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The Mittag-Leffler–Laguerre polynomials and their properties

Maged G. Bin-Saad* **and Waleed K. Mohammed**

Department of Mathematics, University of Aden, Aden, Yemen

* Correspondence: mgbinsaad@yahoo.com

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Abstract: The main goal of this paper is to provide a logical advancement in the mathematical properties and representations related to Mittag-Leffler–Laguerre polynomials. Generating relations, finite summations, integral representations, and integral transforms for these polynomials are established. Some particular cases and consequences of the main results are also considered.

Keywords: symbolic operators, Mittag-Leffler function, Laguerre polynomials, generating functions, finite summations, integral representation, integral transform

MSC: 33C45, 33E12, 26A33.

1. Introduction

The Mittag-Leffler function is an important function that is commonly used in fractional calculus. It is possible to solve a wide range of differential and integral equations involving fractional derivatives by using Mittag-Leffler functions. The Mittag-Leffler function may also be used to address boundary value problems. For further details, we refer to the studies of Kilbas and Saigo [1], Gorenflo and Mainard [2], and Kilbas et al. [3]. The Mittag-Leffler functions of one, two, and three parameters are defined by (see, e.g., [4–7])

$$E_\alpha(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\alpha n+1)},\qquad E_{\alpha,\beta}(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\alpha n+\beta)},\qquad E^{\delta}_{\alpha,\beta}(x)=\sum_{n=0}^{\infty}\frac{(\delta)_n}{\Gamma(\alpha n+\beta)}\frac{x^n}{n!}, \tag{1}$$

where, as usual, $\Gamma(\lambda)$ denotes the Gamma function,

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)},\quad n\geq 0,\quad \lambda\neq 0,-1,-2,\dots,$$

is the Pochhammer symbol [8], and

$$\alpha,\beta,\delta\in\mathbb{C},\qquad \{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta)\}>0.$$

The polynomials

$$L_n^{(\kappa)}(x)=\frac{(\kappa+1)_n}{n!}{}_1F_1\left(-n;\kappa+1;x\right)=\sum_{s=0}^{n}\frac{(-1)^s\,\Gamma(\kappa+n+1)\,x^s}{s!\,(n-s)!\,\Gamma(\kappa+s+1)},\quad \left(\mathrm{Re}(\kappa)\geq 0\right), \tag{2}$$

are called the associated Laguerre polynomials (see [9]), and they can be specified by the generating relation

$$\sum_{n=0}^{\infty}L_n^{(\kappa)}(x)\,t^n=(1-t)^{-\kappa-1}\exp\left(\frac{-xt}{1-t}\right). \tag{3}$$

The generating function (3) holds for

$$|t|<1,\qquad \kappa\in\mathbb{C},\qquad x\in\mathbb{C}.$$

In [10], Dattoli et al. introduced interesting Laguerre polynomials $L_n^{(\kappa)}(x,y)$ of two variables in the form

$$L_n^{(\kappa)}(x,y) = (\kappa+n)! \sum_{s=0}^{n} \frac{(-1)^s\, y^{n-s}\, x^s}{s!\,(n-s)!\,(\kappa+s)!}. \tag{4}$$

Formula (4) is valid under the following conditions:

$$n \in \mathbb{N}_0, \qquad \kappa \in \mathbb{C} \setminus \{-1,-2,-3,\dots\}, \qquad x,y \in \mathbb{C}.$$

Certain novel and well-known special functions can be introduced and studied effectively by using symbolic techniques. The symbolic technique was developed in [11] as a result of this effort. For example, Babusci et al. [12] obtained several lacunary generating functions for the Laguerre polynomials using the symbolic technique. Dattoli et al. [12,13] introduced a symbolic operator $\hat{c}$, which operates on a vacuum function $\phi_z = \frac{1}{\Gamma(z+1)}$ as follows [5,13]:

$$\hat{c}^{\alpha}\phi_z = \frac{1}{\Gamma(z+\alpha+1)}, \qquad z+\alpha \notin \{-1,-2,-3,\dots\}, \tag{5}$$

which clearly satisfies the properties

$$\hat{c}^{\alpha}\hat{c}^{\beta} = \hat{c}^{\alpha+\beta} \quad \text{and} \quad (\hat{c}^{\alpha})^r = \hat{c}^{\alpha r}. \tag{6}$$

Considering Eq. (5), we have

$$\hat{c}^{\alpha}\phi_0 = \frac{1}{\Gamma(\alpha+1)}. \tag{7}$$

A new symbolic approach for studying special functions through the derivation of specific operators, known as symbolic operators, was introduced by Babusci et al. and colleagues in [12]. In their work, Dattoli et al. [14] introduced a symbolic operator denoted by $\hat{d}_{(\alpha,\beta)}$, with $(\alpha,\beta) \in \mathbb{R}^+$. The following equation describes how this operator acts on the vacuum function φ_0:

$$\hat{d}^{k}_{(\alpha,\beta)}\varphi_0 = \frac{\Gamma(k+1)}{\Gamma(\alpha k+\beta)}, \qquad \alpha k+\beta \notin \{0,-1,-2,\dots\}, \tag{8}$$

and

$$\hat{d}^{k+\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\varphi_0 = \frac{\Gamma(k+\delta)}{\Gamma(\alpha k+\beta)}, \tag{9}$$

where $k \in \mathbb{R}$. Notably, when $k=0$, Eq. (8) yields

$$\varphi_0 = \frac{1}{\Gamma(\beta)}. \tag{10}$$

It should be noted that Eq. (9) reduces to Eq. (8) when $\delta = 1$.

The operator given in Eq. (8) satisfies the following properties:

$$\hat{d}^{k}_{(\alpha,\beta)}\hat{d}^{m}_{(\alpha,\beta)} = \hat{d}^{k+m}_{(\alpha,\beta)} \quad \text{and} \quad \left(\hat{d}^{k}_{(\alpha,\beta)}\right)^r = \hat{d}^{kr}_{(\alpha,\beta)}. \tag{11}$$

The Mittag-Leffler function (1) may be represented symbolically using Eqs. (8) and (9) as follows:

$$E^{\delta}_{\alpha,\beta}(x) = e^{x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}}\,\hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}, \tag{12}$$

$$E^{\delta}_{\alpha,\beta}(x) = {}_1F_1\left[\delta;1;x\hat{d}_{(\alpha,\beta)}\right]\varphi_0, \tag{13}$$

where ${}_1F_1$ denotes the confluent hypergeometric function defined by [8]

$$ {}_1F_1[\alpha, \beta; x] = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n} \frac{x^n}{n!}, \qquad \beta \notin \{0, -1, -2, \dots\}, \qquad x \in \mathbb{C}. \tag{14} $$

For the purpose of this work, we recall the definitions of the generalized hypergeometric series and the Fox–Wright function as follows (see [8], pp. 42 and 50):

$$ {}_pF_q \left[\begin{array}{c} \alpha_1, \dots, \alpha_p; \\ \\ \beta_1, \dots, \beta_q; \end{array} z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n}{(\beta_1)_n \dots (\beta_q)_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{\prod_{l=1}^{p} (\alpha_l)_n}{\prod_{j=1}^{q} (\beta_j)_n} \frac{z^n}{n!} $$
$$ = {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z), \tag{15} $$

where

$$ \beta_j \notin \{0, -1, -2, \dots\} \quad (j = 1, \dots, q), \qquad z \in \mathbb{C}, $$

with convergence for all z if $p \leq q$, and for $|z| < 1$ if $p = q + 1$, and

$$ {}_p\Psi_q \left[\begin{array}{c} (\alpha_1, A_1), \dots, (\alpha_p, A_p); \\ \\ (\beta_1, B_1), \dots, (\beta_q, B_q); \end{array} z \right] = \sum_{n=0}^{\infty} \frac{\prod_{l=1}^{p} \Gamma(\alpha_l + A_l n)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j n)} \frac{z^n}{n!}, \tag{16} $$

where

$$ \beta_j + B_j n \notin \{0, -1, -2, \dots\} \quad (j = 1, \dots, q), \qquad z \in \mathbb{C}, $$

and it converges if

$$ \sum_{j=1}^{q} B_j - \sum_{l=1}^{p} A_l > -1. $$

In [13], Dattoli and Torry introduced Laguerre polynomials of two variables in the form

$$ L_n(x, y) = n! \sum_{s=0}^{n} \frac{(-1)^s \, y^{n-s} \, x^s}{(s!)^2 \, (n-s)!}, \qquad n \in \mathbb{N}_0, \quad x, y \in \mathbb{C}, \tag{17} $$

together with the use of operational techniques combined with the principle of monomials, which provided a new analytical tool for deriving solutions of large classes of partial differential equations frequently encountered in physical problems. In addition, two interesting unifications and generalizations of the Laguerre polynomials $L_n(x, y)$ were considered by Dattoli et al. [10] in the forms

$$ {}_1L_{n,\rho}(x, y) = n! \sum_{s=0}^{n} \frac{y^{n-s} \, x^{s-\rho}}{s! \, (n-s)! \, \Gamma(\rho + s + 1)}, \tag{18} $$

where

$$ n \in \mathbb{N}_0, \qquad x, y \in \mathbb{C}, \qquad \rho \in \mathbb{C} \setminus \{-1, -2, -3, \dots\}, $$

so that $\Gamma(\rho + s + 1)$ is well defined for all $s = 0, 1, \dots, n$, and

$$ L_n^{(m)}(x, y) = (m+n)! \sum_{s=0}^{n} \frac{(-1)^s \, y^{n-s} \, x^s}{s! \, (n-s)! \, (m+s)!}, \tag{19} $$

where

$$ n \in \mathbb{N}_0, \qquad x, y \in \mathbb{C}, \qquad m \in \mathbb{C} \setminus \{-1, -2, -3, \dots\}, $$

so that $(m+s)! = \Gamma(m+s+1)$ is well defined for all $s = 0, 1, \dots, n$.

For the present study, we recall the following explicit expression for the Konhauser polynomials $Z_n^{\alpha}(x;k)$ [15]:

$$Z_n^{\alpha}(x;k) = \frac{\Gamma(kn+\alpha+1)}{n!}\sum_{s=0}^{n}(-1)^s \begin{pmatrix} n \\ s \end{pmatrix} \frac{x^{ks}}{\Gamma(ks+\alpha+1)}, \tag{20}$$

where

$$\alpha > -1, \qquad k = 1,2,\ldots.$$

2. Mittag-Leffler–Laguerre polynomials: Basic properties

In this section, we introduce and investigate some basic properties of the Mittag-Leffler–Laguerre polynomials (MLLPs), denoted by ${}_E L_n^{(\kappa,\delta)}(x;\alpha,\beta)$. These polynomials are defined symbolically, based on the description of the corresponding Laguerre polynomials in (2) and the symbolic operator (9), as follows:

$${}_E L_n^{(\kappa,\delta)}(x;\alpha,\beta) = L_n^{(\kappa)}\left(x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)\hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}. \tag{21}$$

Using (2), we obtain from (21) the following symbolic series representation:

$${}_E L_n^{(\kappa,\delta)}(x;\alpha,\beta) = \sum_{s=0}^{n}\frac{(-1)^s\,\Gamma(\kappa+n+1)\,x^s}{s!\,(n-s)!\,\Gamma(\kappa+s+1)}\hat{d}^{s+\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}. \tag{22}$$

Moreover, according to the symbolic operator (9), we can rewrite ${}_E L_n^{(\kappa,\delta)}(x;\alpha,\beta)$ in the following series representation:

$${}_E L_n^{(\kappa,\delta)}(x;\alpha,\beta) = \sum_{s=0}^{n}\frac{\Gamma(\kappa+n+1)\,(-1)^s\,(\delta)_s\,x^s}{s!\,(n-s)!\,\Gamma(\kappa+s+1)\,\Gamma(\alpha s+\beta)}. \tag{23}$$

It may be of interest to point out that the series representation (23), in particular, yields the following relationships:

$${}_E L_n^{(\kappa,2)}(x;1,2) = {}_E L_n^{(\kappa,1)}(x;1,1) = L_n^{(\kappa)}(x), \tag{24}$$

$${}_E L_n^{(\delta-1,\delta)}(x/y;1,m+1) = \frac{(\delta)_n}{y^n\,(m+n)!}L_n^{(m)}(x,y), \qquad m\in\mathbb{N}, \tag{25}$$

$${}_E L_n^{(\delta-1,\delta)}(x^{\alpha};\alpha,\beta+1) = \frac{(\delta)_n}{\Gamma(\alpha n+\beta+1)}Z_n^{\beta}(x;\alpha), \tag{26}$$

$${}_E L_n^{(\delta-1,\delta)}(-x;\alpha,\beta) = \frac{(\delta)_n}{n!}E^{\delta+n}_{\alpha,\beta}(x), \tag{27}$$

$${}_E L_n^{(\delta-1,\delta)}(x;\alpha,\beta) = \frac{(\delta)_n}{n!}E^{-n}_{\alpha,\beta}(x), \tag{28}$$

$$y^n\,{}_E L_n^{(0,1)}(x/y;1,1) = L_n(x,y), \tag{29}$$

$${}_E L_n^{(\delta-1,\delta)}(-x/y;1,\beta+1) = \frac{(\delta)_n\,x^{\beta}}{y^n\,n!}\,{}_1L_{n,\beta}(x,y). \tag{30}$$

In particular, for $\alpha\in\mathbb{N}$, we have the explicit representation

$${}_E L_n^{(\kappa,\delta)}(x;\alpha,\beta) = \frac{(1+\kappa)_n}{n!\,\Gamma(\beta)}\,{}_2F_{1+\alpha}\left[\begin{matrix} -n,\delta; \\ \\ 1+\kappa,\Delta(\alpha;\beta); \end{matrix}\quad \frac{x}{\alpha^{\alpha}}\right], \tag{31}$$

where $\Delta(\alpha;\beta)$ denotes the array

$$\frac{\beta}{\alpha},\frac{\beta+1}{\alpha},\ldots,\frac{\beta+\alpha-1}{\alpha}, \qquad \alpha\geq 1.$$

The main novelty of this work lies in the construction of a unified operational framework that embeds Mittag–Leffler-type Gamma-shift operators into the Laguerre polynomial structure, producing a new class of Mittag–Leffler–Laguerre polynomials. The resulting family simultaneously generalizes several classical

Laguerre-type polynomials and admits a rich hierarchy of generating functions expressed in terms of hypergeometric, Mittag–Leffler, and Wright-type functions within a consistent operator calculus.

3. Generating relations

Here, we establish some generating functions for the MLLPs ${}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta)$ in the form of the following theorems.

Theorem 1. *Let $\kappa \geq 0$ and $\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta)\} > 0$. Then the following generating function holds:*

$$\sum_{n=0}^{\infty} {}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta)\, t^n = (1-t)^{-1-\kappa}\, E_{\alpha,\beta}^{\delta}\left(\frac{-xt}{1-t}\right), \tag{32}$$

where

$$|t|<1, \qquad \kappa\in\mathbb{C}, \qquad x\in\mathbb{C},$$

so that $(1-t)^{-1-\kappa}$ is well defined and the series converges. Alternatively,

$$\sum_{n=0}^{\infty} {}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta)\, t^n = (1-t)^{-1-\kappa}\, {}_1F_1\left[\delta;1;\frac{-xt}{1-t}\,\hat{d}_{(\alpha,\beta)}\right]\varphi_0, \tag{33}$$

where

$$|t|<1, \qquad \kappa\in\mathbb{C}, \qquad x\in\mathbb{C},$$

so that $(1-t)^{-1-\kappa}$ is well defined and the hypergeometric operator series is convergent.

Proof. We have

$$\begin{aligned}(1-t)^{-1-\kappa}\, E_{\alpha,\beta}^{\delta}\left(\frac{-xt}{1-t}\right) &= (1-t)^{-1-\kappa}\sum_{s=0}^{\infty}\frac{(\delta)_s}{s!\,\Gamma(\alpha s+\beta)}\left(\frac{-xt}{1-t}\right)^s \\ &= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s\,(\delta)_s\,(1+\kappa+s)_n\, x^s}{s!\,n!\,\Gamma(\alpha s+\beta)}\, t^{n+s}.\end{aligned} \tag{34}$$

Letting n be replaced by $n-s$ and considering Definition (23), we obtain the desired result (32). Using (20) in (32), we arrive at the result (33). □

Remark 1. Letting $\delta=-m$, $m\in\mathbb{N}$, in (33) and using the relation [16, p. 395, Eq. (10.38)]

$$L_m(x) = {}_1F_1[-m;1;x], \tag{35}$$

leads to the interesting relation

$$\sum_{n=0}^{\infty} {}_EL_n^{(\kappa,-m)}(x;\alpha,\beta)\, t^n = (1-t)^{-1-\kappa}\, L_m\left(\frac{-x\,t}{1-t}\,\hat{d}_{(\alpha,\beta)}\right)\varphi_0, \tag{36}$$

where $L_m(x)$ denotes the Laguerre polynomial [9, p. 200, Eq. (3)].

Theorem 2. *The following generating functions hold:*

$$\sum_{n=0}^{\infty} {}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta)\, t^n = (1-t)^{-1-\kappa}\exp\left(\frac{-x\,t}{1-t}\,\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)\hat{d}^{\,\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}, \qquad |t|<1, \tag{37}$$

and

$$\sum_{n=0}^{\infty} {}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta)\, t^n = (1-t)^{-1-\kappa}\exp\left(x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)E_{\alpha,\beta}^{\delta}\left(\frac{-x}{1-t}\right), \qquad |t|<1. \tag{38}$$

Proof. Using (32), we obtain

$$\sum_{n=0}^{\infty} {}_E L_n^{(\kappa,\delta)}(x;\alpha,\beta)\, t^n = \sum_{n=0}^{\infty} L_n^{(\kappa)}\left(x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right) \hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}^{\delta-1} \left\{\frac{\varphi_0}{\Gamma(\delta)}\right\} t^n. \tag{39}$$

Using (3) on the right-hand side of the above equation gives (37).

Regarding the Lie bracket $[\hat{A},\hat{B}]$, defined by

$$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A},$$

we obtain

$$\left[x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}, \frac{-x}{1-t}\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right] = 0.$$

Consequently, using the Weyl decoupling identity

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{\frac{-k}{2}}, \qquad k = \left[\hat{A},\hat{B}\right], \quad k \in \mathbb{C}, \tag{40}$$

we obtain assertion (38) by applying (40) to the right-hand side of (37) and then using (12) in the resulting equation. □

Theorem 3. *The following generating function holds:*

$$\sum_{n=0}^{\infty} {}_E L_n^{(\kappa,\delta)}(x;\alpha,\beta)\, t^n = (1-t)^{-1-\kappa} \left[1 - \hat{d}_{(\alpha,\beta)}\left(\frac{-t}{1-t}\right) D_x^{-1}\right]^{-\delta} \varphi_0, \tag{41}$$

where

$$|t| < 1, \qquad \kappa \in \mathbb{C}, \qquad \delta \in \mathbb{C},$$

so that $(1-t)^{-1-\kappa}$ *is well defined and the operator series*

$$\left[1 - \hat{d}_{(\alpha,\beta)}\left(\frac{-t}{1-t}\right) D_x^{-1}\right]^{-\delta}$$

is convergent.

Proof. Denote the right-hand side of assertion (41) by I. Then

$$\begin{aligned} I &= (1-t)^{-1-\kappa} \sum_{s=0}^{\infty} \frac{(\delta)_s \left(\frac{-t}{1-t}\right)^s D_x^{-s} \hat{d}_{(\alpha,\beta)}^s\, \varphi_0}{s!} \\ &= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^s (\delta)_s (\kappa+s+1)_n\, D_x^{-s}\hat{d}_{(\alpha,\beta)}^s\, \varphi_0}{s!\, n!} t^{n+s} \\ &= \sum_{n=0}^{\infty}\sum_{s=0}^{n} \frac{(-1)^s (\delta)_s (\kappa+s+1)_{n-s}\, D_x^{-s}\hat{d}_{(\alpha,\beta)}^s\, \varphi_0}{s!\, (n-s)!} t^{n}. \end{aligned} \tag{42}$$

Using (8) and (23), and taking into account that

$$\hat{D}_x^{-\varrho} = \frac{x^\varrho}{\varrho!},$$

we arrive at the desired result (41). □

Theorem 4. *The following generating function holds:*

$$\sum_{n=0}^{\infty} {}_E L_n^{(\kappa,\delta)}(x;\alpha,\beta)\, t^n = (1-t)^{-1-\kappa} \left[1 - \left(\frac{-x\,t}{1-t}\right)\hat{c}^{\alpha}\right]^{-\delta} \hat{c}^{\beta-1}\varphi_0, \tag{43}$$

where

$$|t| < 1, \qquad \kappa, \delta \in \mathbb{C}, \qquad x \in \mathbb{C},$$

so that $(1-t)^{-1-\kappa}$ *is well defined and the operator series*

$$\left[1 - \left(\frac{-x\,t}{1-t}\right)\hat{c}^{\alpha}\right]^{-\delta}$$

is convergent.

Proof. Denote the right-hand side of assertion (43) by I. Then

$$\begin{aligned} I &= (1-t)^{-1-\kappa} \sum_{s=0}^{\infty} \frac{(\delta)_s \left(\frac{-x\,t}{1-t}\right)^s \hat{c}^{\alpha s+\beta-1}\,\phi_0}{s!} \\ &= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^s\,(\delta)_s\,(\kappa+s+1)_n\;x^s\hat{c}^{\alpha s+\beta-1}\,\phi_0}{s!\,n!}\,t^{n+s} \\ &= \sum_{n=0}^{\infty}\sum_{s=0}^{n} \frac{(-1)^s\,(\delta)_s\,(\kappa+s+1)_{n-s}\;x^s\hat{c}^{\alpha s+\beta-1}\,\phi_0}{s!\,(n-s)!}\,t^{n}. \end{aligned} \tag{44}$$

Using (7) and (23), we arrive at the desired result (43). □

Theorem 5. *Let* $\kappa \geq 0$ *and* $\{\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\delta)\} > 0$. *Then the following generating function holds:*

$$\sum_{n=0}^{\infty} \frac{{}_E L_n^{(\kappa,\delta)}(x;\alpha,\beta)\,t^n}{(1+\kappa)_n} = e^t\,{}_0F_1\left(-;1+\kappa;-x\,t\,\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)\hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}, \tag{45}$$

where

$$|t| < 1, \qquad \kappa \notin \{-1,-2,-3,\dots\}, \qquad \delta \in \mathbb{C},$$

so that $(1+\kappa)_n$ *and the hypergeometric operator series are well defined and convergent.*

Proof. Directly from (22), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{{}_E L_n^{(\kappa,\delta)}(x;\alpha,\beta)\,t^n}{(1+\kappa)_n} &= \sum_{n=0}^{\infty}\sum_{s=0}^{n} \frac{(-1)^s\,x^s\,t^n}{s!\,(n-s)!\,(1+\kappa)_s}\hat{d}^{s+\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\} \\ &= \left(\sum_{n=0}^{\infty}\frac{t^n}{n!}\right)\left(\sum_{s=0}^{\infty}\frac{\left(-x\,\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)^s}{s!\,(1+\kappa)_s}\right)\hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}. \end{aligned} \tag{46}$$

Using (15), we obtain the desired result (45). □

From the right-hand side of (45), we are also led to write the following equivalent, although less compact, form:

$$\sum_{n=0}^{\infty} \frac{{}_E L_n^{(\kappa,\delta)}(x;\alpha,\beta)\,t^n}{(1+\kappa)_n} = \Gamma(\kappa+1)\left(xt\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)^{-\frac{\kappa}{2}} e^t\,J_\kappa\left(3\sqrt{xt\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}}\right)\hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}, \tag{47}$$

where $J_\kappa(\cdot)$ is the Bessel function [9, p.108].

Remark 2. If we set $\alpha = \beta = \delta = 1$ in (45) and (47) and use the relation (24), then we obtain the known results [9, p.201, Eqs. (1), (3)].

Theorem 6. *Let c be arbitrary, $\kappa \geq 0$, and $\{\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\delta)\} > 0$. Then the following generating function holds:*

$$\sum_{n=0}^{\infty} \frac{(c)_n \, {}_E L_n^{(\kappa,\delta)}(x;\alpha,\beta)\, t^n}{(1+\kappa)_n} = \frac{1}{(1-t)^c} \, {}_1F_1 \left[\begin{matrix} c; \\ \\ 1+\kappa; \end{matrix} \quad \frac{-x\, t\, \hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}}{1-t} \right] \hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)} \left\{ \frac{\varphi_0}{\Gamma(\delta)} \right\}, \tag{48}$$

where

$$|t| < 1, \qquad \kappa \notin \{-1,-2,-3,\dots\}, \qquad c \in \mathbb{C}, \qquad \delta \in \mathbb{C},$$

so that $(1+\kappa)_n$ is well defined and the hypergeometric operator series is convergent.

Proof. From (22), we carry out the following steps:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(c)_n \, {}_E L_n^{(\kappa,\delta)}(x;\alpha,\beta)\, t^n}{(1+\kappa)_n} &= \sum_{n=0}^{\infty} \sum_{s=0}^{n} \frac{(c)_n \, (-x)^s \, t^n}{s!\,(n-s)!\,(1+\kappa)_s} \hat{d}^{s+\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)} \left\{ \frac{\varphi_0}{\Gamma(\delta)} \right\} \\
&= \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{(c)_{n+s} \, (-x)^s \, t^{n+s}}{s!\,n!\,(1+\kappa)_s} \hat{d}^{s+\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)} \left\{ \frac{\varphi_0}{\Gamma(\delta)} \right\} \\
&= \sum_{s=0}^{\infty} \sum_{n=0}^{\infty} \frac{(c+s)_n \, t^n}{n!} \frac{(c)_s \left(-x\, t\, \hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)^s}{s!\,(1+\kappa)_s} \hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)} \left\{ \frac{\varphi_0}{\Gamma(\delta)} \right\} \\
&= \sum_{s=0}^{\infty} \frac{(c)_s \left(-x\, t\, \hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)^s}{s!\,(1+\kappa)_s\,(1-t)^{c+s}} \hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)} \left\{ \frac{\varphi_0}{\Gamma(\delta)} \right\}.
\end{aligned} \tag{49}$$

Using (15), we obtain the desired result (48). □

Remark 3. *(i)* If we replace c by $1+\kappa$ in (48), then we obtain Eq. (37).

(ii) If we set $\alpha = \beta = \delta = 1$ in (48) and use the relation (24), then we obtain the known result [9, p.202, Eq. (3)].

Theorem 7. *Let $\kappa, \xi \in \mathbb{C}$ with $\mathrm{Re}(\kappa) > -1$ and $|t| < 1$. Then*

$$\sum_{n=0}^{\infty} \frac{(\xi)_n}{(1+\kappa)_n} \, {}_E L_n^{(\kappa,\delta)}(x;\alpha,\beta)\, t^n = \frac{\Gamma(1+\kappa)\,(1-t)^{-\xi}}{\Gamma(\delta)\,\Gamma(\xi)} \, {}_2\Psi_2 \left[\begin{matrix} (\xi,1), (\delta,1); \\ \\ (\beta,\alpha), (1+\kappa,1); \end{matrix} \quad \frac{xt}{t-1} \right], \tag{50}$$

where

$$|t| < 1, \qquad \kappa \notin \{-1,-2,-3,\dots\}, \qquad \xi \in \mathbb{C}, \qquad \delta \in \mathbb{C},$$

so that $(1+\kappa)_n$ is well defined and the Wright-type series is convergent.

Proof. From (23), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(\xi)_n}{(1+\kappa)_n} \, {}_E L_n^{(\kappa,\delta)}(x;\alpha,\beta)\, t^n &= \sum_{n=0}^{\infty} \frac{(\xi)_n}{(1+\kappa)_n} \sum_{s=0}^{n} \frac{(-1)^s \,(\delta)_s \,(1+\kappa)_n \, x^s \, t^n}{s!\,(n-s)!\,(1+\kappa)_s\,\Gamma(\alpha s+\beta)} \\
&= \sum_{n=0}^{\infty} \sum_{s=0}^{n} \frac{(-1)^s \,(\delta)_s \,(\xi)_n \, x^s \, t^n}{s!\,(n-s)!\,(1+\kappa)_s\,\Gamma(\alpha s+\beta)}.
\end{aligned} \tag{51}$$

Using the result [9, p.57, Eq. (3)]

$$\sum_{n=0}^{\infty} \sum_{s=0}^{n} f(s,n) = \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} f(s,n+s), \tag{52}$$

we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(\xi)_n}{(1+\kappa)_n}\,{}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta)\,t^n &= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s\,(\delta)_s\,(\xi)_{n+s}\,x^s\,t^{n+s}}{s!\,n!\,(1+\kappa)_s\,\Gamma(\alpha s+\beta)}\\ &= \sum_{s=0}^{\infty}\frac{(\delta)_s\,(\xi)_s\,(-xt)^s}{\Gamma(\alpha s+\beta)\,(1+\kappa)_s\,s!}\sum_{n=0}^{\infty}\frac{(\xi+s)_n}{n!}\,t^n\\ &= \frac{\Gamma(1+\kappa)\,(1-t)^{-\xi}}{\Gamma(\delta)\,\Gamma(\xi)}\sum_{s=0}^{\infty}\frac{\Gamma(\xi+s)\,\Gamma(\delta+s)\,\left(\frac{-xt}{1-t}\right)^s}{\Gamma(\alpha s+\beta)\,\Gamma(\kappa+s+1)\,s!}. \end{aligned} \tag{53}$$

Using (16), we obtain the desired result (50). □

Theorem 8. *Let* $\mathrm{Re}(\kappa)\geq 0$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$, *and* $\mathrm{Re}(\delta)>-1$. *Then*

$$\sum_{n=0}^{\infty}\frac{\Gamma(\delta)\,{}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta)\,t^n}{\Gamma(\kappa+n+1)} = e^t\,{}_1\Psi_2\left[\begin{matrix}(\delta,1);\\ (\beta,\alpha),(\kappa+1,1);\end{matrix}\;-xt\right]. \tag{54}$$

Proof. The proof follows from the proof of Theorem 7. □

Theorem 9. *Let* $\alpha\in\mathbb{N}$ *and* $\kappa\in\mathbb{C}$ *with* $\mathrm{Re}(\kappa)\geq 0$. *Then*

$$\sum_{n=0}^{\infty}{}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta)\,t^n = \frac{(1-t)^{-\kappa-1}}{\Gamma(\beta)}\,{}_1F_\alpha\left[\begin{matrix}\delta;\\ \Delta(\alpha;\beta);\end{matrix}\;\frac{xt}{\alpha^\alpha(t-1)}\right], \tag{55}$$

where

$$\mathrm{Re}\left(\sum_{j=1}^{\alpha}\frac{\beta+j-1}{\alpha}-\delta\right)>0.$$

Proof. The proof follows from the proof of Theorem 7. □

Theorem 10. *Let*

$$\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta),\mathrm{Re}(\lambda)\}>0$$

and

$$\{\mathrm{Re}(\kappa),\mathrm{Re}(\mu)\}>-1.$$

Then the following bilateral generating function involving the Mittag-Leffler–Laguerre polynomials and a terminating ${}_1\Psi_1$ *holds:*

$$\sum_{n=0}^{\infty}\frac{\Gamma(\delta)\,{}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta)}{\Gamma(\kappa+n+1)}\,{}_1\Psi_1\left[\begin{matrix}(\lambda+n,1);\\ (\mu+n+1,1);\end{matrix}\;-t\right]t^n = {}_2\Psi_3\left[\begin{matrix}(\delta,1),(\lambda,1);\\ (\beta,\alpha),(\kappa+1,1),(\mu+1,1);\end{matrix}\;-xt\right]. \tag{56}$$

Proof. From (54), we have

$$\sum_{n=0}^{\infty}\frac{\Gamma(\delta)\,{}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta)\,t^n}{\Gamma(\kappa+n+1)}\,e^{-t} = \sum_{n=0}^{\infty}\frac{\Gamma(\delta+n)\,(-xt)^n}{\Gamma(\beta+\alpha n)\,\Gamma(\kappa+n+1)\,n!}. \tag{57}$$

Multiplying both sides by $t^{\lambda-1}$ and then applying the operator $\hat{D}_t^{\lambda-\mu-1}$, we get

$$\sum_{n=0}^{\infty}\frac{\Gamma(\delta)\,{}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta)}{\Gamma(\kappa+n+1)}\,\hat{D}_t^{\lambda-\mu-1}\left(e^{-t}\,t^{n+\lambda-1}\right) = \sum_{n=0}^{\infty}\frac{\Gamma(\delta+n)\,(-x)^n}{\Gamma(\beta+\alpha n)\,\Gamma(\kappa+n+1)\,n!}\hat{D}_t^{\lambda-\mu-1}\left(t^{n+\lambda-1}\right). \tag{58}$$

Note that

$$\hat{D}_t^{\lambda-\mu-1}\left(e^{-t}\,t^{n+\lambda-1}\right)=\hat{D}_t^{\lambda-\mu-1}\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}t^{n+\lambda+k-1}\right). \tag{59}$$

After simplification, we get

$$\hat{D}_t^{\lambda-\mu-1}\left(e^{-t}\,t^{n+\lambda-1}\right)=\sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma(n+k+\lambda)}{k!\,\Gamma(n+k+\mu+1)}\,t^{n+k+\mu}. \tag{60}$$

Substituting (60) into (58), we obtain

$$\sum_{n=0}^{\infty}\frac{\Gamma(\delta)\,{}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta)}{\Gamma(\kappa+n+1)}\sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma(n+k+\lambda)}{k!\,\Gamma(n+k+\mu+1)}\,t^{n+k}=\sum_{n=0}^{\infty}\frac{\Gamma(\delta+n)\,\Gamma(n+\lambda)\,(-x)^n}{\Gamma(\beta+\alpha n)\,\Gamma(\kappa+n+1)\,\Gamma(n+\mu+1)\,n!}\,t^n. \tag{61}$$

Using (16), we obtain the desired result (56). □

4. Finite summations

In this section, the following summation formulae for the MLLPs ${}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta)$ are established in the form of the following theorems.

Theorem 11. *For arbitrary κ and ξ with $\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta)\}>0$, we have*

$$ {}_EL_n^{(\kappa+\xi+1,\delta)}(x+y;\alpha,\beta)\,\hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}=\sum_{m=0}^{n}{}_EL_m^{(\kappa,\delta)}(x;\alpha,\beta)\,{}_EL_{n-m}^{(\xi,\delta)}(y;\alpha,\beta). \tag{62}$$

Proof. Using the fact that

$$\begin{aligned}&(1-t)^{-1-\kappa}\exp\left(\frac{-x\,t}{1-t}\,\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)\hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}\\&\times(1-t)^{-1-\xi}\exp\left(\frac{-y\,t}{1-t}\,\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)\hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}\\&=(1-t)^{-1-(\kappa+\xi+1)}\exp\left(\frac{-(x+y)\,t}{1-t}\,\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)\hat{d}^{2\delta-2}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}^2.\end{aligned} \tag{63}$$

Using (37), we get

$$\sum_{n=0}^{\infty}{}_EL_n^{(\kappa+\xi+1,\delta)}(x+y;\alpha,\beta)\,\hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}t^n=\sum_{m=0}^{\infty}{}_EL_m^{(\kappa,\delta)}(x;\alpha,\beta)\,t^m\sum_{n=0}^{\infty}{}_EL_n^{(\xi,\delta)}(y;\alpha,\beta)\,t^n. \tag{64}$$

Comparing the coefficients of t^n gives the desired result (62). □

Remark 4. If, in (62), we set $\alpha=\beta=\delta=1$ and use the relation (24), then we obtain the known result [9, p.209, Eq. (3)].

Theorem 12. *For arbitrary κ with $\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta)\}>0$, we have*

$$ {}_EL_n^{(\kappa,\delta)}\left(xy\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)};\alpha,\beta\right)=\sum_{s=0}^{n}\frac{(1+\kappa)_n\,(1-y)^{n-s}\,y^s\,{}_EL_s^{(\kappa,\delta)}\left(x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)};\alpha,\beta\right)}{(n-s)!\,(1+\kappa)_s}. \tag{65}$$

Proof. Based on Eq. (45) and the relation

$$\begin{aligned}&e^t\,{}_0F_1\left(-;1+\kappa;-xyt\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)\hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}\\&=e^{(1-y)t}e^{yt}\,{}_0F_1\left(-;1+\kappa;-x(yt)\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}\right)\hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\},\end{aligned} \tag{66}$$

we obtain

$$\sum_{n=0}^{\infty} \frac{{}_EL_n^{(\kappa,\delta)}\left(xy\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)};\alpha,\beta\right)t^n}{(1+\kappa)_n} = \left(\sum_{n=0}^{\infty}\frac{(1-y)^n\,t^n}{n!}\right)\left(\sum_{s=0}^{\infty}\frac{{}_EL_s^{(\kappa,\delta)}(x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)};\alpha,\beta)\,(yt)^s}{(1+\kappa)_s}\right). \tag{67}$$

Comparing the coefficients of t^n gives the desired result (65). $\square$

Theorem 13. *For arbitrary κ and c, where c is not zero or a negative integer, with $\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta)\}>0$, we have*

$${}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta) = \frac{(1+\kappa)_n}{(c)_n}\sum_{s=0}^{n}\frac{L_{n-s}^{(3c-\kappa-2)}(x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)})\,(1+\kappa-c)_s\,{}_EL_s^{(\kappa,\delta)}\left(-x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)};\alpha,\beta\right)}{(1+\kappa)_s}. \tag{68}$$

Proof. We know that, for arbitrary c,

$$(1-t)^{-c}\,{}_1F_1\left[\begin{matrix} c; & \\ & \dfrac{-x\,t\,\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}}{1-t} \\ 1+\kappa; & \end{matrix}\right]\hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\} = \sum_{n=0}^{\infty}\frac{(c)_n\,{}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta)\,t^n}{(1+\kappa)_n}. \tag{69}$$

Using Kummer's formula [9, p.125],

$${}_1F_1\left[\begin{matrix} c; & \\ & \dfrac{-x\,t\,\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}}{1-t} \\ 1+\kappa; & \end{matrix}\right] = \exp\left(\frac{-x\,t\,\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}}{1-t}\right){}_1F_1\left[\begin{matrix} 1+\kappa-c; & \\ & \dfrac{x\,t\,\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}}{1-t} \\ 1+\kappa; & \end{matrix}\right]. \tag{70}$$

Using (69) and (70), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(c)_n\,{}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta)\,t^n}{(1+\kappa)_n} \\
&= (1-t)^{-c}\exp\left(\frac{-x\,t\,\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}}{1-t}\right){}_1F_1\left[\begin{matrix} 1+\kappa-c; & \\ & \dfrac{x\,t\,\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}}{1-t} \\ 1+\kappa; & \end{matrix}\right]\hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\} \\
&= (1-t)^{-1-(3c-\kappa-2)}\exp\left(\frac{-x\,t\,\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}}{1-t}\right) \\
&\quad\times(1-t)^{-(1+\kappa-c)}\,{}_1F_1\left[\begin{matrix} 1+\kappa-c; & \\ & \dfrac{x\,t\,\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)}}{1-t} \\ 1+\kappa; & \end{matrix}\right]\hat{d}^{\delta-1}_{(\alpha,\alpha(1-\delta)+\beta)}\left\{\frac{\varphi_0}{\Gamma(\delta)}\right\}.
\end{aligned} \tag{71}$$

By using (3) and (69), we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(c)_n\,{}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta)\,t^n}{(1+\kappa)_n} &= \left[\sum_{n=0}^{\infty}L_n^{(3c-\kappa-2)}(x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)})\,t^n\right] \\
&\quad\times\left[\sum_{s=0}^{\infty}\frac{(1+\kappa-c)_s\,{}_EL_s^{(\kappa,\delta)}\left(-x\hat{d}_{(\alpha,\alpha(1-\delta)+\beta)};\alpha,\beta\right)t^s}{(1+\kappa)_s}\right].
\end{aligned} \tag{72}$$

Comparing the coefficients of t^n gives the desired result (68). $\square$

Remark 5. If, in (68), we set $\alpha=\beta=\delta=1$ and use the relation (24), then we obtain the known result [9, p.210, Eq. (8)].

Theorem 14. *Let* $\text{Re}(\kappa) \geq 0$ *and* $\{\text{Re}(\xi), \text{Re}(\alpha), \text{Re}(\beta), \text{Re}(\delta)\} > 0$. *Then*

$$ {}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta) = \sum_{s=0}^{n} \frac{(\kappa-\xi)_s \, {}_EL_{n-s}^{(\xi,\delta)}(x;\alpha,\beta)}{s!}. \tag{73} $$

Proof. From the right-hand side of (32), we have

$$ (1-t)^{-1-\kappa} E_{\alpha,\beta}^{\delta}\left(\frac{-xt}{1-t}\right) = (1-t)^{-(\kappa-\xi)} (1-t)^{-1-\xi} E_{\alpha,\beta}^{\delta}\left(\frac{-xt}{1-t}\right). \tag{74} $$

Using (32) and the result [9]

$$ \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} f(s,n) = \sum_{n=0}^{\infty}\sum_{s=0}^{n} f(s,n-s), \tag{75} $$

we get

$$ \sum_{n=0}^{\infty} {}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta)\, t^n = \sum_{n=0}^{\infty}\sum_{s=0}^{n} \frac{(\kappa-\xi)_s \, {}_EL_{n-s}^{(\xi,\delta)}(x;\alpha,\beta)\, t^n}{s!}. \tag{76} $$

Comparing the coefficients of t^n gives the desired result (73). □

Remark 6. If, in (73), we set $\alpha = \beta = \delta = 1$ and use the relation (24), then we obtain the known result [9, p.209, Eq. (3)].

Theorem 15. *Let* $\{\text{Re}(\alpha), \text{Re}(\beta), \text{Re}(\delta)\} > 0$ *and* $\text{Re}(\kappa) > -1$. *Then*

$$ {}_EL_n^{(\kappa,\delta)}(xy;\alpha,\beta) = \sum_{s=0}^{n} \frac{(1+\kappa)_n (1-y)^{n-s} y^s \, {}_EL_s^{(\kappa,\delta)}(x;\alpha,\beta)}{(n-s)!\,(1+\kappa)_s}. \tag{77} $$

Proof. In (54), replace x by xy to obtain

$$ \begin{aligned} \sum_{n=0}^{\infty} \frac{\Gamma(\delta)\, {}_EL_n^{(\kappa,\delta)}(xy;\alpha,\beta)\, t^n}{\Gamma(\kappa+n+1)} &= e^t {}_1\Psi_2 \left[\begin{matrix} (\delta,1); & \\ & -xyt \\ (\beta,\alpha),(\kappa+1,1); & \end{matrix} \right] \\ &= e^{(1-y)t} e^{yt} {}_1\Psi_2 \left[\begin{matrix} (\delta,1); & \\ & -x(yt) \\ (\beta,\alpha),(\kappa+1,1); & \end{matrix} \right]. \end{aligned} \tag{78} $$

Once more, by using (54), we arrive at

$$ \sum_{n=0}^{\infty} \frac{\Gamma(\delta)\, {}_EL_n^{(\kappa,\delta)}(xy;\alpha,\beta)\, t^n}{\Gamma(\kappa+n+1)} = \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} \frac{\Gamma(\delta)\,(1-y)^n y^s \, {}_EL_s^{(\kappa,\delta)}(x;\alpha,\beta)\, t^{n+s}}{n!\,\Gamma(\kappa+s+1)}. \tag{79} $$

Using formula (75) and equating the coefficients of t^n gives the desired result (77). □

Remark 7. If, in (65) and (77), we set $\alpha = \beta = \delta = 1$ and use the relation (24), then we obtain the known result [9, p.209, Eq. (5)].

Theorem 16. *Let* $\{\text{Re}(\alpha), \text{Re}(\beta), \text{Re}(\delta)\} > 0$ *and* $\text{Re}(\kappa) > -1$. *Then*

$$ {}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta) = \left(\frac{x}{y}\right)^n \sum_{s=0}^{n} \frac{(1+\kappa)_n \left(\frac{y}{x}-1\right)^s \, {}_EL_{n-s}^{(\kappa,\delta)}(y;\alpha,\beta)}{(1+\kappa)_{n-s}\, s!}. \tag{80} $$

Proof. Letting $t = yz$ in (54), we obtain

$$\sum_{n=0}^{\infty} \frac{\Gamma(\delta)\, {}_E L_n^{(\kappa,\delta)}(x;\alpha,\beta)\,(yz)^n}{\Gamma(\kappa+n+1)} = e^{(yz)}\, {}_1\Psi_2 \left[\begin{array}{cc} (\delta,1); & \\ & -xyz \\ (\beta,\alpha),(\kappa+1,1); & \end{array} \right]. \tag{81}$$

After substituting x for y in (81) and vice versa, we compare the resulting expression with (81) and obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\Gamma(\delta)\, {}_E L_n^{(\kappa,\delta)}(x;\alpha,\beta)\,(yz)^n}{\Gamma(\kappa+n+1)} &= e^{(yz-xz)} \sum_{n=0}^{\infty} \frac{\Gamma(\delta)\, {}_E L_n^{(\kappa,\delta)}(y;\alpha,\beta)\,(xz)^n}{\Gamma(\kappa+n+1)} \\ &= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} \frac{\left(\frac{y}{x}-1\right)^s \Gamma(\delta)\, {}_E L_n^{(\kappa,\delta)}(y;\alpha,\beta)\,(xz)^{n+s}}{s!\,\Gamma(\kappa+n+1)}. \end{aligned} \tag{82}$$

Based on (75), we get

$$\sum_{n=0}^{\infty} \frac{\Gamma(\delta)\, {}_E L_n^{(\kappa,\delta)}(x;\alpha,\beta)\,(yz)^n}{\Gamma(\kappa+n+1)} = \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} \frac{\left(\frac{y}{x}-1\right)^s \Gamma(\delta)\, {}_E L_{n-s}^{(\kappa,\delta)}(y;\alpha,\beta)\,(xz)^{n}}{s!\,\Gamma(\kappa+n-s+1)}. \tag{83}$$

Comparing the coefficients of z^n in the above equation and simplifying gives the desired result (80). □

5. Integral representations

In this section, we present some integral representations for the polynomials ${}_E L_n^{(\kappa,\delta)}(x;\alpha,\beta)$.

Theorem 17. *Let $\kappa, \xi \in \mathbb{C}$ with $\mathrm{Re}(\kappa) > \mathrm{Re}(\xi) > 0$. Then*

$${}_E L_n^{(\kappa,\delta)}(x;\alpha,\beta) = \frac{\Gamma(\kappa+n+1)}{\Gamma(\xi+n+1)\,\Gamma(\kappa-\xi)\,x^\kappa} \int_0^x (x-u)^{\kappa-\xi-1}\, u^{\xi}\, {}_E L_n^{(\xi,\delta)}(u;\alpha,\beta)\, du. \tag{84}$$

Proof. Denoting the right-hand side of (84) by I and using (23), we get

$$I = \frac{\Gamma(\kappa+n+1)}{\Gamma(\xi+n+1)\,\Gamma(\kappa-\xi)\,x^\kappa} \int_0^x (x-u)^{\kappa-\xi-1}\, u^{\xi} \sum_{s=0}^{n} \frac{\Gamma(\xi+n+1)\,(-1)^s\,(\delta)_s\, u^s}{s!\,(n-s)!\,\Gamma(\xi+s+1)\,\Gamma(\alpha s+\beta)}\, du. \tag{85}$$

Interchanging the order of integration and summation gives

$$I = \frac{\Gamma(\kappa+n+1)}{\Gamma(\xi+n+1)\,\Gamma(\kappa-\xi)\,x^\kappa} \sum_{s=0}^{n} \frac{\Gamma(\xi+n+1)\,(-1)^s\,(\delta)_s}{s!\,(n-s)!\,\Gamma(\xi+s+1)\,\Gamma(\alpha s+\beta)} \int_0^x (x-u)^{\kappa-\xi-1}\, u^{\xi+s}\, du. \tag{86}$$

Now, using the correct relation

$$\int_0^x (x-u)^{a-1}\, u^{b-1}\, du = x^{a+b-1} B(a,b), \tag{87}$$

we get

$$\begin{aligned} I &= \frac{\Gamma(\kappa+n+1)}{\Gamma(\kappa-\xi)\,x^\kappa} \sum_{s=0}^{n} \frac{(-1)^s\,(\delta)_s\, x^{\kappa-\xi+\xi+s+1-1}}{s!\,(n-s)!\,\Gamma(\xi+s+1)\,\Gamma(\alpha s+\beta)} B(\kappa-\xi,\xi+s+1) \\ &= \sum_{s=0}^{n} \frac{\Gamma(\kappa+n+1)\,(-1)^s\,(\delta)_s\, x^s}{s!\,(n-s)!\,\Gamma(\kappa+s+1)\,\Gamma(\alpha s+\beta)}, \end{aligned} \tag{88}$$

which gives the desired result (84). □

Remark 8. If, in (84), we set $\alpha = \beta = \delta = 1$ and use (24), then we obtain a known result due to [17, p.642, Eq. (34)].

Theorem 18. *Let $\kappa, \xi \in \mathbb{C}$ with $\mathrm{Re}(\kappa) > \mathrm{Re}(\xi) > -1$. Then*

$$ {}_E L_n^{(\kappa,\delta)}(x-t;\alpha,\beta) = \frac{\Gamma(\kappa+n+1)}{(x-t)^{\kappa}\,\Gamma(\kappa-\xi+n+1)\,\Gamma(\xi)} \int_t^x (x-u)^{\xi-1}\,(u-t)^{\kappa-\xi}\,{}_E L_n^{(\kappa-\xi,\delta)}(u-t;\alpha,\beta)\,du. \tag{89} $$

Proof. The proof follows from the proof of result (84). □

Theorem 19. *Let $\kappa, \alpha, \beta, \delta, \sigma, x \in \mathbb{C}$ with $\mathrm{Re}(\kappa) \geq 0$ and*

$$ \{\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\delta), \mathrm{Re}(\sigma), \mathrm{Re}(x)\} > 0, \qquad m > l, \quad s > l. $$

Then

$$ {}_E L_n^{(\kappa,\delta)}\left(x(m-l)^{\alpha};\alpha,\beta+\sigma\right) = \frac{(m-l)^{1-\sigma-\beta}}{\Gamma(\sigma)} \int_l^m (m-s)^{\sigma-1}(s-l)^{\beta-1}\,{}_E L_n^{(\kappa,\delta)}\left(x(s-l)^{\alpha};\alpha,\beta\right) ds. \tag{90} $$

Proof. Consider the following relation [17, p.803, Eq. (3.4.2)], for $q = 1$:

$$ (m-l)^{\sigma+\beta-1}\, E^{\delta}_{\alpha,\beta+\sigma}\left[z(m-l)^{\alpha}\right] = \frac{1}{\Gamma(\sigma)} \int_l^m (m-s)^{\sigma-1}(s-l)^{\beta-1} E^{\delta}_{\alpha,\beta}\left[z(s-l)^{\alpha}\right] ds. \tag{91} $$

Replacing z by $\dfrac{-xt}{1-t}$ in Eq. (91) and multiplying both sides by $(1-t)^{-\kappa-1}$, we get

$$ \begin{aligned} &(m-l)^{\sigma+\beta-1}\,(1-t)^{-\kappa-1}\,E^{\delta}_{\alpha,\beta+\sigma}\left[\frac{-xt\,(m-l)^{\alpha}}{1-t}\right] \\ &= \frac{1}{\Gamma(\sigma)} \int_l^m (m-s)^{\sigma-1}\,(s-l)^{\beta-1}\,(1-t)^{-\kappa-1}\,E^{\delta}_{\alpha,\beta}\left[\frac{-xt\,(s-l)^{\alpha}}{1-t}\right] ds. \end{aligned} \tag{92} $$

Using (32), we obtain

$$ \begin{aligned} &(m-l)^{\sigma+\beta-1} \sum_{n=0}^{\infty} {}_E L_n^{(\kappa,\delta)}\left(x\,(m-l)^{\alpha};\alpha,\beta+\sigma\right) t^n \\ &= \frac{1}{\Gamma(\sigma)} \int_l^m (m-s)^{\sigma-1}\,(s-l)^{\beta-1} \sum_{n=0}^{\infty} {}_E L_n^{(\kappa,\delta)}\left(x\,(s-l)^{\alpha};\alpha,\beta\right) t^n\, ds. \end{aligned} \tag{93} $$

Interchanging the order of integration and summation gives

$$ \begin{aligned} &(m-l)^{\sigma+\beta-1} \sum_{n=0}^{\infty} {}_E L_n^{(\kappa,\delta)}\left(x\,(m-l)^{\alpha};\alpha,\beta+\sigma\right) t^n \\ &= \frac{1}{\Gamma(\sigma)} \sum_{n=0}^{\infty} \int_l^m (m-s)^{\sigma-1}\,(s-l)^{\beta-1}\,{}_E L_n^{(\kappa,\delta)}\left(x\,(s-l)^{\alpha};\alpha,\beta\right) ds\, t^n. \end{aligned} \tag{94} $$

Comparing the coefficients of t^n gives the desired result (90). □

Moreover, with the help of Euler's integral [8, p.19, Eq. (1)],

$$ \Gamma(z) = \int_0^{\infty} u^{z-1} e^{-u} du, \qquad \mathrm{Re}(z) > 0, \tag{95} $$

we can establish the following theorem.

Theorem 20. *Let $\alpha, \beta, \delta, \kappa, u \in \mathbb{C}$ with*

$$ \{\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\delta), \mathrm{Re}(\kappa)\} > 0. $$

Then

$$ {}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta) = \frac{1}{n!\,\Gamma(\delta)} \int_0^\infty u^{\delta-1} e^{-u} {}_1\Psi_2 \left[\begin{array}{cc} (\kappa+n+1,1); & \\ & -xu \\ (\beta,\alpha),(\kappa+1,1); & \end{array} \right] du. \tag{96} $$

Proof. Using (23) and the following result

$$ (\delta)_m = \frac{1}{\Gamma(\delta)} \int_0^\infty u^{\delta+m-1} e^{-u} du, \tag{97} $$

we obtain

$$ {}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta) = \sum_{s=0}^{n} \int_0^\infty \frac{\Gamma(\kappa+n+1)\,(-1)^s\, x^s u^{\delta+s-1}}{s!\,(n-s)!\,\Gamma(\kappa+s+1)\,\Gamma(\alpha s+\beta)\,\Gamma(\delta)} e^{-u} du. \tag{98} $$

Interchanging the order of integration and summation, and then replacing n by $n+s$ on the right-hand side, we get

$$ {}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta) = \frac{1}{n!\,\Gamma(\delta)} \int_0^\infty \sum_{s=0}^{\infty} \frac{\Gamma(\kappa+n+s+1)\,(-xu)^s}{s!\,\Gamma(\kappa+s+1)\,\Gamma(\alpha s+\beta)} u^{\delta-1} e^{-u} du. \tag{99} $$

Using (16), we obtain (96). □

Additionally, using Hankel's definition of the complex integral [18]

$$ \frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^t\, t^{-z} dt, \qquad (|\arg(z)| \leq \pi), \tag{100} $$

where the Hankel contour $-\infty^{(0_+)}$ consists of the lower edge $\arg(t) = -\pi$, a small circle around 0, and the upper edge $\arg(t) = +\pi$, with the branch cut on $(-\infty, 0]$, and where termwise integration is justified by dominated convergence and absolute integrability along the contour, we establish the following theorems.

Theorem 21. *Let $\kappa, \alpha, \beta, \delta, t \in \mathbb{C}$ with $\mathrm{Re}(\kappa) \geq 0$ and*

$$ \{\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\delta)\} > 0. $$

Then

$$ {}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta) = \frac{(1+\kappa)_n}{2\pi i\, n!} \int_{-\infty}^{(0+)} e^t\, t^{-\beta} {}_2F_1 \left[\begin{array}{cc} -n,\delta; & \\ & xt^{-\alpha} \\ 1+\kappa; & \end{array} \right] dt. \tag{101} $$

Proof. Using (23) and then applying the result (100), we get

$$ {}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta) = \frac{(1+\kappa)_n}{2\pi i\, n!} \sum_{s=0}^{\infty} \int_{-\infty}^{(0+)} \frac{(-n)_s\,(\delta)_s\, x^s}{s!\,(1+\kappa)_s} t^{-(\alpha s+\beta)} e^t\, dt. \tag{102} $$

Interchanging the order of integration and summation and using definition (15), we obtain (101). □

Theorem 22. *Let $\kappa, \alpha, \beta, \delta, t \in \mathbb{C}$ with $\mathrm{Re}(\kappa) \geq 0$ and*

$$ \{\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\delta)\} > 0. $$

Then

$$ {}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta) = \frac{\Gamma(\kappa+n+1)}{2\pi i\, n!\,\Gamma(\beta)} \int_{-\infty}^{(0+)} e^t\, t^{-(\kappa+1)} {}_2F_\alpha \left[\begin{array}{cc} -n,\delta; & \\ & \dfrac{x}{\alpha^\alpha t} \\ \Delta(\alpha;\beta); & \end{array} \right] dt. \tag{103} $$

Proof. Using (23) and then applying the result (100), we get

$$ {}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta) = \frac{\Gamma(\kappa+n+1)}{2\pi i\, n!\,\Gamma(\beta)} \sum_{s=0}^{\infty} \int_{-\infty}^{(0_+)} \frac{(-n)_s\,(\delta)_s\,x^s}{s!\,(\beta)_{\alpha s}} t^{-(\kappa+s+1)} e^t\,dt. \tag{104} $$

Interchanging the order of integration and summation and using definition (15), we obtain (103). □

Theorem 23. *Let $\alpha, \beta, \delta, \kappa, u \in \mathbb{C}$ with*

$$ \{\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\delta), \mathrm{Re}(\kappa)\} > 0. $$

Then

$$ {}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta) = \frac{1}{\Gamma(\delta)} \int_0^\infty e^{-u}\,u^{\delta-1}\,L_n^{(\kappa)}\left(xu\hat{c}^{\alpha}\right) \hat{c}^{\beta-1}\phi_0\,du. \tag{105} $$

Proof. Denote the right-hand side of assertion (105) by I. Using (2), we obtain

$$ I = \frac{1}{\Gamma(\delta)} \int_0^\infty e^{-u}\,u^{\delta-1} \sum_{s=0}^{n} \frac{(-1)^s\,\Gamma(\kappa+n+1)x^s\,u^s\,\hat{c}^{\alpha s+\beta-1}}{s!\,(n-s)!\,\Gamma(\kappa+s+1)} \phi_0\,du. \tag{106} $$

Interchanging the order of integration and summation and then using (7) and (95), we arrive at the desired result (105). □

6. Integral transforms

In this section, we derive some integral transforms for ${}_EL_n^{(\kappa,\delta)}(x;\alpha,\beta)$ by applying the Laplace transform and the Euler transform, also known as the Beta transform.

Theorem 24. *Let $\sigma, \alpha \in \mathbb{N}$, $\kappa, \nu, s \in \mathbb{C}$, $\sigma \le \alpha - 1$, with*

$$ \{\mathrm{Re}(\kappa), \mathrm{Re}(\nu)\} > -1 \quad \text{and} \quad \mathrm{Re}(s) > 0. $$

Then

$$ L\left\{t^{\nu}\,{}_EL_n^{(\kappa,\delta)}\left(xt^{\sigma};\alpha,\beta\right);s\right\} = \frac{(1+\kappa)_n\,\Gamma(1+\nu)}{\Gamma(\beta)\,n!\,s^{1+\nu}}\,{}_{2+\sigma}F_{1+\alpha}\left[\begin{matrix} -n,\delta,\Delta(\sigma;1+\nu); & \\ & \dfrac{\sigma^{\sigma}x}{\alpha^{\alpha}s^{\sigma}} \\ 1+\kappa,\Delta(\alpha;\beta); & \end{matrix}\right]. \tag{107} $$

Proof. We know that the Laplace transform of $f(t)$ is defined as [19]

$$ L\{f(t);s\} = \int_0^\infty e^{-st}\,f(t)\,dt, \qquad \mathrm{Re}(s) > 0. \tag{108} $$

Using (108) and applying definition (23) to the left-hand side of (107), we get

$$ \begin{aligned} L\left\{t^{\nu}\,{}_EL_n^{(\kappa,\delta)}\left(xt^{\sigma};\alpha,\beta\right);s\right\} &= \int_0^\infty e^{-st}\,t^{\nu}\,{}_EL_n^{(\kappa,\delta)}\left(xt^{\sigma};\alpha,\beta\right)dt \\ &= \sum_{m=0}^{n} \frac{\Gamma(\kappa+n+1)\,(-1)^m\,(\delta)_m\,x^m}{m!\,(n-m)!\,\Gamma(\kappa+m+1)\,\Gamma(\alpha m+\beta)} \int_0^\infty e^{-st}\,t^{\sigma m+\nu}dt \\ &= \frac{(1+\kappa)_n}{\Gamma(\beta)\,n!} \sum_{m=0}^{n} \frac{(-n)_m\,(\delta)_m\,x^m}{(\beta)_{\alpha m}\,(1+\kappa)_m\,m!} \frac{\Gamma(1+\nu+\sigma m)}{s^{1+\nu+\sigma m}} \\ &= \frac{(1+\kappa)_n\,\Gamma(1+\nu)}{\Gamma(\beta)\,n!\,s^{1+\nu}} \sum_{m=0}^{n} \frac{(1+\nu)_{\sigma m}\,(-n)_m\,(\delta)_m\left(\frac{x}{s^{\sigma}}\right)^m}{(\beta)_{\alpha m}\,(1+\kappa)_m\,m!}. \end{aligned} \tag{109} $$

Using definition (15), we obtain the desired result (107). □

Remark 9. In particular, for $\nu = 0$, $\sigma = 1$, and $x = 1$, assertion (107) gives

$$L\left\{{}_EL_n^{(\kappa,\delta)}(t;\alpha,\beta);s\right\} = \frac{(1+\kappa)_n}{\Gamma(\beta)\,n!\,s}\,{}_3F_{1+\alpha}\left[\begin{array}{c} 1,-n,\delta; \\ \\ 1+\kappa,\Delta(\alpha;\beta); \end{array} \frac{1}{\alpha^\alpha s}\right]. \tag{110}$$

Theorem 25. *Let $l, m, \kappa \in \mathbb{C}$ and $\alpha, \sigma \in \mathbb{N}$ with*

$$\{\mathrm{Re}(l), \mathrm{Re}(m)\} > 0.$$

Then

$$B\left\{{}_EL_n^{(\kappa,\delta)}\left(tx^\sigma;\alpha,\beta\right);l,m\right\} = \frac{(1+\kappa)_n\,B(l,m)}{\Gamma(\beta)\,n!}\,{}_{2+\sigma}F_{1+\alpha+\sigma}\left[\begin{array}{c} -n,\delta,\Delta(\sigma;l); \\ \\ 1+\kappa,\Delta(\alpha;\beta),\Delta(\sigma;l+m); \end{array} \frac{t}{\alpha^\alpha}\right]. \tag{111}$$

Proof. We know that the Beta transform is defined by [19]

$$B\{f(x):l,m\} = \int_0^1 x^{l-1}(1-x)^{m-1}f(x)\,dx. \tag{112}$$

Using (112) and applying Definition (23) to the left-hand side of Eq. (111), we get

$$\begin{aligned} B\left\{{}_EL_n^{(\kappa,\delta)}\left(tx^\sigma;\alpha,\beta\right);l,m\right\} &= \int_0^1 x^{l-1}(1-x)^{m-1}\,{}_EL_n^{(\kappa,\delta)}\left(tx^\sigma;\alpha,\beta\right)dx \\ &= \sum_{s=0}^{n} \frac{\Gamma(\kappa+n+1)\,(-1)^s\,(\delta)_s\,t^s}{s!\,(n-s)!\,\Gamma(\kappa+s+1)\,\Gamma(\alpha s+\beta)} \int_0^1 x^{\sigma s+l-1}(1-x)^{m-1}dx \\ &= \frac{(1+\kappa)_n}{\Gamma(\beta)\,n!} \sum_{s=0}^{n} \frac{(-n)_s\,(\delta)_s\,t^s}{(\beta)_{\alpha s}\,(1+\kappa)_s\,s!} \frac{\Gamma(\sigma s+l)\,\Gamma(m)}{\Gamma(\sigma s+l+m)} \\ &= \frac{(1+\kappa)_n\,\Gamma(m)\,\Gamma(l)}{\Gamma(\beta)\,n!\,\Gamma(l+m)} \sum_{s=0}^{n} \frac{(-n)_s\,(\delta)_s\,t^s}{(\beta)_{\alpha s}\,(1+\kappa)_s\,s!} \frac{(l)_{\sigma s}}{(l+m)_{\sigma s}}. \end{aligned} \tag{113}$$

Using definition (15), we obtain (111). □

Remark 10. In particular, for $\sigma = 1$ and $l = 1+\kappa$, from (111) we infer that

$$B\left\{{}_EL_n^{(\kappa,\delta)}\left(tx;\alpha,\beta\right);1+\kappa,m\right\} = \frac{(1+\kappa)_n\,B(1+\kappa,m)}{\Gamma(\beta)\,n!}\,{}_2F_{1+\alpha}\left[\begin{array}{c} -n,\delta; \\ \\ 1+\kappa+m,\Delta(\alpha;\beta); \end{array} \frac{t}{\alpha^\alpha}\right], \tag{114}$$

which, on using (31), gives

$$B\left\{{}_EL_n^{(\kappa,\delta)}(xt;\alpha,\beta);1+\kappa,m\right\} = \frac{\Gamma(\kappa+n+1)\,\Gamma(m)}{\Gamma(\kappa+n+m+1)}\,{}_EL_n^{(\kappa+m,\delta)}(t;\alpha,\beta). \tag{115}$$

If we take $m = 1$, we obtain

$$B\left\{{}_EL_n^{(\kappa,\delta)}(xt;\alpha,\beta);1+\kappa,1\right\} = \frac{1}{\kappa+n+1}\,{}_EL_n^{(\kappa+1,\delta)}(t;\alpha,\beta). \tag{116}$$

Replacing $1+\kappa$ by κ in (116), we get

$${}_EL_n^{(\kappa,\delta)}(t;\alpha,\beta) = (\kappa+n)B\left\{{}_EL_n^{(\kappa-1,\delta)}(xt;\alpha,\beta);\kappa,1\right\}. \tag{117}$$

Remark 11. Eq. (117) can be written in the form

$$ {}_E L_n^{(\kappa,\delta)}(t;\alpha,\beta) = (\kappa+n)\int_0^1 x^{\kappa-1}\,{}_E L_n^{(\kappa-1,\delta)}(xt;\alpha,\beta)\,dx. \quad (118) $$

7. Conclusion

In this paper, we have developed and studied a new class of Mittag–Leffler–Laguerre polynomials constructed through a symbolic operator framework. The main objective was to extend classical Laguerre polynomials by incorporating Mittag–Leffler-type structures, thereby enriching their analytical properties and providing a unified setting that connects special functions arising in fractional calculus and operational analysis.

We derived several fundamental properties of the introduced polynomials, including generating relations, explicit finite summation formulas, integral representations, and integral transforms. These results demonstrate that the proposed family preserves the structural richness of both Mittag–Leffler functions and Laguerre polynomials while allowing additional flexibility through the operational approach.

The main novelties of this work can be summarized as follows:

- Introduction of a new Mittag–Leffler–Laguerre polynomial family using a symbolic, or umbral-type, operator approach.
- Construction of unified generating functions that generalize classical Laguerre polynomial generating relations.
- Derivation of new finite summation formulas and explicit series representations.
- Establishment of integral representations and integral transform formulas associated with the new polynomials.
- Recovery of several known classical and generalized Laguerre-type and Mittag–Leffler-type results as special or limiting cases.

Comparison with the existing literature shows that most previous works treat Mittag–Leffler functions and Laguerre polynomials either separately or through partial extensions involving fractional parameters or deformation techniques. For instance, classical Laguerre-type generalizations focus mainly on orthogonal polynomial structures, while Mittag–Leffler generalizations are typically developed within the framework of fractional calculus and special function theory.

In contrast, the present work provides a unified operational framework that simultaneously incorporates both structures. The proposed Mittag–Leffler–Laguerre polynomials extend previous models by embedding Mittag–Leffler-type behavior into Laguerre polynomial systems through symbolic operators. This leads to richer analytic representations and a broader class of transform and summation identities that are not available in earlier studies.

Therefore, the introduced framework not only generalizes several known results but also offers a flexible tool for further investigations in special functions, operational calculus, and applications related to fractional differential equations.

Finally, the obtained results suggest several possible directions for future work, including orthogonality analysis, asymptotic behavior, and applications in fractional and integral equations.

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