M(q^{-k}z)\begin{pmatrix} R_k(z) \\ D_k(z) \end{pmatrix}\ k \geq 0, \tag{18}$$

with

$$C_0 = 1, R_0 = -1, B_0 = D_0 = 0. \tag{19}$$

Moreover

$$\begin{pmatrix} B_{k+1}(z) \\ C_{k+1}(z) \end{pmatrix} = M(q^{-k}z)M(q^{-(k-1)}z)...M(z)\begin{pmatrix} 0 \\ 1 \end{pmatrix}, k \geq 0. \tag{20}$$

$$\begin{pmatrix} R_{k+1}(z) \\ D_{k+1}(z) \end{pmatrix} = M(q^{-k}z)M(q^{-(k-1)}z)...M(z)\begin{pmatrix} -1 \\ 0 \end{pmatrix}, k \geq 0. \tag{21}$$

In addition u_q is also q^{-k-1}-Laguerre-Hahn form satisfying the q^{k+1}-Ricatti equation

$$\begin{aligned}\varphi_{k+1}(q^{-k-1}z)H_{q^{-k-1}}\left(S(z,u_q)\right) = &- B_{k+1}(z)S(z,u_q)S(q^{-k-1}z,u_q) \\ &- \left(R_{k+1}(z) + C_{k+1}(z)\right)S(z,u_q) - D_{k+1}(z).\end{aligned} \tag{22}$$

where

$$\varphi_{k+1}(z) = (1-q^{k+1})zR_{k+1}(q^{k+1}z). \tag{23}$$

Proof. By induction on $k \in \mathbb{N}$.

For $k=0$ it is the q-difference Eq. (12).

Assume that for an integer $k \geq 0$ we have

$$B_k(z)S(q^{-k}z,u_q)S(z,u_q) + R_k(z)S(q^{-k}z,u_q) + C_k(z)S(z,u_q) + D_k(z) = 0,\ k \in \mathbb{N}. \tag{24}$$

From the Eq. (12), we get

$$S(z,u_q) = -\frac{R_1(z)S(q^{-1}z,u_q) + D_1(z)}{B_1(z)S(q^{-1}z,u_q) + C_1(z)}.$$

Therefore

$$S(q^{-k}z,u_q) = -\frac{R_1(q^{-k}z)S(q^{-k-1}z,u_q) + D_1(q^{-k}z)}{B_1(q^{-k}z)S(q^{-k-1}z,u_q) + C_1(q^{-k}z)}.$$

Based on the previous relation, (24) becomes

$$\begin{aligned}-B_k(z)S(z,u_q)&\frac{R_1(q^{-k}z)S(q^{-k-1}z,u_q) + D_1(q^{-k}z)}{B_1(q^{-k}z)S(q^{-k-1}z,u_q) + C_1(q^{-k}z)} - R_k(z)\frac{R_1(q^{-k}z)S(q^{-k-1}z,u_q) + D_1(q^{-k}z)}{B_1(q^{-k}z)S(q^{-k-1}z,u_q) + C_1(q^{-k}z)} \\ &+ C_k(z)S(z,u_q) + D_k(z) = 0.\end{aligned}$$

Equivalently

$$B_{k+1}(z)S(z,u_q)S(q^{-k-1}z,u_q) + R_{k+1}(z)S(q^{-k-1}z,u_q) + C_{k+1}(z)S(z,u_q) + D_{k+1}(z) = 0, \tag{25}$$

where

$$\begin{aligned}B_{k+1}(z) &= B_1(q^{-k}z)C_k(z) - R_1(q^{-k}z)B_k(z), \\ C_{k+1}(z) &= C_1(q^{-k}z)C_k(z) - D_1(q^{-k}z)B_k(z), \\ R_{k+1}(z) &= B_1(q^{-k}z)D_k(z) - R_1(q^{-k}z)R_k(z), \\ D_{k+1}(z) &= C_1(q^{-k}z)D_k(z) - D_1(q^{-k}z)R_k(z).\end{aligned} \tag{26}$$

Thus we get (17) with (18).

Based on (26), we deduce (20) and (21).

Eq. (22) can be deduced from (17) and Lemma 1. □

Remark 3. Based on Eq. (22), we see that the form u_q is also q^{k+1}-Laguerre-Hahn and its class depends on the integer k with respect to the operator $H_{q^{-k-1}}$. Of course the form $u_{q^{k+1}}$ remains of class $s(u_q)$.

Lemma 2. *When $B=0$ i.e $B_1=0$, we have*

$$M(q^{-k}z)M(q^{-(k-1}z)...M(z) = \begin{pmatrix} (-1)^{k+1}\prod_{n=0}^{k} R_1(q^{-n}z) & 0 \\ A_k(z) & \prod_{n=0}^{k} C_1(q^{-n}z) \end{pmatrix}, k \geq 0, \tag{27}$$

with

$$A_k(z) = -\left(\prod_{n=0}^{k} C_1(q^{-n}z)\right)\sum_{n=0}^{k}(-1)^n\frac{D_1(q^{-n}z)}{R_1(q^{-n}z)}\prod_{\mu=0}^{n}\frac{R_1(q^{-\mu}z)}{C_1(q^{-\mu}z)}, k \geq 0. \tag{28}$$

Proof. Since $B_1 = 0$, from Lemma 1, we get $C_1R_1 \neq 0$.

By induction on k, $k \in \mathbb{N}$.

For $k = 0$, it is (16).

Suppose that for an integer k, we have (27). We may write

$$M(q^{-(k+1)}z)M(q^{-k}z)...M(z) = M(q^{-(k+1)}z)\left(M(q^{-k}z)...M(z)\right),$$

using the hypothesis of the induction and (16), we get

$$\begin{aligned} &M(q^{-(k+1)}z)\left(M(q^{-k}z)...M(z)\right) \\ &= \begin{pmatrix} -R_1(q^{-(k+1)}z) & B_1(q^{-(k+1)}z) \\ -D_1(q^{-(k+1)}z) & C_1(q^{-(k+1)}z) \end{pmatrix}\begin{pmatrix} (-1)^{k+1}\prod_{n=0}^{k} R_1(q^{-n}z) & 0 \\ A_k(z) & \prod_{n=0}^{k} C_1(q^{-n}z) \end{pmatrix} \\ &= \begin{pmatrix} (-1)^{k}\prod_{n=0}^{k+1} R_1(q^{-n}z) & 0 \\ x_k(z) & \prod_{n=0}^{k+1} C_1(q^{-n}z) \end{pmatrix}, \end{aligned}$$

with $x_k(z) = C_1(q^{-k-1}z)A_k(z) + (-1)^k D_1(q^{-k-1}z)\prod_{n=0}^{k} C_1(q^{-n}z)$.

From (28), it follows that

$$\begin{aligned} x_k(z) &= (-1)^k D_1(q^{-k-1}z)\prod_{n=0}^{k} R_1(q^{-n}z) - C_1(q^{-k-1}z)\left(\prod_{n=0}^{k} C_1(q^{-n}z)\right)\sum_{n=0}^{k}(-1)^n\frac{D_1(q^{-n}z)}{R_1(q^{-n}z)}\prod_{\mu=0}^{n}\frac{R_1(q^{-\mu}z)}{C_1(q^{-\mu}z)} \\ &= -\left(\prod_{n=0}^{k+1} C_1(q^{-n}z)\right)\sum_{n=0}^{k+1}(-1)^n\frac{D_1(q^{-n}z)}{R_1(q^{-n}z)}\prod_{\mu=0}^{n}\frac{R_1(q^{-\mu}z)}{C_1(q^{-\mu}z)}. \end{aligned}$$

Hence the desired result. □

Corollary 1. *When u_q is H_q-semi-classical form ($B_1 = 0$), we have*

$$R_{k+1}(z)S(q^{-k-1}z, u_q) + C_{k+1}(z)S(z, u_q) + D_{k+1}(z) = 0, k \in \mathbb{N}, \tag{29}$$

where

$$R_{k+1}(z) = (-1)^k\prod_{n=0}^{k} R_1(q^{-n}z), \tag{30}$$

$$C_{k+1}(z) = \prod_{n=0}^{k} C_1(q^{-n}z), \tag{31}$$

$$D_{k+1}(z) = \left(\prod_{n=0}^{k} C_1(q^{-n}z)\right)\sum_{n=0}^{k}(-1)^n\frac{D_1(q^{-n}z)}{R_1(q^{-n}z)}\left(\prod_{\mu=0}^{n}\frac{R_1(q^{-\mu}z)}{C_1(q^{-\mu}z)}\right), k \in \mathbb{N}. \tag{32}$$

Moreover, the form u_q is also $H_{q^{k+1}}$-semi-classical, satisfying

$$\varphi_{k+1}(z)H_{q^{-k-1}}\left(S(z,u_q)\right) = -\left(R_{k+1}(z)+C_{k+1}(z)\right)S(z,u_q) - D_{k+1}(z), \tag{33}$$

with

$$\varphi_{k+1}(z) = (q^{-k-1}-1)zR_{k+1}(z). \tag{34}$$

Proof. Since $B_1 = 0$, by (26) and Proposition 1, we get successively

$$\begin{pmatrix} B_{k+1}(z) \\ C_{k+1}(z) \end{pmatrix} = \begin{pmatrix} (-1)^{k+1}\prod\limits_{n=0}^{k} R_1(q^{-n}z) & 0 \\ A_k(z) & \prod\limits_{n=0}^{k} C_1(q^{-n}z) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, k \geq 0,$$

and

$$\begin{pmatrix} R_{k+1}(z) \\ D_{k+1}(z) \end{pmatrix} = \begin{pmatrix} (-1)^{k+1}\prod\limits_{n=0}^{k} R_1(q^{-n}z) & 0 \\ A_k(z) & \prod\limits_{n=0}^{k} C_1(q^{-n}z) \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix}, k \geq 0.$$

Which implies (29)-(32) and $B_{k+1} = 0$, $k \geq 0$.
Based on (29) and Proposition 1, we get (33). Hence the desired results. □

In the following we study some examples of H_q-semi-classical and q-Laguerre-Hahn forms.

4. Some examples

4.1. Case where u_q is the q-Chebyshev form of the first kind

Let $u_q := \mathcal{T}_q$, we know that $\mathcal{T}_q$ it is of class $s = 0$ and we have [9]

$$\begin{cases} \gamma_1^{\mathcal{T}_q} = \frac{q}{q+1}, \ \gamma_{n+1}^{\mathcal{T}_q} = \frac{q^{n+1}}{(q^n+1)(q^{n+1}+1)}, \ n \geq 1, \\ (z^2-q^2)H_{q^{-1}}(S(z,\mathcal{T}_q)) + qzS(z,\mathcal{T}_q) = 0. \end{cases} \tag{35}$$

Proposition 2. *For $k \geq 0$, the Stieltjes function $S(.,\mathcal{T}_q)$ satisfies the following q^{k+1}-difference equation*

$$\varphi_{k+}(q^{-k-1}z,\mathcal{T}_q)H_{q^{-k-1}}(S(z,\mathcal{T}_q)) + X_{k+1}(z,\mathcal{T}_q)S(z,\mathcal{T}_q) = 0, \tag{36}$$

where

$$\begin{aligned} &\varphi_{k+1}(z,\mathcal{T}_q) = (z^2,q^2)_{k+1}, \\ &X_{k+1}(z,\mathcal{T}_q) = \frac{1}{(q^{-k-1}-1)z}\left\{(q^{-2}z^2,q^{-2})_{k+1} - (q^{-1}z^2,q^{-2})_{k+1}\right\}, \ k \geq 0. \end{aligned} \tag{37}$$

Furthermore, the form $\mathcal{T}_q$ it is $H_{q^{k+1}}$-semi-classical form of class $2k$.

Proof. From (35) and Lemma 1, we get

$$q^2(1-q^{-2}z^2)S(q^{-1}z,\mathcal{T}_q) - q^2(1-q^{-1}z^2)S(z,\mathcal{T}_q) = 0.$$

Which implies that $S(z,\mathcal{T}_q)$ satisfies (12) with

$$R_1(z) = 1-q^{-2}z^2, C_1(z) = -(1-q^{-1}z^2), \ D_1(z) = 0, \ B_1(z) = 0. \tag{38}$$

Taking into account Corollary 1, we obtain the following q^{k+1}-difference equation

$$R_{k+1}(z)S(q^{-k-1}z,\mathcal{T}_q)+C_{k+1}(z)S(z,\mathcal{T}_q)=0. \tag{39}$$

with

$$\begin{aligned} R_{k+1}(z) &= (-1)^k(q^{-2}z^2,q^{-2})_{k+1},\\ C_{k+1}(z) &= (-1)^{k+1}(q^{-1}z^2,q^{-2})_{k+1},\ k\geq 0. \end{aligned} \tag{40}$$

Based on (39), (40) and Corollary 1, we deduce (36) with (37).
Let c be zero of $\varphi_{2k+2}(x,\mathcal{T}_q)$, then $c^2=q^{-2m}$, $0\leq m\leq k$.
We have $X_{k+1}(cq^{k+1},\mathcal{T}_q)=X_{k+1}(q^{k+1-2m},\mathcal{T}_q)$, using (37) we obtain

$$\begin{aligned} X_{k+1}(cq^{k+1},\mathcal{T}_q) &= \frac{1}{(1-q^{k+1})c}\left\{\prod_{n=0}^{k}(1-q^{2(k-n-m)})-\prod_{n=0}^{k}(1-q^{2(k-n-m)+1})\right\}\\ &= -\frac{\prod_{n=0}^{k}(1-q^{2(n-m)+1})}{(1-q^{k+1})c}\neq 0. \end{aligned}$$

From (7), we can not simplify Eq. (36). Since $\deg(X_{k+1}(x,\mathcal{T}_q))=2k+1$, we deduce that $\mathcal{T}_q$ is $H_{q^{k+1}}$-semi-classical of class $s=\deg(X_{k+1}(x,\mathcal{T}_q))-1=2k$, $k\geq 0$. Hence the desired result. □

4.2. Case where u_q is the q-Chebyshev form of the second kind

Let $u_q:=\mathcal{U}_q$, we know that $\mathcal{U}_q$ it is of class $s=0$ and we have [9]

$$\begin{cases} \gamma_{n+1}^{\mathcal{U}_q}=\frac{q^{n+2}}{(q^{n+1}+1)(q^{n+2}+1)},\ n\geq 0,\\ (z^2-1)H_{q^{-1}}(S(z,\mathcal{U}_q))-zS(z,\mathcal{U}_q)-q-1=0. \end{cases} \tag{41}$$

Proposition 3. *One has*

$$h_{q^{-k-1}}\varphi_{k+1}(z,\mathcal{U}_q)H_{q^{-k-1}}(S(z,\mathcal{U}_q))+X_{k+1}(z,\mathcal{U}_q)S(z,\mathcal{U}_q)+T_{k+1}(z,\mathcal{U}_q)=0, \tag{42}$$

where

$$\begin{aligned} \varphi_{k+1}(z,\mathcal{U}_q) &= (qz^2,q^2)_{k+1},\\ X_{k+1}(z,\mathcal{U}_q) &= \frac{1}{(q^{-k-1}-1)z}\left\{(q^{-1}z^2,q^{-2})_{k+1}-(q^{-2}z^2,q^{-2})_{k+1})\right\},\\ T_{k+1}(z,\mathcal{U}_q) &= \frac{1-q^{-2}}{q^{-k-1}-1}(q^{-2}z^2,q^{-2})_{k+1}\times\sum_{n=0}^{k}\frac{(-1)^nq^{-n}}{1-q^{-2n-1}z^2}\frac{(q^{-1}z^2,q^{-2})_{n+1}}{(q^{-2}z^2,q^{-2})_{n+1}},\ k\geq 0. \end{aligned} \tag{43}$$

In addition, the form $\mathcal{U}_q$ it is $H_{q^{k+1}}$-semi-classical form of class $2k, k\geq 0$.

Proof. Based on (41) and Lemma 1, we get

$$R_1(z)S(q^{-1}z,\mathcal{U}_q)+C_1(z)S(z,\mathcal{U}_q)+D_1(z)=0, \tag{44}$$

with

$$R_1(z)=1-q^{-1}z^2,\ C_1(z)=-(1-q^{-2}z^2),\ D_1(z)=(q^{-2}-1)z. \tag{45}$$

Indeed by virtue of Corollary 1, we get

$$
\begin{aligned}
R_{k+1}(z) &= (-1)^k (q^{-1}z^2, q^{-2})_{k+1}, \\
C_{k+1}(z) &= (-1)^{k+1} (q^{-2}z^2, q^{-2})_{k+1}, \\
D_{k+1}(z) &= (-1)^{k+1}(1-q^{-2}) z (q^{-2}z^2, q^{-2})_{k+1} \times \sum_{n=0}^{k} \frac{q^{-n}}{1-q^{-2n-1}z^2} \frac{(q^{-1}z^2, q^{-2})_{n+1}}{(q^{-2}z^2, q^{-2})_{n+1}}, \; k \geq 0,
\end{aligned} \tag{46}
$$

and

$$R_{k+1}(z) S(q^{-k-1}z, \mathcal{U}_q) + C_{k+1}(z) S(z, \mathcal{U}_q) + D_{k+1}(z) = 0, \; k \geq 0. \tag{47}$$

Taking into account (46), (47) and Corollary 1, we obtain (42) with (43).

Let c be zero of $\phi_{2k+2}(x, \mathcal{U}_q)$, by (43), we get $c^2 = q^{-2m-1}$, $0 \leq m \leq k$.

From (43), we obtain

$$
\begin{aligned}
X_{k+1}(q^{k+1}c, \mathcal{U}_q) &= \frac{-1}{(q^{-k-1}-1)cq^{k+1}} \left\{ (q^2, q^{-2m})_{k+1} - (q^2, q^{-2m+1})_{k+1} \right\} \\
&= \frac{(q^2, q^{-2m+1})_{k+1}}{(q^{-k-1}-1)cq^{k+1}} \neq 0, \; k \geq 0.
\end{aligned}
$$

Based on (7), (42) cannot be simplified. Then the class of $\mathcal{U}_q$ is $2k$, $k \geq 0$ since $\deg(X_{k+1}(x, \mathcal{U}_q)) = 2k+1$ and $\deg(T_{k+1}(x, \mathcal{U}_q)) = 2k$, $k \geq 0$. □

4.3. Case where u_q is the symmetric form of Brenk type

Let $u_q(\omega)$ be the symmetric form of Brenk type which is H_q-semi-classical form of class one [5, pp.20] [8, pp.134]

$$
\begin{cases}
\gamma_{2n+1}^{u_q(\omega)} = q^{-4n-3}(1-\omega q^{2n}), \; \gamma_{2n+2}^{u_q(\omega)} = q^{-4n-5}(1-q^{2n+2}), \; n \geq 0, \\
\omega \neq q, \, \omega \neq q^{2n}, \, n \geq 0, \\
\phi(q^{-1}z) H_{q^{-1}}(S(z, u_q(\omega))) = C(z) S(z, u_q(\omega)) + D(z), \\
\phi(z) = z(z^2 + \omega q^{-3}), \; C(z) = q^{-3}(q-1)^{-1}(qz^2 + \omega - q), \\
D(z) = q^{-1}(q-1)^{-1} z.
\end{cases} \tag{48}
$$

We recall the Jacobi triple product identity [10,11]:

$$\sum_{n=-\infty}^{+\infty} q^{-n^2} z^n = (q^{-2}, -\frac{z}{q}, -\frac{1}{qz}; q^{-2})_\infty, \; | q |> 1, \, z \in \mathbb{C} - \{0\}. \tag{49}$$

Proposition 4. *On has*

$$S(z, u_q(0)) = -\frac{1}{z} \sum_{n=0}^{+\infty} q^{-n^2} \left(\frac{1}{q^2 z^2} \right)^n, \; | q |> 1, \, z \in \mathbb{C} - \{0\}, \tag{50}$$

$$-zS(z, u_q(0)) - \frac{1}{q^2 z} S(\frac{1}{q^2 z}, u_q(0)) - 1 = (q^{-2}, -\frac{1}{q^2 z^2}, -qz^2; q^{-2})_\infty \; | q |> 1, \, z \in \mathbb{C} - \{0\}, \tag{51}$$

$$S(z, u_q(\omega)) = -\frac{1}{z} \sum_{n=0}^{+\infty} (-\omega q^{-3})^n \frac{(\omega^{-1}, q^{-2})_n}{z^{2n}}, \; | q |> 1, \; | z |>| \omega q^{-3} |^{\frac{1}{2}}, \; \omega \neq 0. \tag{52}$$

Proof. From (48), we may write

$$\left\langle H_q \left(x(x^2 + \omega q^{-3}) u_q \right) - (q-1)^{-1} \left(x^2 + (\omega - 1) q^{-3} \right) u_q, x^{2n} \right\rangle = 0, \; n \geq 0.$$

Equivalently

$$(q^{2n}-1)\left\langle u_q, x^{2n-1}(x(x^2+\omega q^{-3})\right\rangle + \left\langle u_q, x^{2n}\left(x^2+(\omega-1)q^{-3}\right)\right\rangle = 0,\ n\geq 0.$$

Thus

$$(u_q(\omega))_{2n+2} = (q^{-2n-3}-\omega q^{-3})(u_q(\omega))_{2n},\ n\geq 0.$$

Since $(u_q(\omega))_0 = 1$, then

$$(u_q(\omega))_{2n+2} = \prod_{k=0}^{n}(q^{-2k-3}-\omega q^{-3}),\ n\geq 0. \tag{53}$$

When $\omega = 0$, by (53), we get

$$(u_q(0))_{2n} = \prod_{k=0}^{n-1} q^{-2k-3} = q^{-n^2-2n},\ n\geq 0. \tag{54}$$

From (54) and the fact that $(u_q(0))_{2n+1} = 0$, $n\geq 0$, we obtain (50).
By (49), we may write

$$\sum_{n=-\infty}^{+\infty}\frac{q^{-n^2}}{(q^2z^2)^n} = (q^{-2}, -\frac{1}{q^3z^2}, -qz^2; q^{-2})_\infty,\ |\ q\ |>1,\ z\in\mathbb{C}-\{0\}.$$

Which is equivalent to

$$\sum_{n=0}^{+\infty}\frac{q^{-n^2}}{(q^2z^2)^n} + \sum_{n=0}^{+\infty} q^{-n^2}(q^2z^2)^n - 1 = (q^{-2}, -\frac{1}{q^3z^2}, -qz^2; q^{-2})_\infty,\ |\ q\ |>1,\ z\in\mathbb{C}-\{0\}, \tag{55}$$

and by (50), we get

$$-zS(z,u_q(0)) + \sum_{n=0}^{+\infty} q^{-n^2}(q^2z^2)^n - 1 = (q^{-2}, -\tfrac{1}{q^3z^2}, -qz^2; q^{-2})_\infty,\ |\ q\ |>1,\ z\in\mathbb{C}-\{0\}.$$

Based on (50), we have $\sum_{n=0}^{+\infty} q^{-n^2}(q^2z^2)^n = -\frac{1}{q^2z}S(\frac{1}{q^2z}, u_q(0))$, which provides (51).
In the case where $\omega\neq 0$, (53), implies that

$$(u_q(\omega))_{2n} = (-q^{-3}\omega)^n(\omega^{-1}, q^{-2})_n,\ n\geq 0.$$

Then, we obtain (52).
Let $a_n = \frac{(-q^{-3}\omega)^n(\omega^{-1},q^{-2})_n}{z^{2n}}$, $n\geq 0$. Then

$$\lim_{n\longrightarrow+\infty}\left|\ \frac{a_{n+1}}{a_n}\ \right| = \lim_{n\longrightarrow+\infty}\frac{|\ \omega q^{-3}\ |}{|\ z\ |^2}(1-\omega^{-1}q^{-2n}) = \frac{|\ \omega q^{-3}\ |}{|\ z\ |^2},\ |\ q\ |>1.$$

By the D'Alembert criterion, we conclude the convergence of the series (52) when $|\ z\ |>|\ \omega q^{-3}\ |^{\frac{1}{2}}$. □

Proposition 5. *The Stieltjes function $S(z,u_q(\omega))$ satisfies the following q^{-k-1} -difference equation for $k\geq 0$*

$$X_{k+1}(z,u_q(\omega))S(q^{-k-1}z,u_q(\omega)) + Y_{k+1}(z,u_q(\omega))S(z,u_q(\omega)) + Z_{k+1}(z,u_q(\omega)) = 0, \tag{56}$$

where

$$X_{k+1}(z,u_q(\omega)) = \prod_{n=0}^{k}\left(\omega+q^{-2n+1}z^2\right),\ Y_{k+1}(z,u_q(\omega)) = -q^{k+1},$$
$$Z_{k+1}(z,u_q(\omega)) = q^{k+1}z\sum_{n=0}^{k}\frac{q^{-2n+1}}{\omega+q^{-2n+1}z^2}\prod_{\mu=0}^{n}\left(\omega+q^{-2\mu+1}z^2\right). \tag{57}$$

Moreover, the form $u_q(\omega)$ it is $H_{q^{k+1}}$-semi-classical form of class $2k+1$, $k\geq 0$ satisfying the following equation:

$$\varphi_{k+1}(q^{-k-1}z,u_q(\omega))H_{q^{-k-1}}\left(S(z,u_q(\omega))\right) = -\left(X_{k+1}(z,u_q(\omega))+Y_{k+1}(z,u_q(\omega))\right)S(z,u_q(\omega)) - Z_{k+1}(z,u_q(\omega)), \tag{58}$$

where

$$\varphi_{k+1}(z,u_q(\omega)) = (1-q^{k+1})zX_{k+1}(q^{k+1}z,u_q(\omega)). \tag{59}$$

Proof. From (48) and Lemma 1, we get

$$-q^{-4}z(qz^2+\omega)S(q^{-1}z,u_q(\omega))+q^{-3}zS(z,u_q(\omega))-q^{-2}z^2=0.$$

Then

$$R_1(z)S(q^{-1}z,u_q(\omega))+C_1(z)S(z,u_q(\omega))+D_1(z)=0, \tag{60}$$

where

$$R_1(z)=qz^2+\omega,\ C_1(z)=-q,\ D_1(z)=q^2z. \tag{61}$$

By virtue of Corollary 1, it follows that

$$R_{k+1}(z)S(q^{-k-1}z,u_q(\omega))+C_{k+1}(z)S(z,u_q(\omega))+D_{k+1}(z)=0,\ k\in\mathbb{N}, \tag{62}$$

where

$$R_{k+1}(z)=(-1)^k\prod_{n=0}^{k}(\omega+q^{-2n+1}z^2),\ C_{k+1}(z)=(-1)^{k+1}q^{k+1},$$
$$D_{k+1}(z)=(-1)^kq^{k+1}z\sum_{n=0}^{k}\frac{q^{-2n+1}}{q^{-2n+1}z^2+\omega}\prod_{\mu=0}^{n}(q^{-2\mu+1}z^2+\omega),\ k\in\mathbb{N}. \tag{63}$$

Taking into account (62) and (63), we obtain (56) with (57).

From (57) and Corollary 1, we get (58).

Let c be a zero of $\varphi_{k+1}(z,u_q(\omega))$, by (34), we have $c=0$, or $X_{k+1}(q^{k+1}z,u_q(\omega))=0$.

If $c=0$, we have $X_{k+1}(0,u_q(\omega))+Y_{k+1}(0,u_q(\omega))=\omega^{k+1}-q^{k+1}\neq 0$. By (7), we can not simplify (58) by the factor z.

In the case where $X_{k+1}(q^{k+1}c,u_q(\omega))=0$, (57) implies that $Y_{k+1}(q^{k+1}c,u_q(\omega))=-q^{k+1}\neq 0$, then, (58) is not simplified by the factor $z-cq^{k+1}$ by virtue of (7).

Moreover, we have $\deg(X_{k+1}+Y_{k+1})=2k+2$, $\deg(Z_{k+1})=2k+1$, then $u_q(\omega)$ $H_{q^{k+1}}$-semi-classical form of class $2k+1$. □

In the following, we study the first associated form $u_q^{(1)}(\omega)$.

Proposition 6. *For $k\geq 0$, the form $u_q^{(1)}(\omega)$ it is q^{k+1}-Laguerre-Hahn form of class $2k+1$ and its Stieltjes function satisfy the following q^{k+1}-Ricati equation*

$$\begin{aligned}&\varphi^{(1)}_{k+1}(q^{-k-1}z,u_q^{(1)}(\omega))H_{q^{-k-1}}(S(z,u_q^{(1)}(\omega)))\\&=B^{(1)}_{k+1}(z,u_q^{(1)}(\omega))S(z,u_q^{(1)}(\omega))S(q^{-k-1}z,u_q^{(1)}(\omega))+X^{(1)}_{k+1}(z,u_q^{(1)}(\omega))S(z,u_q^{(1)}(\omega))+Y^{(1)}_{k+1}(z,u_q^{(1)}(\omega)),\end{aligned} \tag{64}$$

with

$$\begin{aligned}
\varphi^{(1)}_{k+1}(z, u^{(1)}_q(\omega)) =&\gamma_1^{u_q(\omega)}\Big\{\varphi_{k+1}(z, u_q(\omega)) + (q^{k+1}-1)z\big(X_{k+1}(q^{k+1}z, u_q(\omega))\\
&+ Y_{k+1}(q^{k+1}z, u_q(\omega)) - q^{k+1}zZ_{k+1}(q^{k+1}z, u_q(\omega))\big)\Big\},\\
B^{(1)}_{k+1}(z, u^{(1)}_q(\omega)) =& -(\gamma_1^{u_q(\omega)})^2 Z_{k+1}(z, u_q(\omega)),\\
X^{(1)}_{k+1}(z) =&\gamma_1^{u_q(\omega)}\Big\{X_{k+1}(z, u_q(\omega)) + Y_{k+1}(z, u_q(\omega)) - (1+q^{-k-1})zZ_{k+1}(z, u_q(\omega))\Big\},\\
Y^{(1)}_{k+1}(z, u^{(1)}_q(\omega)) =&q^{-k-1}z\Big\{X_{k+1}(z, u_q(\omega)) + Y_{k+1}(z, u_q(\omega)) - zZ_{k+1}(z, u_q(\omega))\Big\} - \varphi_{k+1}(q^{-k-1}z, u_q(\omega)), \quad (65)
\end{aligned}$$

where $X_{k+1}(z, u_q(\omega))$, $Y_{k+1}(z, u_q(\omega))$, $Z_{k+1}(z, u_q(\omega))$ *and* $\varphi_{k+1}(z, u_q(\omega))$ *are the polynomials defined by* (57) *and* (59).

Proof. We know that [12]

$$S(z, u_q(\omega)) = \frac{-1}{\gamma_1^{u_q(\omega)} S(z, u^{(1)}_q(\omega)) + z}. \quad (66)$$

Applying the operator $H_{q^{-k-1}}$ to (66), we get

$$H_{q^{-k-1}}(S(z, u_q(\omega))) = \frac{\gamma_1^{u_q(\omega)} H_{q^{-k-1}}(S(z, u^{(1)}_q(\omega))) + 1}{\left(\gamma_1^{u_q(\omega)} S(z, u^{(1)}_q(\omega)) + z\right)\left(\gamma_1^{u_q(\omega)} S(q^{-k-1}z, u^{(1)}_q(\omega)) + q^{-k-1}z\right)}.$$

Multiplying both sides identities by $\phi(q^{-k-1}z)$, and using (58), we get

$$\begin{aligned}
&-\left(X_{k+1}(z, u_q(\omega)) + Y_{k+1}(z, u_q(\omega))\right) S(z, u_q(\omega)) - Z_{k+1}(z, u_q(\omega))\\
&= \frac{\gamma_1^{u_q(\omega)}\phi(q^{-k-1}z) H_{q^{-k-1}}(S(z, u^{(1)}_q(\omega))) + \phi(q^{-k-1}z)}{\left(\gamma_1^{u_q(\omega)} S(z, u^{(1)}_q(\omega)) + z\right)\left(\gamma_1^{u_q(\omega)} S(q^{-k-1}z, u^{(1)}_q(\omega)) + q^{-k-1}z\right)},
\end{aligned}$$

and by (66), it follows that

$$\begin{aligned}
&\frac{X_{k+1}(z, u_q(\omega)) + Y_{k+1}(z, u_q(\omega))}{\gamma_1^{u_q(\omega)} S(z, u^{(1)}_q(\omega)) + z} - Z_{k+1}(z, u_q(\omega))\\
&= \frac{\gamma_1^{u_q(\omega)}\phi(q^{-k-1}z) H_{q^{-k-1}}(S(z, u^{(1)}_q(\omega))) + \phi(q^{-k-1}z)}{\left(\gamma_1^{u_q(\omega)} S(z, u^{(1)}_q(\omega)) + z\right)\left(\gamma_1^{u_q(\omega)} S(q^{-k-1}z, u^{(1)}_q(\omega)) + q^{-k-1}z\right)}.
\end{aligned}$$

Equivalently

$$\begin{aligned}
\gamma_1^{u_q(\omega)}\phi_{k+1}(q^{-k-1}z) H_{q^{-k-1}}(S(z, u^{(1)}_q(\omega))) =&\gamma_1^{u_q(\omega)}\left(X_{k+1}(z) + Y_{k+1}(z) - zZ_{k+1}(z)\right) S(q^{-k-1}z, u^{(1)}_q(\omega))\\
&- (\gamma_1^{u_q(\omega)})^2 Z_{k+1}(z) S(z, u^{(1)}_q(\omega)) S(q^{-k-1}z, u^{(1)}_q(\omega))\\
&- \gamma_1^{u_q(\omega)} q^{-k-1}zZ_{k+1}(z) S(z, u^{(1)}_q(\omega))\\
&+ q^{-k-1}z\left(X_{k+1}(z) + Y_{k+1}(z) - zZ_{k+1}(z)\right) - \phi_{k+1}(q^{-k-1}z).
\end{aligned}$$

Since $S(q^{-k-1}z, u^{(1)}_q(\omega)) = (q^{-k-1}-1)zH_{q^{-k-1}}(S(z, u^{(1)}_q(\omega))) + S(z, u^{(1)}_q(\omega))$. We deduce deduce (64) with (65).

Let c be a zero of $\varphi^{(1)}_{k+1}(z, u^{(1)}_q)$, and suppose that we can simplify (64) by the factor $z - cq^{k+1}$. From (7), we get

$$\varphi_{k+1}(c, u_q(\omega)) + (q^{k+1} - 1)c + \Big(X_{k+1}(q^{k+1}c, u_q(\omega)) + Y_{k+1}(q^{k+1}c, u_q(\omega)) - q^{k+1}cZ_{k+1}(q^{k+1}c, u_q(\omega))\Big) = 0,$$
$$Z_{k+1}(q^{k+1}c, u_q(\omega)) = 0,$$
$$X_{k+1}(q^{k+1}c, u_q(\omega)) + Y_{k+1}(q^{k+1}c, u_q(\omega)) - (1 + q^{k+1})cZ_{k+1}(q^{k+1}c, u_q(\omega)),$$
$$c\Big\{X_{k+1}(q^{k+1}c, u_q(\omega)) + Y_{k+1}(q^{k+1}c, u_q(\omega)) - q^{k+1}cZ_{k+1}(q^{k+1}c, u_q(\omega))\Big\} - \varphi_{k+1}(c, u_q(\omega)) = 0.$$

Equivalently

$$\phi_{k+1}(c) = 0,\ Z_{k+1}(q^{k+1}c) = 0,\ X_{k+1}(q^{k+1}c) + Y_{k+1}(q^{k+1}c) = 0.$$

Which means that (58) can be simplified by the factor $z - cq^{k+1}$. Which is contradictory by virtue of (7). From (57), we may write for $k \geq 0$

$$\begin{aligned} X_{k+1}(z, u_q(\omega)) &= q^{-k^2+1}z^{2k+2} + ..., \\ Y_{k+1}(z, u_q(\omega)) &= -q^{k+1}, \\ Z_{k+1}(z, u_q(\omega)) &= q^{-k^2+k+2}z^{2k+1} + ..., \\ \varphi_{k+1}(z, u_q(\omega)) &= (1 - q^{k+1})q^{k^2+4k+3}z^{2k+1} + ... \end{aligned} \tag{67}$$

By (65), we have

$$\deg(B^{(1)}_{k+1}, u^{(1)}_q(\omega)) = 2k + 1. \tag{68}$$

Always by (65) and (67) we have $X_{k+1}(z, u_q(\omega)) = -(\gamma_1^{u_q(\omega)})^2 q^{-k^2+k+2}z^{2k+2}....$ Then

$$\deg(X^{(1)}_{k+1})(z, u^{(1)}_q(\omega)) = 2k + 2. \tag{69}$$

From (65) and (69), we deduce that $\deg(Y^{(1)}_{k+1}) \leq 2k + 2$, but $Y^{(1)}_{k+1}$ is an odd polynomial, then

$$\deg(Y^{(1)}_{k+1}) \leq 2k + 1. \tag{70}$$

Finally, by virtue of (67)-(70) and (7), we deduce that the class of $u_q(\omega)$ is $2k + 1$. Hence the desired result. □

Remark 4. When $k = 0$, based on (65), (48), we obtain

$$\varphi^{(1)}_1(z, u^{(1)}_q, \omega)) = \gamma_1^{u_q(\omega)} q^4(1 - q)z(q^{-3} + z^2),\ B^{(1)}_1(z, u^{(1)}_q, \omega)) = -(\gamma_1^{u_q(\omega)})^2 qz,$$
$$X^{(1)}_1(z, u^{(1)}_q, \omega)) = -\gamma_1^{u_q(\omega)}(q - \omega + q^2z^2),\ Y^{(1)}_1(z, u^{(1)}_q, \omega)) = (\omega - 1)z,$$

and by (64), we get

$$\begin{aligned} q^{-1}z(q^{-3} + q^{-2}z^2)H_{q^{-1}}(S(z, u^{(1)}_q(\omega)) =& q^{-5}\frac{1-\omega}{q-1}S(z, u^{(1)}_q(\omega)S(q^{-1}z, u^{(1)}_q(\omega) \\ &+ \frac{q^{-4}}{q-1}(q - \omega + q^2z^2)S(z, u^{(1)}_q(\omega) + \frac{z}{q(q-1)}. \end{aligned} \tag{71}$$

Thus we recover the case studied in [8, pp.134] that $u^{(1)}_q(\omega)$ it is $H_{q^{-1}}$-Laguerre forms of class one. But there are some misprints in formula (4.9) given in [6] and it must be written as (71).

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